AGING UNCOUPLED CONTINUOUS TIME RANDOM WALK LIMITS

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Abstract. Aging is a prevalent phenomenon in physics, chemistry and many other fields. In this paper we consider the aging process of uncoupled Continuous Time Random Walk Limits (CTRWLs) which are Levy processes time changed by the inverse stable subordinator of index $0 < \alpha < 1$. We apply a recent method developed by Meerschaert and Straka of finding the finite dimensional distributions of CTRWL, to obtaining the aging process’s finite dimensional distributions, self-similarity-like property, asymptotic behavior and its Fractional Fokker-Planck equation (FFPE).

1. Introduction

Continuous time random walks (CTRW) are widely used in physics and mathematical finance to model a random walk for which the waiting times between jumps are random which in many cases better describes phenomena in these fields. CTRWLs are used to model anomalous diffusion, where the squared averaged distance of the process from the origin is no longer proportional to the time index $t$. A related concept and widely studied ([36, 32]) in statistical physics, is aging. Suppose the CTRW $X_t$ starts at $t = 0$ and evolves until time $t_0 > 0$ when we then start to measure it. One can consider the varying dynamics of the new process $X_{t_0}^t = X_{t_0 + t} - X_{t_0}$ as $t_0$ varies and the process ages. In [26] Monthus and Bouchaud studied a CTRW with aging properties. In [6] Barkai and Cheng considered the Aging Continuous Time Random Walk (ACTRW) which is an uncoupled CTRW with iid power law waiting times, that started at $t = 0$ and is observed at $t = t_0$. They found the one dimensional distribution of the process $X_{t_0}^t$ which they referred to as the ACTRW, for $t_0$ and $t$ large. In [5], Barkai found the Fractional Fokker-Planck Equation (FFPE) for the unnormalized pdf of the process $X_{t_0}^t$ for $t_0$ and $t$ large.

In this paper we wish to give analogous results to the ones given in [6, 5] as well as new ones for a large class of CTRWLs which hopefully will lay the foundation for further study of their aging. We consider the class that consists of all processes of the form $Y_t = A_t \mathcal{E}_t$ where $A_t$ is a Levy process that is time changed by the inverse of an independent stable subordinator of index $0 < \alpha < 1$; we denote this class by $\mathcal{S}$. We denote the aging process by $Y_{t_0}^t = Y_{t+t_0} - Y_{t_0} = A_{t+t_0} - A_{t_0}$ (note that $Y_{t_0}^0 = Y_t$). Section 2 is devoted to a brief review of the theory and method introduced by Meerschaert and Straka in [25] and [23] upon which we base our results. In Section 3 we give the main result of this paper, that the finite dimensional distributions of the process $Y_{t_0}^t$ can be obtained by a convolution in time of the finite dimensional distributions of $Y_t$ and a generalized beta prime distribution. The self-similarity-like property of the process $Y_{t_0}^t$ is obtained in Section 4. In Section 5 we obtain results on the asymptotic behavior of the distribution of $Y_{t_0}^t$ when $t_0$ is far from the origin as well as when $\alpha \to 1$ and the governing equation of $Y_{t_0}^t$. 

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One example for a process that lies in $S$ is the Fractional Poisson Process (FPP) which we denote by $N^\alpha_t$. The FPP is a renewal process with interarrival times $W_n$ such that $P(W_1 > t) = e^{-\lambda t^\alpha}$.

\[
E_\alpha (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}
\]

is the Mittag-Leffler function. Since the interarrival times are not exponentially distributed the process $N^\alpha_t = \sup \{ k : T_k \leq t \}$, where $T_k = \sum_{i=1}^k W_i$ are the arrival times, is not Markovian and the calculation of the finite dimensional distributions of $N^\alpha_t$ is no longer straightforward. The FPP was first studied in [16],[13] and [17, 18]. In [8] an integral representation of the one dimensional distribution of the FPP was given and was used in [28] to find and simulate the finite dimensional distributions of the FPP. In [19], it was shown that $N^\alpha_t = N_t^0$, where $N_t$ is a Poisson process and $E_t$ is the inverse of a standard stable subordinator of index $0 < \alpha < 1$ independent of $N_t$.

