School of Mathematics Spring 2020

QUANTUM COMPUTATION

Exercise sheet 5

Ashley Montanaro, University of Bristol
ashley.montanaro@bristol.ac.uk

1. **Pauli matrices.**

   (a) Show that any 2×2 matrix M can be written as $M = \alpha I + \beta X + \gamma Y + \delta Z$ for some coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

   (b) For $s \in \{I, X, Y, Z\}^n$, let σ_s denote the matrix which is a tensor product of the corresponding Pauli matrices, $\sigma_s = s_1 \otimes s_2 \otimes \cdots \otimes s_n$. Using part (a), show that any $2^n \times 2^n$ matrix M can be written as

   $$M = \sum_{s \in \{I,X,Y,Z\}^n} \alpha_s \sigma_s$$

   for some coefficients $\alpha_s \in \mathbb{C}$.

   (c) Show that if M is Hermitian, we can assume that $\alpha_s \in \mathbb{R}$. [Hint: consider $\frac{1}{2}(M + M^\dagger)$.]

2. **More efficient quantum simulation.**

   (a) Let A and B be Hermitian operators with $\|A\| \leq \delta$, $\|B\| \leq \delta$ for some $\delta \leq 1$. Show that

   $$e^{-iA/2} e^{-iB} e^{-iA/2} = e^{-i(A+B)} + O(\delta^3)$$

   (this is the so-called *Strang splitting*). Use this to give a more efficient quantum algorithm for simulating k-local Hamiltonians than the algorithm discussed in the lecture, and calculate its complexity.

   (b) Let H be a Hamiltonian on n qubits which can be written as $H = UDU^\dagger$, where U is a unitary matrix that can be implemented by a quantum circuit running in time poly(n), and $D = \sum_x d(x)|x\rangle\langle x|$ is a diagonal matrix such that the map $|x\rangle \mapsto e^{-id(x)t}|x\rangle$ can be implemented in time poly(n) for all x. Show that e^{-iHt} can be implemented in time poly(n).

3. **The amplitude damping channel.** The amplitude damping channel $\mathcal{E}_{\text{AD}}$ has Kraus operators (with respect to the standard basis)

   $$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

   for some γ.

(a) What is the result of applying the amplitude damping channel to the pure state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?

(b) Show that, when applied to the Pauli matrices X, Y, Z, $\mathcal{E}_{\mathrm{AD}}$ rescales each one by a factor depending on γ, and determine what these factors are.

(c) Hence determine the representation of the amplitude-damping channel as an affine map $v \mapsto Av + b$ on the Bloch sphere.

(d) What does this channel "look like" geometrically in terms of its effect on the Bloch sphere?

4. **General quantum channels.**

(a) Given two channels $\mathcal{E}_1$, $\mathcal{E}_2$, with Kraus operators $\{E_k^{(1)}\}$, $\{E_k^{(2)}\}$, what is the Kraus representation of the composite channel $\mathcal{E}_2 \circ \mathcal{E}_1$ which is formed by first applying $\mathcal{E}_1$, then applying $\mathcal{E}_2$?

(b) Determine a Kraus representation for the channel Tr which maps $\rho \mapsto \operatorname{tr} \rho$ for a mixed quantum state ρ in d dimensions.

(c) Let $\mathcal{E}$ and $\mathcal{F}$ be quantum channels with d Kraus operators each, E_k and F_k (respectively), such that for all j, $F_j = \sum_{k=1}^{d} U_{jk} E_k$ for some unitary matrix U. Show that $\mathcal{E}$ and $\mathcal{F}$ are actually the same quantum channel.