AGING UNCOUPLED CONTINUOUS TIME RANDOM WALK LIMITS

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Abstract. Aging is a prevalent phenomenon in physics, chemistry and many other fields. In this paper we consider the aging process of uncoupled Continuous Time Random Walk Limits (CTRWLs) which are Levy processes time changed by the inverse stable subordinator of index $0 < \alpha < 1$. We apply a recent method developed by Meerscheart and Straka of finding the finite dimensional distributions of CTRWL, to obtaining the aging process's finite dimensional distributions, self-similarity-like property, asymptotic behavior and its Fractional Fokker-Planck equation(FFPE).

1. Introduction

Continuous time random walks(CTRW) are widely used in physics and mathematical finance to model a random walk for which the waiting times between jumps are random which in many cases better describes phenomena in these fields. CTRWLs are used to model anomalous diffusion, where the squared averaged distance of the process from the origin is no longer proportional to the time index t. A related concept and widely studied ([36, 32]) in statistical physics, is aging. Suppose the CTRW X_t starts at $t = 0$ and evolves until time $t_0 > 0$ when we then start to measure it. One can consider the varying dynamics of the new process $X_t^{t_0} = X_{t+t_0} - X_{t_0}$ as t_0 varies and the process ages. In [26] Monthus and Bouchaud studied a CTRW with aging properties. In [6] Barkai and Cheng considered the Aging Continuous Time Random Walk (ACTRW) which is an uncoupled CTRW with iid power law waiting times, that started at $t = 0$ and is observed at $t = t_0$. They found the one dimensional distribution of the process $X_t^{t_0}$ which they referred to as the ACTRW, for t_0 and t large. In [5], Barkai found the Fractional Fokker-Planck Equation (FFPE) for the unnormalized pdf of the process $X_t^{t_0}$ for t_0 and t large.

In this paper we wish to give analogous results to the ones given in [6, 5] as well as new ones for a large class of CTRWLs which hopefully will lay the foundation for further study of their aging. We consider the class that consists of all processes of the form $Y_t = A_{E_t}$ where A_t is a Levy process that is time changed by the inverse of an independent stable subordinator of index $0 < \alpha < 1$; we denote this class by $\mathcal{S}$. We denote the aging process by $Y_t^{t_0} = Y_{t+t_0} - Y_{t_0} = A_{E_{t+t_0}} - A_{E_{t_0}}$ (note that $Y_t^0 = Y_t$). Section 2 is devoted to a brief review of the theory and method introduced by Meerschaert and Straka in [25] and [23] upon which we base our results. In Section 3 we give the main result of this paper, that the finite dimensional distributions of the process $Y_t^{t_0}$ can be obtained by a convolution in time of the finite dimensional distributions of Y_t and a generalized beta prime distribution. The self-similarity-like property of the process $Y_t^{t_0}$ is obtained in Section 4. In Section 5 we obtain results on the asymptotic behavior of the distribution of $Y_t^{t_0}$ when t_0 is far from the origin as well as when $\alpha \to 1$ and the governing equation of $Y_t^{t_0}$.

One example for a process that lies in $\mathcal{S}$ is the Fractional Poisson Process(FPP) which we denote by N_t^{α}. The FPP is a renewal process with interarrival times W_n such that $P(W_1 > t) = E_\alpha(-\lambda t^\alpha)$ where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

is the Mittag-Leffler function. Since the interarrival times are not exponentially distributed the process $N_t^\alpha = \sup\{k : T_k \le t\}$, where $T_k = \sum_{i=1}^{k} W_i$ are the arrival times, is not Markovian and the calculation of the finite dimensional distributions of N_t^α is no longer straightforward. The FPP was first studied in [16],[13] and [17, 18]. In [8] an integral representation of the one dimensional distribution of the FPP was given and was used in [28] to find and simulate the finite dimensional distributions of the FPP. In [19], it was shown that $N_t^\alpha = N_{E_t}$ where N_t is a Poisson process and E_t is the inverse of a standard stable subordinator of index $0 < \alpha < 1$ independent of N_t.

Since the distribution of the increments (and therefore the aging process) of the CTRWL is closely related to the two dimensional distributions, their study is quite cumbersome. In a recent paper ([25]), Meerscheart and Straka found a way of embedding CTRWLs in a larger state space that renders these processes Markovian. We use this method to find the finite dimensional distributions of the process $Y_t^{t_0}$, its asymptotic behavior, self-similarity-like property and its FFPE.

2. Finite dimensional distribution of CTRWL

CTRWL are usually not Markovian, a fact that makes the calculation of their finite dimensional distributions quite difficult. It is therefore that the distribution of the increments (which can be obtained by the finite dimensional distributions) of the CTRWL is not well understood.
Although the method in [25] is very general we focus only on uncoupled CTRWLs which are Levy processes time changed by the inverse of an independent stable subordinator. In order to facilitate reading of this section and referring to the original paper we retain most of the notation in [25]. The uncoupled CTRW we consider consist of two independent sequences of iid r.vs, $\{W_n^c\}$ and $\{J_n^c\}$. The parameter c is the convergence parameter as in [20] which allows us to construct infinitesimal triangular arrays. Here, $\{J_n^c\}$ represents the size of the jumps of a particle in space, while $\{W_n^c\}$ represents the waiting times between jumps. Hence, the time elapsed by the particle's k'th jump is $T_k^c = D_0^c + \sum_{i=1}^{k} W_i^c$ and the position of the particle is $S_k^c = A_0^c + \sum_{i=1}^{k} J_i^c$. Let $L_t^c = \sup\{k : T_k^c \le t\}$ be the number of jumps until time t, then the CTRW Y_t^c is

$$Y_t^c = A_0^c + \sum_{i=1}^{L_t^c} J_i^c.$$

Assume we have

$$\left(S_{[cu]}^c, T_{[cu]}^c\right) = (A_0^c, D_0^c) + \sum_{i=1}^{[cu]} (J_i^c, W_i^c) \Rightarrow (A_u, D_u) \tag{2.1}$$

where $\Rightarrow$ denotes convergence in the Skorokhod J_1 topology. In this paper we assume D_u is a stable subordinator of index $0 < \alpha < 1$ starting from D_0, i.e, $E\left(e^{-s(D_u - D_0)}\right) = e^{-uCs^\alpha}$, where C is a constant. This can be achieved by assuming $W_i^c = c^{-\frac{1}{\alpha}} W_i$ where $\{W_i\}$ are independent random variables that are in the strict domain of attraction of $D_1 - D_0$. Note that $A_t - A_0$ is a Lévy process as it is the limit of a triangular array. Now, let $E_t = \inf\{s : D_s > t\}$ be the first hitting time of D_t, also called the inverse of D_t. By [34, Theorem 2.4.3] applied to the case of independent space and time jumps we have

$$Y_t^c \Rightarrow Y_t = A_{E_t}, \tag{2.2}$$

as $c \to \infty$ where convergence is in the Skorokhod J_1 topology, see also [35, theorem 3.6] and [14, Theorem 3.1]. Since (S_k^c, T_k^c) is a Markov chain for all $c > 0$ it follows that the CTRWL Y_t is a semi-Markov process and it is possible to embed it in a process of larger state space that includes the time to regeneration, the *remaining life time process* R_t. More precisely, let $\mathbb{D}\left([0,\infty),\mathbb{R}^2\right)$ be the space of càdlàg functions $f : [0,\infty) \to \mathbb{R}^2$ with the J_1 Skorokhod topology which is endowed with transition operators T_u, $u > 0$ and hence a probability measure $P^{\chi,\tau}$ such that trajectories start at point (χ,τ) with probability one. Thus, we have a stochastic basis $\left(\Omega, \mathcal{F}_\infty, (\mathcal{F}_u)_{u\geq 0}, P^{\chi,\tau}\right)$, where each element of Ω is in $\mathbb{D}\left([0,\infty),\mathbb{R}^2\right)$, $\mathcal{F}_u = \sigma\left(\left(A_u(\omega), D_u(\omega)\right)\right)$ and $\mathcal{F}_\infty = \vee_{u>0}\mathcal{F}_u$. The process $(A,D)_t$ has a generator of the form