Since the distribution of the increments (and therefore the aging process) of the CTRWL is closely related to the two dimensional distributions, their study is quite cumbersome. In a recent paper ([25]), Meerscheart and Straka found a way of embedding CTRWLs in a larger state space that renders these processes Markovian. We use this method to find the finite dimensional distributions of the process $Y^\alpha_t$, its asymptotic behavior, self-similarity-like property and its FFPE.

2. Finite dimensional distribution of CTRWL

CTRWL are usually not Markovian, a fact that makes the calculation of their finite dimensional distributions quite difficult. It is therefore that the distribution of the increments (which can be obtained by the finite dimensional distributions) of the CTRWL is not well understood. Although the method in [25] is very general we focus only on uncoupled CTRWLs which are Levy processes time changed by the inverse of an independent stable subordinator. In order to facilitate reading of this section and referring to the original paper we retain most of the notation in [25]. The uncoupled CTRW we consider consist of two independent sequences of iid r.v.s, $\{W^\alpha_n\}$ and $\{J^\alpha_n\}$. The parameter $c$ is the convergence parameter as in [20] which allows us to construct infinitesimal triangular arrays. Here, $\{J^\alpha_n\}$ represents the size of the jumps of a particle in space, while $\{W^\alpha_n\}$ represents the waiting times between jumps. Hence, the time elapsed by the particle’s $k$’th jump is $T^\alpha_k = D^\alpha_0 + \sum_{i=1}^k W^\alpha_i$ and the position of the particle is $S^\alpha_k = A^\alpha_0 + \sum_{i=1}^k J^\alpha_i$. Let $L^\alpha_t = \sup \{ k : T^\alpha_k \leq t \}$ be the number of jumps until time $t$, then the CTRW $Y^\alpha_t$ is

\[
Y^\alpha_t = A^\alpha_0 + L^\alpha_t J^\alpha_t.
\]

Assume we have

\[
( S^\alpha_{[cu]}, T^\alpha_{[cu]} ) = ( A^\alpha_0, D^\alpha_0 ) + \sum_{i=1}^{[cu]} ( J^\alpha_i, W^\alpha_i ) \Rightarrow (A_u, D_u)
\]

where $\Rightarrow$ denotes convergence in the Skorokhod $J_1$ topology. In this paper we assume $D_u$ is a stable subordinator of index $0 < \alpha < 1$ starting from $D_0$, i.e, $E \left( e^{-s(D_u - D_0)} \right) = e^{-Cs^\alpha}$, where $C$ is a constant. This can be achieved by assuming $W^\alpha_i = c^{-\frac{1}{\alpha}} W_i$ where $\{W_i\}$ are independent random variables that are in the strict domain of attraction of $D_1 - D_0$. Note that $A_t - A_0$ is a Lévy process as it is the limit of a triangular array. Now, let $E_t = \inf \{ s : D_s > t \}$ be the first hitting time of $D_t$,
also called the inverse of \( D_t \). By \cite[Theorem 2.4.3]{34} applied to the case of independent space and time jumps we have

\[
Y_t^c = Y_t = A_{E_t},
\]  

(2.2)

as \( c \to \infty \) where convergence is in the Skorokhod \( J_1 \) topology, see also \cite[Theorem 3.6]{35} and \cite[Theorem 3.1]{14}. Since \((S^c_k, T_k^c)\) is a Markov chain for all \( c > 0 \) it follows that the CTRWL \( Y_t \) is a semi-Markov process and it is possible to embed it in a process of larger state space that includes the time to regeneration, the remaining life time process \( R_t \). More precisely, let \(\mathcal{D} (\{0, \infty\}, \mathbb{R}^2)\) be the space of càdlàg functions \( f : \{0, \infty\} \to \mathbb{R}^2 \) with the \( J_1 \) Skorokhod topology which is endowed with transition operators \( T_u, u > 0 \) and hence a probability measure \( P^{X, \tau} \) such that trajectories start at point \((\chi, \tau)\) with probability one. Thus, we have a stochastic basis \( (\Omega, \mathcal{F}_\infty, (\mathcal{F}_u)_{u \geq 0}, P^{X, \tau}) \), where each element of \( \Omega \) is in \( \mathcal{D} (\{0, \infty\}, \mathbb{R}^2) \), \( \mathcal{F}_u = \sigma ((A_u (\omega), D_u (\omega))) \) and \( \mathcal{F}_\infty = \vee_{u > 0} \mathcal{F}_u \). The process \((A, D)_t\) has a generator of the form