$$\begin{aligned} \mathcal{A}(f)(x,t) &= b\frac{\partial f(x,t)}{\partial x} - \frac{1}{2}a\frac{\partial^2 f(x,t)}{\partial x^2} \\ &\quad + \int_{\mathbb{R}^2} \left(f(x+y,t+w) - f(x,t) - y\frac{\partial f(x,t)}{\partial x}1_{\{|(y,w)|<1\}}\right) K(dy,dw), \end{aligned} \tag{2.3}$$

where $a > 0$ and $b \in \mathbb{R}$ and $K(dy,dw)$ is a Lévy measure. The *occupation time measure* of the process $(A,D)_t$ is the average time spent by the process in a given Borel set in $\mathbb{R}^2$, i.e

$$\int f(x,t)\, U^{\chi,\tau}(dx,dt) = \mathbb{E}^{\chi,\tau}\left(\int_0^\infty f(A_u, D_u)\, du\right) = \int_0^\infty T_u f(\chi,\tau)\, du.$$

Let us now define the *remaining life time process* R_t

$$R_t = D_{E_t} - t,$$

which is the time left for the process Y_t to leave its current state. It was proven in [25, Theorem 2.3] that

$$\begin{aligned} &E^{\chi,\tau}\left(f(Y_t,R_t)\right) = \\ &\qquad \int_{x\in\mathbb{R}} \int_{s\in[\tau,t]} U^{\chi,\tau}(dx,ds) \int_{y\in\mathbb{R}} \int_{w\in[t-s,\infty)} K(dy,dw)\, f(x+y, w-(t-s)). \end{aligned} \tag{2.4}$$

In [25], a more general CTRWL is considered and hence a more general form of (2.3) where the coefficients a and b as well as the Lévy measure $K(dy,dw)$ are allowed to be dependent on the position of the CTRWL in space and time, that is, we have $b(x,t)$, $a(x,t)$ and $K(x,t;dy,dw)$. As was noted in [25, section 4], when these coefficients do not depend on t (as in our case), the process (Y_t,R_t) is a homogeneous Markov process. More precisely, we define

$$Q_t[f](y,0) = E^{y,0}\left(f(Y_t,R_t)\right) \tag{2.5}$$

$$Q_t[f](y,r) = 1_{\{0\leq t<r\}} f(y, r-t) + 1_{\{0\leq r\leq t\}} Q_{t-r}[f](y,0) \qquad r > 0, \tag{2.6}$$

for every f bounded and measurable on $\mathbb{R}\times[0,\infty)$. Q_t is the transition operator of the Markov process (Y_t,R_t) starting at χ,τ, i.e

$$E^{\chi,\tau}\left(f(Y_{t+h},R_{t+h}) \mid \sigma\left((Y_r,R_r), t\geq r\geq 0\right)\right) = Q_h[f](Y_t,R_t). \tag{2.7}$$

One can use the Chapman-Kolmogorov's equation to obtain the finite dimensional distributions of the process Y_t. For example, suppose $(Y_0, R_0) = (0,0)$ a.s, then for the two dimensional distribution of the process Y_t at times $t_1 < t_2$ we have

$$\begin{aligned} P\left(Y_{t_1} \in B_1, Y_{t_2} \in B_2\right) &= P\left(\left(Y_{t_1} \in B_1, R_{t_1} \in [0,\infty)\right), \left(Y_{t_2} \in B_2, R_{t_2} \in [0,\infty)\right)\right) \\ &= Q_{t_1}\left[1_{\{B_1\times\mathbb{R}\}}\left(y_1, r_1\right) Q_{t_2-t_1}\left[1_{\{B_2\times\mathbb{R}\}}\left(y_2, r_2\right)\right]\left(y_1, r_1\right)\right](0,0), \end{aligned} \tag{2.8}$$

where $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ are Borel sets.

Remark 1. In [25] a result stronger than (2.7) was shown. Indeed, the process (Y_t, R_t) is a strong Markov process with respect to a filtration larger than the natural filtration. For the sake of brevity and the fact that the Markov property is adequate for our work we brought the result in a weaker form.

3. Aging

Let us assume (2.1) holds with $\chi = \tau = 0$ so A_t is a Levy process with CDF $P_t(x) = P(A_t \in (-\infty, x])$ and with Levy triplet (μ, A, ϕ), i.e

$$E\left(e^{iuA_t}\right) = \exp\left[t\left(i\mu u - \frac{1}{2}Au^2 + \int_{\mathbb{R}}\left(e^{iuy} - 1 - iuy1_{\{|y|<1\}}\right)\phi(dy)\right)\right].$$

Also assume D_t is a stable subordinator of index $0 < \alpha < 1$ with Laplace transform (LT) $E\left(e^{-uD_t}\right) = e^{-tcu^\alpha}$ independent of A_t. Then (2.3) holds with $b = \mu$, $a = A$ and (see [7, Corollary 2.3])

$$K(dy, dw) = \phi(dy)\,\delta_0(dw) + \delta_0(dy)\frac{c\alpha}{\Gamma(1-\alpha)}w^{-1-\alpha}1_{\{w>0\}}dw. \tag{3.1}$$

Next, we wish to find the occupation measure of the process $(A, D)_t$. We have for $f(y,w) = 1_{\{(-\infty,x]\times(-\infty,t]\}}(y,w)$

$$\begin{aligned} \int f(y,w)\,U^{\chi,\tau}(dy,dw) &= \mathbb{E}^{\chi,\tau}\left(\int_0^\infty f(A_u, D_u)\,du\right) \\ = \int_0^\infty T_u f(\chi,\tau)\,du &= \int_0^\infty T_u 1_{\{(-\infty,x]\times(-\infty,t]\}}(\chi,\tau)\,du \\ &= \int_0^\infty \int_{w\in\mathbb{R}}\int_{y\in\mathbb{R}} 1_{\{(-\infty,x]\times(-\infty,t]\}}(y+\chi, w+\tau)\,q_u(dy,dw)du, \end{aligned}$$

where q_t is the distribution of the process $(A,D)_t$ cf. [1, Eq. 3.11]. By independence of A_t and D_t we have

$$\int f(x,t)\,U^{\chi,\tau}(dx,dt) = \int_0^\infty P(A_u \in (-\infty, x-\chi])\,P(D_u \in (-\infty, t-\tau])\,du$$

$$= \int_0^\infty P_u(x-\chi) \int_{-\infty}^{t-\tau} g(w,u)\,dw\,du, \tag{3.2}$$

where $g(x,t)$ is the pdf of D_t, i.e $g(x,t)\,dx = P(D_t \in dx)$ and is known to be absolutely continuous with respect to the Lebesgue measure [38, Section 2.4].