\[
A(f) (x, t) = b \frac{\partial f(x, t)}{\partial x} - \frac{1}{2} a \frac{\partial^2 f(x, t)}{\partial x^2} + \int_{\mathbb{R}^2} \left( f(x + y, t + w) - f(x, t) - y \frac{\partial f(x, t)}{\partial x} 1_{\{|(y,w)|<1\}} \right) K(dy, dw),
\]

where \( a > 0 \) and \( b \in \mathbb{R} \) and \( K(dy, dw) \) is a Lévy measure. The occupation time measure of the process \((A, D)_t\) is the average time spent by the process in a given Borel set in \( \mathbb{R}^2 \), i.e

\[
\int f(x, t) U^{X, \tau} (dx, dt) = \mathbb{E}^{X, \tau} \left( \int_0^\infty f(A_u, D_u) \, du \right) = \int_0^\infty T_u f(\chi, \tau) \, du.
\]

Let us now define the remaining life time process \( R_t \)

\[
R_t = D_{E_t} - t,
\]

which is the time left for the process \( Y_t \) to leave its current state. It was proven in \cite[Theorem 2.3]{25} that

\[
E^{X, \tau} (f(Y_t, R_t)) = \int \int \int \int \int K(dy, dw) f(x + y, w - (t - s)).
\]

In \cite[Theorem 2.3]{25}, a more general CTRWL is considered and hence a more general form of (2.3) where the coefficients \( a \) and \( b \) as well as the Lévy measure \( K(dy, dw) \) are allowed to be dependent on the position of the CTRWL in space and time, that is, we have \( b(x, t), a(x, t) \) and \( K(x, t; dy, dw) \). As was noted in \cite[section 4]{25}, when these coefficients do not depend on \( t \) (as in our case), the process \((Y_t, R_t)\) is the transition operator of the Markov process \((Y_t, R_t)\) starting at \( \chi, \tau \), i.e

\[
Q_t [f] (y, 0) = E^{y, 0} (f(Y_t, R_t))
\]

\[
Q_t [f] (y, r) = 1_{\{0 \leq r < t\}} f(y, r - t) + 1_{\{0 \leq r \leq t\}} Q_{t-r} [f](y, 0) \quad r > 0,
\]

(2.5)

(2.6)

for every \( f \) bounded and measurable on \( \mathbb{R} \times [0, \infty) \). \( Q_t \) is the transition operator of the Markov process \((Y_t, R_t)\) starting at \( \chi, \tau \), i.e

\[
E^{X, \tau} (f(Y_{t+h}, R_{t+h}) | \sigma ((Y_r, R_r), t \geq r \geq 0)) = Q_h [f] (Y_t, R_t).
\]

(2.7)
One can use the Chapman-Kolmogorov’s equation to obtain the finite dimensional distributions of the process $Y_t$. For example, suppose $(Y_0, R_0) = (0, 0)$ a.s, then for the two dimensional distribution of the process $Y_t$ at times $t_1 < t_2$ we have

\begin{equation}
(2.8) \quad P (Y_{t_1} \in B_1, Y_{t_2} \in B_2) = P ((Y_{t_1} \in B_1, R_{t_1} \in [0, \infty)), (Y_{t_2} \in B_2, R_{t_2} \in [0, \infty))) = Q_{t_1} \left[ 1_{\{B_1 \times \mathbb{R}\}} (y_1, r_1) Q_{t_2-t_1} \left[ 1_{\{B_2 \times \mathbb{R}\}} (y_2, r_2) \right] (y_1, r_1) \right] (0, 0),
\end{equation}

where $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ are Borel sets.

Remark 1. In [25] a result stronger than (2.7) was shown. Indeed, the process $(Y_t, R_t)$ is a strong Markov process with respect to a filtration larger than the natural filtration. For the sake of brevity and the fact that the Markov property is adequate for our work we brought the result in a weaker form.