Since $(A,D)_t$ is a Levy process the coefficients in (2.3) are independent of t and therefore the process $(A,D)_t$ is a Markov additive process [25, Section 4] and the occupation measure is of the form

$$U^y(dx,dt) = \int_0^\infty P_u(dx-y)\,g(t,u)\,du\,dt. \tag{3.3}$$

Furthermore, one may choose $\tau = 0$ and plug (3.1) and (3.3) in (2.4) to obtain

(3.4)

$$E^{\chi,0}(f(Y_t,R_t)) = \int_{x\in\mathbb{R}} \int_{s\in[0,t]} \left(\int_{u\in\mathbb{R}^+} P_u(dx-\chi)\,g(s,u)\,du \right)$$
$$\times \int_{y\in\mathbb{R}} \int_{w\in[t-s,\infty)} \left(\phi(dy)\,\delta_0(dw) + \delta_0(dy)\frac{c\alpha}{\Gamma(1-\alpha)} w^{-1-\alpha} dw \right) f(x+y, w-(t-s))\,ds$$
$$= \int_{x\in\mathbb{R}} \int_{s\in[0,t]} \left(\int_{u\in\mathbb{R}^+} P_u(dx-\chi)\,g(s,u)\,du \right)$$
$$\times \int_{w\in[t-s,\infty)} f(x, w-(t-s))\frac{c\alpha}{\Gamma(1-\alpha)} w^{-1-\alpha} dw\,ds,$$

for $Y_t \in \mathcal{S}$ and its time to regeneration R_t.
We say that the r.v X has beta distribution with parameters $\mu,\nu > 0$ if it has pdf of the form

$$f(x,\mu,\nu) = \frac{x^{\mu-1}(1-x)^{\nu-1}}{B[\mu,\nu]} \qquad x \in (0,1)$$

where $B[\mu,\nu] = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$ is the Beta function and we write $X \sim B(\mu,\nu)$. We say that the r.v X has beta prime distribution with parameters $\mu,\nu > 0$ if it has pdf of the form

$$f(x,\mu,\nu) = \frac{x^{\mu-1}(1+x)^{-\mu-\nu}}{B[\mu,\nu]} \qquad x > 0 \tag{3.5}$$

and we write $X \sim B'(\mu,\nu)$. It was noted in [12, II.4] that if $X \sim B(\mu,\nu)$ then $\frac{X}{1-X} \sim B'(\mu,\nu)$. The distribution (3.5) can be further generalized to the so called *generalized Beta prime distribution* also known as the general Beta of the second kind distribution whose pdf is

$$f(x,\mu,\nu,h)=\frac{\left(\frac{x}{h}\right)^{\mu-1}\left(1+\frac{x}{h}\right)^{-\mu-\nu}}{h\cdot B[\mu,\nu]}\qquad x>0 \tag{3.6}$$

with $h,\mu,\nu>0$. If X has generalized Beta prime distribution of the form (3.6) then we write $X\sim GB2(\mu,\nu,h)$.

Theorem 1. *Let $Y_t^{t_0}=A_{E_{t+t_0}}-A_{E_{t_0}}$ where $t_0>0$ be the aging process. Let $B_1,B_2,...,B_k$ be Borel sets such that $0\notin B_1$. Let $p_{t_0}(r)=f(r,1-\alpha,\alpha,t_0)$ be a generalized beta prime distribution as in (3.6). Then we have for $0<t_1<t_2<\cdots<t_k$*

$$P\left(Y_{t_1}^{t_0}\in B_1,Y_{t_2}^{t_0}\in B_2,...,Y_{t_k}^{t_0}\in B_k\right)=\int_0^{t_1}P\left(Y_{t_1-r}\in B_1,Y_{t_2-r}\in B_2,...,Y_{t_k-r}\in B_k\right)p_{t_0}(r)\,dr. \tag{3.7}$$

Proof. For simplicity, we proof the result for $k=2$, the proof for $k>2$ is similar. We have

$$\begin{aligned}P\left(Y_{t_1}^{t_0}\in B_1,Y_{t_2}^{t_0}\in B_2\right)&=Q_{t_0}\left[1_{\{\mathbb{R}\times\mathbb{R}\}}(y_0,r_0)\right.\\&\quad\times Q_{t_1}\left[1_{\{B_1+y_0\times\mathbb{R}\}}(y_1,r_1)\,Q_{t_2-t_1}\left[1_{\{B_2+y_0\times\mathbb{R}\}}(y_2,r_2)\right](y_1,r_1)\right]\left.(y_0,r_0)\right](0,0).\end{aligned} \tag{3.8}$$

It is easy to see that by (3.4) the semi-group operator Q_t is translation invariant with respect to the space variable when $r=0$, i.e, $Q_t[f](y+a,0)=Q[g](y,0)$ where $g(y,r)=f(y+a,r)$. Moreover,

$$\begin{aligned}Q_t[f](y+a,r)&=1_{\{0\le t<r\}}f(y+a,r-t)+1_{\{0\le r\le t\}}Q_{t-r}[f](y+a,0)\\&=1_{\{0\le t<r\}}g(y,r-t)+1_{\{0\le r\le t\}}Q_{t-r}[g](y,0)\\&=Q_t[g](y,r).\end{aligned}$$

Hence, Q_t is translation invariant with respect to the space variable. Consequently, since $0\notin B_1$, by (2.6) we have

$$\begin{aligned}&Q_{t_1}\left[1_{\{B_1+y_0\times\mathbb{R}\}}(y_1,r_1)\,Q_{t_2-t_1}\left[1_{\{B_2+y_0\times\mathbb{R}\}}(y_2,r_2)\right](y_1,r_1)\right](y_0,r_0)\\&=1_{\{0\le r_0\le t_1\}}Q_{t_1-r_0}\left[1_{\{B_1+y_0\times\mathbb{R}\}}(y_1+y_0,r_1)\,Q_{t_2-t_1}\left[1_{\{B_2+y_0\times\mathbb{R}\}}(y_2,r_2)\right](y_1+y_0,r_1)\right](0,0)\\&=1_{\{0\le r_0\le t_1\}}Q_{t_1-r_0}\left[1_{\{B_1\times\mathbb{R}\}}(y_1,r_1)\,Q_{t_2-t_1}\left[1_{\{B_2+y_0\times\mathbb{R}\}}(y_2+y_0,r_2)\right](y_1,r_1)\right](0,0)\\&=1_{\{0\le r_0\le t_1\}}P\left(Y_{t_1-r_0}\in B_1,Y_{t_2-r_0}\in B_2\right).\end{aligned} \tag{3.9}$$

For ease of notation we write $P\left(Y_{t_1-r_0}\in B_1,Y_{t_2-r_0}\in B_2\right)=f(r_0)$. Plug (3.9) in (3.8) and use (3.4) to obtain,

$$\begin{aligned}P\left(Y_{t_1}^{t_0}\in B_1,Y_{t_2}^{t_0}\in B_2\right)=&\int\limits_{s'\in[0,t_0]}\left(\int\limits_{u'\in\mathbb{R}^+}g(s',u')\,du'\right)\\&\times\int\limits_{w'\in[t_0-s',\infty)}\frac{c\alpha}{\Gamma(1-\alpha)}w^{-1-\alpha}dwds'\\&\times\left[1_{\{0\le w'-(t_0-s')\le t_1\}}\times f\left(w'-(t_0-s')\right)\right]\end{aligned} \tag{3.10}$$

$$= \int_{s'\in[0,t_0]} \left(\int_{u'\in\mathbb{R}^+} g(s',u')\,du' \right) \times \int_{w'\in[t_0-s',t_1+t_0-s')} f(w'-(t_0-s'))$$
$$\times \frac{c\alpha}{\Gamma(1-\alpha)} w'^{-1-\alpha} dw' ds'.$$

By [31, Eq. 37.12] if D_t is a stable subordinator of index $0<\alpha<1$ with $E\left(e^{-uX_t}\right)=e^{-tcu^\alpha}$ and probability distribution $P(D_t\in dx)=g(x,t)\,dx$ then its potential density is given by

$$v(s)=\int_{u\in\mathbb{R}^+} g(s,u)\,du=\frac{1}{c\Gamma(\alpha)}s^{\alpha-1} \qquad s>0. \tag{3.11}$$

Substitute (3.11) in (3.10) and apply the change of variables $r=w'+s'-t_0$ to obtain

$$P\left(Y^{t_0}_{t_1}\in B_1, Y^{t_0}_{t_2}\in B_2\right)=\int_0^{t_1}\int_{s'\in[0,t_0]} \frac{s'^{\alpha-1}}{c\Gamma(\alpha)} f(r)\frac{c\alpha}{\Gamma(1-\alpha)}\left(r-s'+t_0\right)^{-1-\alpha} ds' dr. \tag{3.12}$$

Now apply the change of variables $v=s'\left(r-s'+t_0\right)^{-1}$ to compute the integral with respect to s' and to obtain

$$P\left(Y^{t_0}_{t_1}\in B_1, Y^{t_0}_{t_2}\in B_2\right)=\int_0^{t_1} f(r)\frac{\left(\frac{r}{t_0}\right)^{-\alpha}\left(1+\frac{r}{t_0}\right)^{-1}}{t_0\cdot B[\alpha,1-\alpha]}dr$$
$$=\int_0^{t_1} P\left(Y_{t_1-r}\in B_1, Y_{t_2-r}\in B_2\right)p_{t_1}(r)\,dr.$$

□

Remark 2. It follows from Theorem 1 that

$$P\left(Y^{t_0}_t=0\right)=\int_t^\infty p_{t_0}(r)\,dr+\int_0^t P\left(Y_{t-r}=0\right)p_{t_0}(r)\,dr>0. \tag{3.13}$$

Therefore, the distribution of $Y^{t_0}_t$ has an atom at the origin for every t. More interesting is the fact that if $P(A_t=0)=0$(this is true for all processes with pdf) then $P\left(Y^{t_0}_t=0\right)$ does not depend on the choice of the process A_t. On the other hand it can be easily seen that for every t the process $Y^{t_0}_t$ has density on $\mathbb{R}\setminus\{0\}$ given by $p_{t_0}(x,t)=\int_0^t p(x,t-r)\,p_{t_0}(r)\,dr$ whenever A_t has pdf $p(x,t)$. Furthermore, note that the finite dimensional distributions of the process Y^{t_0} on Borel sets $B_1,...,B_k$ such that $0\notin B_1$, determine completely the finite dimensional distributions of the process Y^{t_0}. We demonstrate this for $k=2$; if B_2 is a Borel set then

$$P\left(Y^{t_0}_{t_1}=0, Y^{t_0}_{t_2}\in B_2\right)=P\left(Y^{t_0}_{t_2}\in B_2\right)-P\left(Y^{t_0}_{t_1}\in\mathbb{R}/\{0\}, Y^{t_0}_{t_2}\in B_2\right),$$

which by (3.13) determines the two dimensional distributions completely.

Remark 3. In [6], a result similar to Theorem 1 for the one dimensional distribution is obtained for CTRW for large t_0 and t. The proof in [6] sheds light on our result, as it was derived from showing that the distribution of the first epoch τ_1 of the aging CTRW $X_t^{t_0}$ has beta prime distribution, i.e $\tau_1 \sim B'(1-\alpha,\alpha,t_0)$. This can be shown by a result by Dynkin on renewal processes ([9, Theorem 8.6.3]). Interestingly, the distribution of the first epoch τ_1 does not scale out as we move to the limit and obtain the process $Y_t^{t_0}$. Indeed, one can show (similarly to the proof of Theorem 1) that the distribution of the process R_t, the time left before the next regeneration at time t, is

$$f_{R_t}(r) = \frac{\left(\frac{r}{t}\right)^{-\alpha}\left(1+\frac{r}{t}\right)^{-1}}{t\cdot B[\alpha,1-\alpha]} \qquad r>0. \tag{3.14}$$

Since it was noted in [25] that the process Y_t starts afresh at time $H_t = D_{E_t} = t$ depending only on the position of Y_t, and by the fact that in our case the process Y_t is homogeneous in space, it follows that once the process $Y_t^{t_0}$ leaves the state 0 it behaves like the process Y_t from that point on. Now, condition the probability $P\left(Y_{t_1}^{t_0}\in B_1, Y_{t_2}^{t_0}\in B_2,\ldots,Y_{t_k}^{t_0}\in B_k\right)$ on the event $\{R_{t_0}=r\}$and integrate with respect to r to obtain (3.7). It should be clear now why $0\notin B_1$ as we would like to make sure that the system is mobilized before time t_1.

Remark 4. Let X_t be a renewal process with interarrival times $\{W_i\}$ whose tail distribution $1-F(x)\in R(-\alpha)$ for $0<\alpha<1$, namely, there exists a slowly varying function $L(x)$ such that $1-F(x)\sim x^{-\alpha}L(x)$ when $x\to\infty$. Define the arrival times $T_n=\sum_{i=1}^n W_i$ and let $S_t = t - T_{X_t}$ be the *age process,* the time spent at the current state. It was shown in [9, Theorem 8.6.3] that the distribution of $\frac{S_t}{t}$ converges, as $t\to\infty$. The limit is the so called Generalized Beta of the first kind distribution $GB1(1-\alpha,\alpha,1)$ whose pdf equals f_{V_1}, where

$$f_{V_t}(v) = \frac{\left(\frac{v}{t}\right)^{-\alpha}\left(1-\frac{v}{t}\right)^{\alpha-1}}{tB[\alpha,1-\alpha]} \qquad 0<v<t. \tag{3.15}$$

In [25], the analogous process $V_t = t - D_{E_t-}$ was defined to track the time that has passed since the last regeneration of the process Y_t. It can be easily shown, along similar lines to the proof of Theorem 1, that the process V_{t-} has distribution $GB1(1-\alpha,\alpha,t)$. Equations 3.14 and 3.15 explain the results of Jurlewicz et al in [14]. There it was proven ([14, Eq. 5.12]) that D_{E_t} has pdf

$$g(r) = \frac{r^{-1}}{B[\alpha,1-\alpha]}\left(\frac{t}{r-t}\right)^{\alpha} \qquad r>t, \tag{3.16}$$

and that D_{E_t-} has pdf ([14, Eq. 5.9])

$$h(v) = \frac{v^{\alpha-1}(t-v)^{-\alpha}}{B[\alpha,1-\alpha]} \qquad 0<v<t. \tag{3.17}$$

Equation 3.16 and 3.17 can be obtained by 3.14 and 3.15 respectively, by translation and reflection.

4. Aging self similarity

Recall that a process X_t is called self-similar if for every $a>0$ there exists $b>0$ such that the finite dimensional distributions of the time scaled process X_{at} equals that of the process bX_t. It is well known ([31, Section 13]) that if X_t is a Lévy process then it is self-similar if and only if X_t is strictly stable, i.e for every $a>0$ there exist $b>0$ such that $E\left(e^{iuX_1}\right)^a = E\left(e^{iubX_1}\right)$. For self-similar non trivial processes that are stochastically continuous at $t=0$, $b=a^H$ ([10, Theorem 1.1.1]), where $H>0$ if and only if $X_t=0$ with probability one. H is sometimes called the Hurst parameter. For example, for fractional Brownian motion $0<H<1$ while the Hurst parameter of the stable subordinator of index $0 < \alpha \leq 2$ is $1/\alpha$. For self-similar processes with stationary increments and finite second moment the Hurst parameter (when it exists) determines long range dependence ([10, Section 3.2]). Throughout this section we consider the process $Y_t^{t_0} = A_{E_{t+t_0}} - A_{E_{t_0}}$ where A_t is a strictly stable process whose Hurst parameter we denote by $1/\beta$ and E_t is the inverse of a stable subordinator of index α. We wish to find whether $Y_t^{t_0}$ has the property of self-similarity or a different property that resembles self-similarity to some extent. From Theorem 1 it is only reasonable that any self-similarity-like property of $Y_t^{t_0}$ should be strongly connected to the self-similarity of the process Y_t.

The next corollary states that although the aging process $Y_t^{t_0}$ is not self-similar it exhibits a self-similar-like behavior. Intuitively it suggests that $Y_{at}^{t_0}$ behaves like a "younger"($a > 1$) scaled version of itself.

Corollary 1. *Let $Y_t^{t_0}$ be an aging process and let B_i for $1 \leq i \leq k$ be Borel sets in $\mathbb{R}$. Then*

$$\left(Y_{at_1}^{t_0}, Y_{at_2}^{t_0}, ..., Y_{at_k}^{t_0}\right) \stackrel{d}{=} \left(a^{\frac{\alpha}{\beta}} Y_{t_1}^{\frac{t_0}{a}}, a^{\frac{\alpha}{\beta}} Y_{t_2}^{\frac{t_0}{a}}, ..., a^{\frac{\alpha}{\beta}} Y_{t_k}^{\frac{t_0}{a}}\right).$$

Proof. For simplicity we only prove the result for $k = 2$ as the proof for $k > 2$ is similar. First assume that $B_1 \subseteq \mathbb{R}$ does not contain zero . By Theorem 1 we have

$$P\left(Y_{at_1}^{t_0} \in B_1, Y_{at_2}^{t_0} \in B_2\right) = \int_0^{at_1} P\left(Y_{at_1-r} \in B_1, Y_{at_2-r} \in B_2\right) p_{t_0}(r)\, dr.$$

Apply the change of variables $r' = \frac{r}{a}$ to obtain

$$P\left(Y_{at_1}^{t_0} \in B_1, Y_{at_2}^{t_0} \in B_2\right) = \int_0^{t_1} P\left(Y_{a(t_1-r')} \in B_1, Y_{a(t_2-r')} \in B_2\right) \frac{\left(\frac{r'a}{t_0}\right)^{-\alpha}\left(1+\frac{r'a}{t_0}\right)^{-1}}{t_0 \cdot B\left[\alpha, 1-\alpha\right]} a dr'.$$

By [21, Corollary 4.1] Y_t is self similar with Hurst parameter $\frac{\alpha}{\beta}$. Therefore we have

$$\begin{aligned} P\left(Y_{at_1}^{t_0} \in B_1, Y_{at_2}^{t_0} \in B_2\right) &= \int_0^{t_1} P\left(a^{\frac{\alpha}{\beta}} Y_{(t_1-r')} \in B_1, a^{\frac{\alpha}{\beta}} Y_{(t_2-r')} \in B_2\right) p_{\frac{t_0}{a}}(r)\, dr \\ &= P\left(a^{\frac{\alpha}{\beta}} Y_{t_1}^{\frac{t_0}{a}} \in B_1, a^{\frac{\alpha}{\beta}} Y_{t_2}^{\frac{t_0}{a}} \in B_2\right). \end{aligned} \tag{4.1}$$

Now, by Remark 2 it follows that (4.1) holds for any Borel sets $B_1, B_2 \subseteq \mathbb{R}$ and the result follows. □

5. Asymptotic behavior and the Fractional Fokker-Planck equation

An easy yet important consequence of Theorem 1 is the following.

Corollary 2. Let $B \subseteq \mathbb{R}$ be a Borel measurable subset such that $0 \notin B$ and $P\left(Y_t^{t_0} \in B\right) \neq 0$, then

$$P\left(Y_t^{t_0} \in B\right) \sim C t_0^{\alpha-1} \qquad t_0 \to \infty \tag{5.1}$$

where $C = \frac{\sin(\pi\alpha)}{\pi} \int_0^t P\left(Y_{t-r} \in B\right) r^{-\alpha} dr$.

Proof. First note that by the continuity of $P(Y_t \in B)$ (see (1)) $C \neq 0 \Leftrightarrow P\left(Y_t^{t_0} \in B\right) \neq 0$. By dominated convergence we then have,

$$\lim_{t_0\to\infty} \frac{P\left(Y_t^{t_0} \in B\right)}{Ct_0^{\alpha-1}} = \lim_{t_0\to\infty} \frac{t_0^{\alpha-1}\frac{\sin(\pi\alpha)}{\pi}\int_0^t P\left(Y_{t-r}\in B\right) r^{-\alpha}\left(1+\frac{r}{t_0}\right)^{-1} dr}{t_0^{\alpha-1}\frac{\sin(\pi\alpha)}{\pi}\int_0^t P\left(Y_{t-r}\in B\right) r^{-\alpha} dr} = 1.$$

□

Remark 5. When the process Y_t is a renewal process (that is the case for the FPP) the convergence of $P\left(Y_t^{t_0} \in B\right)$ to zero is expected by the Renewal Theorem ([12, XI.1]) and the fact that the interarrival times have the Mittag-Leffer distribution with infinite expectation. Interestingly, it was shown by Erickson in [11, Theorem 1], that if Y_t is a renewal process with interarrival times W_n with $F(t) = P(W_1 \leq t)$ such that $1 - F(t) \in R(-\alpha)$ for $0 < \alpha < 1$, i.e $1 - F(t) \sim t^{-\alpha}L(t)$ as $t \to \infty$ where $L(t)$ is a slowly varying function and F is not arithmetic, then

$$E\left(Y_t^{t_0}\right) \sim \frac{\sin(\pi\alpha)}{\pi}\frac{t}{L(t_0)} t_0^{\alpha-1} \qquad t_0 \to \infty. \tag{5.2}$$

We now show how (5.2) can be obtained for the FPP by Corollary 2. First note that by similar arguments as in Corollary 2 we have

$$E\left(Y_t^{t_0}\right) \sim t_0^{\alpha-1}\frac{\sin(\pi\alpha)}{\pi}\int_0^t E\left(Y_{t-r}\right) r^{-\alpha} dr \qquad t_0 \to \infty. \tag{5.3}$$

Let $Y_t = N_t^\alpha$ be the fractional Poisson process with intensity $\lambda = 1$. By [8, Eq. 2.7], $E(Y_{t-r}) = \frac{(t-r)^\alpha}{\Gamma(1+\alpha)}$ and so by (5.3) we have

$$\begin{aligned} E\left(Y_t^{t_0}\right) &\sim t_0^{\alpha-1}\frac{\sin(\pi\alpha)}{\pi}\int_0^t \frac{(t-r)^\alpha}{\Gamma(1+\alpha)} r^{-\alpha} dr \\ &= t_0^{\alpha-1}\frac{\sin(\pi\alpha)}{\pi} t\Gamma(1-\alpha). \end{aligned}$$

To see this, note that $\int_0^t (t-r)^\alpha r^{-\alpha} dr = \frac{\Gamma(1-\alpha)}{(\alpha+1)}\partial_t^\alpha\left[t^{\alpha+1}1_{t\geq 0}\right] = \Gamma(1-\alpha)\Gamma(\alpha+1)t$, where ∂_t^α is the Caputo derivative of index α (5.8). This agrees with (5.2). Indeed, note that by [29] the asymptotic behavior of the Mittag-Leffler distribution pdf is $f^\alpha(t) \sim \frac{t^{-1-\alpha}\alpha}{\Gamma(1-\alpha)}$ as $t \to \infty$ (note that there is a typo there as α should be in the numerator) and by the Karamata Tauberian Theorem ([9, Theorem 1.5.11]) we see that $E^\alpha(-t^\alpha) = \int_t^\infty f(y)\,dy \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ as $t \to \infty$ so $(L(t))^{-1} = \Gamma(1-\alpha)$.

While it is known that generally CTRWL lose their stationarity property for $0 < \alpha < 1$ ([21, Corollary 4.3]), Theorem 1 suggests a way of measuring the stationarity of a process in the class $\mathcal{S}$. The FPP for example has no stationary increments for $0 < \alpha < 1$, however, for $\alpha = 1$ we obtain the Poisson process which is of course stationary as being a Levy process. We proceed with a useful lemma that states that the distribution of the processes in $\mathcal{S}$ is continuous as a function of time.

Lemma 1. *Let $Y_t \in \mathcal{S}$ and $C \subset \mathbb{R}$ a Borel set, then the function $t \mapsto P(Y_t \in C)$ is continuous on $(0,\infty)$.*

Proof. Since $Y_t = A_{E_t}$, by a simple conditioning argument ([24, Eq. (2.7)]) we have

$$P(Y_t \in C) = \int_0^\infty P(A_y \in C)\, h(y,t)\, dy,$$

where $h(x,t)$ is the pdf of the process E_t. Then

$$\limsup_{h\to 0} |P(Y_{t+h} \in C) - P(Y_t \in C)| = \limsup_{h\to 0} \left| \int_0^\infty P(A_y \in C)\, h(y,t+h)\, dy - \int_0^\infty P(A_y \in C)\, h(y,t)\, dy \right|$$
$$\leq \limsup_{h\to 0} \int_0^\infty |h(y,t+h) - h(y,t)|\, dy.$$

It was proved in ([21, Corollary 3.1]) that

$$h(x,t) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} g\left(t x^{-\frac{1}{\alpha}}\right)$$

where $g(x)$ is the pdf of a stable r.v. Since $g(x)$ is smooth it follows that $h(x,t)$ is continuous on $t, x > 0$. Trivially we have

$$\lim_{h\to 0} \int_0^\infty h(y,t+h)\, dy = \int_0^\infty h(y,t)\, dy = 1.$$

Hence, a basic result in analysis [30, Chapter 7, Theorem 7] implies that

$$\lim_{h\to 0} \int_0^\infty |h(y,t+h) - h(y,t)|\, dy = 0,$$

and the result follows. □

The next result states that as $\alpha \to 1$ the process Y_t , in some sense, becomes more stationary.

Proposition 1. *Let $Y_t \in \mathcal{S}$, then for every $t, t_0 > 0$*

$$Y_t^{t_0} = Y_{t+t_0} - Y_{t_0} \stackrel{d}{\to} Y_t \qquad \alpha \to 1.$$

Proof. In [33, eq. 3.1.19] it was shown that

$$U(a,b,s) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-sx} x^{a-1} (1+x)^{b-a-1}\, dx, \tag{5.4}$$

where $U(a,b,s)$ is a hypergeometric function that solves the confluent hypergeometric equation, also known as Kummer's equation

$$s\frac{\partial^2 U}{\partial^2 s} + (b-s)\frac{\partial U}{\partial s} - aU = 0. \tag{5.5}$$

By (5.4) and a simple change of variables we find that the Laplace transform of the generalized Beta prime distribution is given by

$$\hat{p}_{t_0}(s) = \frac{U(1-\alpha, 1-\alpha, st_0)}{\Gamma(\alpha)} \qquad s > 0. \tag{5.6}$$

Using the identity $U(1-\alpha, 1-\alpha, x) = e^x \Gamma(\alpha, x)$ where $\Gamma(\alpha, x)$ is the incomplete gamma function defined by $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$, we can write (5.6) in a more familiar notation

$$\hat{p}_{t_0}(s) = \frac{e^{st_0}\Gamma(\alpha, st_0)}{\Gamma(\alpha)}.$$

Now, by dominated convergence

$$\lim_{\alpha\to 1} \hat{p}_{t_0}(s) = \lim_{\alpha\to 1} \frac{e^{st_0}\int_{st_0}^\infty r^{\alpha-1} e^{-r} dr}{\Gamma(\alpha)} = \frac{e^{st_0} e^{-st_0}}{1} = 1.$$

Therefore, by [15, Theorem 4.3] we have $p_{t_0} \stackrel{w}{\to} \delta$ as $\alpha \to 1$ where $\stackrel{w}{\to}$ denotes weak convergence of probability measures and δ is the Dirac delta measure. For a Borel set B such that $0 \notin B$ define

$$f(r) = \begin{cases} P(Y_{t-r} \in B) & 0 \le r \le t \\ 0 & t < r \end{cases},$$

and note that $P(Y_0 \in dx) = \delta_0(dx)$ and therefore $P(Y_{t-r} \in B) = 0$ at $r = t$. Consequently, Lemma 1 suggests that $f(r)$ is continuous. By the fact that

$$P\left(Y_t^{t_0} \in B\right) = \int_0^\infty f(r) p_{t_0}(r) dr = \int_0^t P(Y_{t-r} \in B) p_{t_0}(r) dr \to P(Y_t \in B),$$

we also have $P\left(Y_t^{t_0} = 0\right) \to P(Y_t = 0)$ and the proof is complete. □

Remark 6. It was shown in [33, eq. 4.1.12] that $U(a, b, s) \sim Cs^{-a}$ as $s \to \infty$. It follows that

$$\hat{p}_{t_0}(s) \sim C(st_0)^{\alpha-1} \qquad t_0 \to \infty \tag{5.7}$$

and therefore $\hat{p}_{t_0}(s) \to 0$ as $t_0 \to \infty$. Hence, $p_{t_0} \stackrel{v}{\to} 0$ as $t_0 \to \infty$ where $\stackrel{v}{\to}$ denotes vague convergence of distributions, and $P\left(Y_t^{t_0} \in dx\right) \stackrel{w}{\to} \delta_0(dx)$, another proof for the fact that $P\left(Y_t^{t_0} \in B\right) \to 0$ as $t_0 \to \infty$ for B such that $0 \notin B$. It is not hard to verify that $\hat{p}_{t_0} \to 0$ as $\alpha \to 0$. Intuitively, this is expected since a small α suggests long waiting times between jumps and that Y_t is very subdiffusive.

Let $p(dx, t)$ be a stochastic kernel, that is, for every $t > 0$ $p(dx, t)$ is a probability measure on $\sigma(\mathbb{R})$ and for each Borel set $B \subseteq \mathbb{R}$ $p(B, \cdot)$ is measureable. Denote the Fourier transform of $p(dx, t)$ by $\widetilde{p}(k, t) = \int_{\mathbb{R}} e^{-ikx} p(dx, t)$, and the Fourier-Laplace transform (FLT) by $\overline{p}(k, s) =$

$\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{-st-ikx}p\left(dx,t\right)dt$. Recall the definition of the Caputo $0<\alpha<1$ fractional derivative of a function $f\left(t\right)$,

$$\partial_t^\alpha f=\frac{1}{\Gamma\left(1-\alpha\right)}\int_0^t\left(t-r\right)^{-\alpha}\frac{\partial f\left(r\right)}{\partial r}dr. \tag{5.8}$$

For $0<\alpha<1$ the Laplace transform of $\partial_t^\alpha f$ is ([22, p. 39])

$$\widehat{\partial_t^\alpha f}=s^\alpha\hat{f}-s^{\alpha-1}f\left(0+\right).$$

A closely related operator is the Riemann Liouville derivative $\mathbb{D}_t^\alpha$ for $0<\alpha<1$, which is defined by

$$\mathbb{D}_t^\alpha f=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{\partial}{\partial t}\int_0^t\left(t-r\right)^{-\alpha}f\left(r\right)dr. \tag{5.9}$$

The LT of (5.9) can be shown to be $\widehat{\mathbb{D}_t^\alpha f}=s^\alpha\hat{f}$. It follows that

$$\partial_t^\alpha f=\mathbb{D}_t^\alpha f-f\left(0+\right)\frac{t^{-\alpha}}{\Gamma\left(1-\alpha\right)}. \tag{5.10}$$

The following is a short summary of results in [3]. Let $V^\omega=L^1_\omega\left(\mathbb{R}\times\mathbb{R}_+\right)$ be the space of real valued measurable functions on $\mathbb{R}\times\mathbb{R}_+$ such that

$$\left\|f\right\|_\omega=\int_0^\infty\int_{\mathbb{R}^d}e^{-\omega t}\left|f\left(x,t\right)\right|dxdt<\infty,$$

for some $\omega>0$. V^ω is a Banach space w.r to $\left\|\cdot\right\|_\omega$. If $\left(A_t,D_t\right)$ is a Lévy process where D_t is a subordinator and s.t $E\left(e^{-ikA_t-sD_t}\right)=e^{t\eta(-k,s)}$, then the distribution of $\left(A_t,D_t\right)$ gives way to a semi group of operators whose infinitesimal generator L' satisfies $\overline{L'f}=\eta\left(-k,s\right)\overline{f}\left(k,s\right)$(for $\omega\leq s$). In fact, f is in the domain of L', $D\left(L'\right)$, iff $\overline{g}\left(k,s\right)=\eta\left(-k,s\right)\overline{f}\left(k,s\right)$ where $\overline{g}\left(k,s\right)$ is the FLT of some $g\in V^\omega$. If the first and second order spatial weak derivatives as well as the first order time weak derivative of f is in V^ω then $f\in D\left(L'\right)$. Let $p^{t_0}\left(dx,t\right)$ be the probability measure of the process $Y_t^{t_0}$, i.e. $p^{t_0}\left(dx,t\right)=P\left(Y_t^{t_0}\in dx\right)$. Suppose A_t has the symbol $\psi\left(k\right)$ and the infinitesimal generator L, and D_t is an independent standard stable subordinator. We then have $\eta\left(-k,s\right)=-s^\alpha+\psi\left(-k\right)$, and $L'=-\mathbb{D}_t^\alpha+L$ (since $f\in V^\omega$, Lf should be understood as $f\left(\cdot,t\right)\in D\left(A\right)$ for every $t>0$). Note that by [4, Theorem 2.2] smooth functions on $\mathbb{R}$ are contained in $D\left(L\right)$. The FLT of $p^0\left(dx,t\right)$ is well known([22, Eq. 4.43]) and given by

$$\overline{p^0}\left(k,s\right)=\frac{s^{\alpha-1}}{-\eta\left(-k,s\right)}=\frac{s^{\alpha-1}}{s^\alpha-\psi\left(-k\right)}, \tag{5.11}$$

which in turn implies that

$$\begin{aligned}\partial_t^\alpha p^0\left(dx,t\right)&=Lp^0\left(dx,t\right)\\ p^0\left(dx,0\right)&=\delta_0\left(dx\right).\end{aligned} \tag{5.12}$$

Equation (5.12) describes the dynamics of $p^0\left(dx,t\right)$ and therefore is called the Fractional Fokker Planck Equation(FFPE) of $p^0\left(dx,t\right)$. Suppose that the process $Y_t^{t_0}$ starts from the random point X_0 with density $p\left(x\right)\in C_c^\infty\left(\mathbb{R}\right)$, that is, smooth with compact support and that X_0 is independent of $Y_t^{t_0}$. The distribution of $Y_t^{t_0} + X_0$ is $C(x,t) = \int_{\mathbb{R}} p(x-y)\, p^{t_0}(dy,t)$ which is again smooth. The next theorem obtains the governing equation of $C(x,t)$.

Theorem 2. *Let $Y_t = A_{E_t}$ have probability measure $p^0(dx,t)$ whose FLT is given by (5.11) for $0 < \alpha < 1$. Let L be the generator of A_t.* Then we have

$$\partial_t^\alpha C(x,t) = L\left(C(x,t) - p(x)\int_t^\infty p_{t_0}(r)\, dr\right) \tag{5.13}$$
$$C(x,0) = p(x).$$

Proof. Let

$$p(x,t) = C(x,t) - p(x)\int_t^\infty p_{t_0}(r)\, dr, \tag{5.14}$$

and note that the FLT of (5.14) is

$$\overline{p}(k,s) = \overline{C}(k,s) - \widetilde{p}(k)\left(\frac{1}{s} - \frac{1}{s}\hat{p}_{t_0}(s)\right). \tag{5.15}$$

By Remark 2 we have

$$p(x,t) = \int_{\mathbb{R}} p(x-y)\int_0^t p^0(dy, t-r)\, p_{t_0}(r)\, dr. \tag{5.16}$$

By a general version of Fubini's Theorem [2, Theorem 2.6.4] we have

$$\int_{\mathbb{R}} e^{-ikx}\int_0^t p^0(dx, t-r)\, p_{t_0}(r)\, dr = \int_0^t \widetilde{p}^0(k, t-r)\, p_{t_0}(r)\, dr. \tag{5.17}$$

Take the LT of both sides of equation (5.17) to obtain

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{-st-ikx}\int_0^t p^0(dx,t-r)\, p_{t_0}(r)\, dr = \overline{p}^0(k,s)\, \widehat{p}_{t_0}(s).$$

It follows that

$$\overline{p}(k,s) = \overline{p}^0(k,s)\, \widehat{p}_{t_0}(s)\, \widetilde{p}(k)$$

Since by (5.11) $\overline{p}^0(k,s) = \frac{s^{\alpha-1}}{s^\alpha - \psi(-k)}$ we have

$$\overline{p}(k,s) = \frac{s^{\alpha-1}}{s^\alpha - \psi(-k)}\widehat{p}_{t_0}(s)\, \widetilde{p}(k). \tag{5.18}$$

Substitute (5.15) in (5.18) to obtain

$$\overline{C}(k,s)\, s^\alpha - \overline{C}(k,s)\, \psi(-k) - \widetilde{p}(k)\left(\frac{1}{s} - \frac{1}{s}\widehat{p}_{t_0}\right)s^\alpha + \widetilde{p}(k)\left(\frac{1}{s} - \frac{1}{s}\widehat{p}_{t_0}\right)\psi(-k) = s^{\alpha-1}\widehat{p}_{t_0}(s)\, \widetilde{p}(k),$$

which can be rearranged to obtain

$$\overline{C}(k,s)(s^{\alpha}-\psi(-k))=\widetilde{p}(k)s^{\alpha-1}-\left(\frac{1}{s}-\frac{1}{s}\widehat{p}_{t_0}\right)\widetilde{p}(k)\psi(-k). \tag{5.19}$$

By the preceding discussion the right hand side of 5.19 inverts to a function in V^{ω}, taking the IFLT of (5.19) we have

$$\mathbb{D}_t^{\alpha}C(x,t)-LC(x,t)=p(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}-Lp(x)\int_t^{\infty}p_{t_0}(r)\,dr. \tag{5.20}$$

Noting that $C(x,0^+)=p(x)$ one can rewrite (5.20) by using (5.10) to arrive at (5.13). □

Remark 7. Although Equation (5.13) is not an abstract Cauchy problem, one may adopt the concept of a mild solution from [27, Chapter 4] and use it in our case. Let $f\in L^1(\mathbb{R})$, we say that a function $C(x,t)=f*p^{t_0}(dx,t)=\int_{\mathbb{R}}f(x-y)p^{t_0}(dy,t)$ is a mild solution of

$$\partial_t^{\alpha}C(x,t)=L\left(C(x,t)-f(x)\int_t^{\infty}p_{t_0}(r)\,dr\right) \tag{5.21}$$
$$C(x,0)=f(x)$$

if there exists a sequence $\phi_n\in D(L)$ s.t $\phi_n\overset{L^1}{\to}f$ (this implies that $\phi_n*p^{t_0}(dx,t)\overset{L^1}{\to}f*p^{t_0}(dx,t)$ uniformly in t on bounded sets as $p^{t_0}(dx,t)$ is a contraction for every t). From Theorem 2 and the fact that $C_c^{\infty}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we conclude that every $f\in L^1(\mathbb{R})$ is a mild solution of (5.21). We then write for simplicity

$$\partial_t^{\alpha}p^{t_0}(dx,t)=L\left(p^{t_0}(dx,t)-\delta_0(dx)\int_t^{\infty}p_{t_0}(r)\,dr\right) \tag{5.22}$$
$$p^{t_0}(dx,0)=\delta_0(dx).$$

Since (5.22) describes the dynamics of the probability kernel $p^{t_0}(dx,t)$ we call it its FFPE.

Remark 8. Theorem 2 shows that the dynamics of $p^{t_0}(dx,t)$ are the same as those of $p^0(dx,t)$ on $\mathbb{R}/\{0\}\times[0,\infty)$. There is a nice intuitive interpretation to equation (5.22) when A_t is a stable process. Equation (5.13) can be explained as the behavior of a plume of particles by arguments of conservation of mass and Fick's law ([22, Remark 2.3] and [37, Section 16.1]). However, note that the portion of the mass of particles that does not diffuse away from point $x=0$ at time t (and therefore does not contribute to the change in $p^{t_0}(dx,t)$ over time) is $\int_t^{\infty}p_{t_0}(r)\,dr$ by Remark 2 and the fact that stable processes have pdf. This accounts for the difference between (5.22) and (5.12).

Remark 9. In [5], a deterministic system was modeled by a CTRW and its aging properties were studied. There, the FFPE was given for the unnormalized distribution (5.14) of the aging process when t_0 and t are large. To see that the results agree, simply plug $\psi(-k)=\frac{-k^2}{2A}$ in (5.18) and take the IFLT of both sides of the equation. Since $\overline{p}(k,s)s^{\alpha}$ is the FLT of the fractional Riemann-Liouville α derivative we obtain [5, eq. 18].

References

[1] David Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.

[2] Robert B. Ash. Real analysis and probability, 1972. *Probability and Mathematical Statistics*.

[3] Boris Baeumer, Mark Meerschaert, and Jeff Mortensen. Space-time fractional derivative operators. *Proceedings of the American Mathematical Society*, 133(8):2273–2282, 2005.

[4] Boris Baeumer and Mark M Meerschaert. Stochastic solutions for fractional cauchy problems. *Fractional Calculus and Applied Analysis*, 4(4):481–500, 2001.

[5] Eli Barkai. Aging in subdiffusion generated by a deterministic dynamical system. *Physical review letters*, 90(10):104101, 2003.

[6] Eli Barkai and Yuan-Chung Cheng. Aging continuous time random walks. *The Journal of chemical physics*, 118(14):6167–6178, 2003.

[7] Peter Becker-Kern, Mark M Meerschaert, and Hans-Peter Scheffler. Limit theorems for coupled continuous time random walks. *Annals of Probability*, pages 730–756, 2004.

[8] Luisa Beghin and Enzo Orsingher. Fractional poisson processes and related planar random motions. *Electron. J. Probab*, 14(61):1790–1826, 2009.

[9] Nicholas H Bingham, Charles M Goldie, and Jef L Teugels. *Regular variation*, volume 27. Cambridge university press, 1989.

[10] Paul Embrechts. *Selfsimilar processes*. Princeton University Press, 2009.

[11] Bruce K. Erickson. Strong renewal theorems with infinite mean. *Transactions of the American Mathematical Society*, 151(1):263–291, 1970.

[12] William Feller. *An Introduction to Probability Theory and Its Applications: Volumes I & II: Second Edition*. John Wiley & Sons, 1971.

[13] Guy Jumarie. Fractional master equation: non-standard analysis and liouville–riemann derivative. *Chaos, Solitons & Fractals*, 12(13):2577–2587, 2001.

[14] Agnieszka Jurlewicz, P Kern, Mark M Meerschaert, and H-P Scheffler. Fractional governing equations for coupled random walks. *Computers & Mathematics with Applications*, 64(10):3021–3036, 2012.

[15] Olav Kallenberg. *Foundations of modern probability*. Springer Science & Business Media, 2002.

[16] Nick Laskin. Fractional poisson process. *Communications in Nonlinear Science and Numerical Simulation*, 8(3):201–213, 2003.

[17] Francesco Mainardi, Rudolf Gorenflo, and Enrico Scalas. A fractional generalization of the poisson processes. *arXiv preprint math/0701454*, 2007.

[18] Francesco Mainardi, Rudolf Gorenflo, and Alessandro Vivoli. Beyond the poisson renewal process: A tutorial survey. *Journal of Computational and Applied Mathematics*, 205(2):725–735, 2007.

[19] Mark M Meerschaert, Erkan Nane, and P Vellaisamy. The fractional poisson process and the inverse stable subordinator. *Electron. J. Probab*, 16(59):1600–1620, 2011.

[20] Mark M Meerschaert and Hans-Peter Scheffler. Triangular array limits for continuous time random walks. *Stochastic processes and their applications*, 118(9):1606–1633, 2008.

[21] Mark M Meerschaert, Hans-Peter Scheffler, et al. Limit theorems for continuous-time random walks with infinite mean waiting times. *Journal of applied probability*, 41(3):623–638, 2004.

[22] Mark M Meerschaert and Alla Sikorskii. *Stochastic models for fractional calculus*, volume 43. Walter de Gruyter, 2011.

[23] Mark M Meerschaert and Peter Straka. Fractional dynamics at multiple times. *Journal of statistical physics*, 149(5):878–886, 2012.

[24] Mark M Meerschaert and Peter Straka. Inverse stable subordinators. *Mathematical modelling of natural phenomena*, 8(02):1–16, 2013.

[25] Mark M Meerschaert, Peter Straka, et al. Semi-markov approach to continuous time random walk limit processes. *The Annals of Probability*, 42(4):1699–1723, 2014.

[26] Cécile Monthus and Jean-Philippe Bouchaud. Models of traps and glass phenomenology. *Journal of Physics A: Mathematical and General*, 29(14):3847, 1996.

[27] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[28] Mauro Politi, Taisei Kaizoji, and Enrico Scalas. Full characterization of the fractional poisson process. *EPL (Europhysics Letters)*, 96(2):20004, 2011.

[29] ON Repin and AI Saichev. Fractional poisson law. *Radiophysics and Quantum Electronics*, 43(9):738–741, 2000.

[30] Halsey Lawrence Royden, Patrick Fitzpatrick, and Prentice Hall. *Real analysis*, volume 198. Macmillan New York, 2010.
[31] Ken-Iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68. Cambridge university press, 1999.
[32] Johannes HP Schulz, Eli Barkai, and Ralf Metzler. Aging renewal theory and application to random walks. *Physical Review X*, 4(1):011028, 2014.
[33] Lucy Joan Slater. *Confluent hypergeometric functions*. University Press Cambridge, 1960.
[34] Peter Straka. *Continuous Time Random Walk Limit Processes - Stochastic Models for Anomalous Diffusion*. Phd dissertation, University of New South Wales, 2011.
[35] Peter Straka and Bruce Ian Henry. Lagging and leading coupled continuous time random walks, renewal times and their joint limits. *Stochastic Processes and their Applications*, 121(2):324–336, 2011.
[36] Leendert Cornelis Elisa Struik. *Physical aging in amorphous polymers and other materials*. 1977.
[37] George M Zaslavsky. *Hamiltonian chaos and fractional dynamics*. Oxford Univ. Press, 2008.
[38] Vladimir M Zolotarev. *One-dimensional stable distributions*, volume 65. American Mathematical Soc., 1986.