3. Aging

Let us assume (2.1) holds with $\chi = \tau = 0$ so $A_t$ is a Levy process with CDF $P_t(x) = P (A_t \in (-\infty, x])$ and with Levy triplet $(\mu, A, \phi)$, i.e.

\[
E (e^{iuA_t}) = \exp \left[ t \left( i\mu u - \frac{1}{2}Au^2 + \int \phi(dy) \right) \right].
\]

Also assume $D_t$ is a stable subordinator of index $0 < \alpha < 1$ with Laplace transform (LT) $E (e^{-uD_t}) = e^{-tcu^\alpha}$ independent of $A_t$. Then (2.3) holds with $b = \mu, a = A$ and (see [7, Corollary 2.3])

\begin{equation}
K(dy, dw) = \phi(dy) \delta_0(dw) + \delta_0(dy) \frac{c\alpha}{\Gamma(1-\alpha)} w^{-1-\alpha} 1_{\{w > 0\}} dw.
\end{equation}

Next, we wish to find the occupation measure of the process $(A, D)_t$. We have for $f(y, w) = 1_{\{(-\infty,x] \times (-\infty,t]\}}(y, w)$

\[
\int f(y, w) U^{x,\tau} (dy, dw) = E^{x,\tau} \left( \int_0^\infty f(A_u, D_u) du \right)
\]

\[
= \int_0^\infty T_u f(\chi, \tau) du = \int_0^\infty T_u 1_{\{(-\infty,x] \times (-\infty,t]\}}(\chi, \tau) du
\]

\[
= \int \int 1_{\{(-\infty,x] \times (-\infty,t]\}}(y + \chi, w + \tau) q_u(dy, dw) du,
\]
where \( q_t \) is the distribution of the process \((A, D)_t\) cf. [1, Eq. 3.11]. By independence of \( A_t \) and \( D_t \) we have

\[
\int f(x, t) U^{x, \tau} (dx, dt) = \int_0^\infty P(A_u \in (-\infty, x - \chi]) P(D_u \in (-\infty, t - \tau]) du
\]

\[
= \int_0^\infty P_u(x - \chi) \int_{-\infty}^{t-\tau} g(w, u) dw du,
\]

(3.2)

where \( g(x, t) \) is the pdf of \( D_t \), i.e. \( g(x, t) dx = P(D_t \in dx) \) and is known to be absolutely continuous with respect to the Lebesgue measure [38, Section 2.4].

Since \((A, D)_t\) is a Levy process the coefficients in (2.3) are independent of \( t \) and therefore the process \((A, D)_t\) is a Markov additive process [25, Section 4] and the occupation measure is of the form

(3.3)

\[
U^n (dx, dt) = \int_0^\infty P_u(dx - y) g(t, u) du dt.
\]

Furthermore, one may choose \( \tau = 0 \) and plug (3.1) and (3.3) in (2.4) to obtain

(3.4)

\[
E^{x,0} (f(Y_t, R_t)) = \int_{x \in \mathbb{R}} \int_{s \in [0, t]} \left( \int_{u \in \mathbb{R}^+} P_u(dx - \chi) g(s, u) du \right)
\times \int_{y \in \mathbb{R}} \int_{w \in [t - s, \infty)} \left( \phi(dy) \delta_0(dw) + \delta_0(dy) \frac{c\alpha}{\Gamma(1 - \alpha)} w^{-1 - \alpha} dw \right) f(x + y, w - (t - s)) ds
\]

\[
= \int_{x \in \mathbb{R}} \int_{s \in [0, t]} \left( \int_{u \in \mathbb{R}^+} P_u(dx - \chi) g(s, u) du \right)
\times \int_{w \in [t - s, \infty)} f(x, w - (t - s)) \frac{c\alpha}{\Gamma(1 - \alpha)} w^{-1 - \alpha} dw ds,
\]

for \( Y_t \in \mathcal{S} \) and its time to regeneration \( R_t \).

We say that the r.v \( X \) has beta distribution with parameters \( \mu, \nu > 0 \) if it has pdf of the form

\[
f(x, \mu, \nu) = \frac{x^{\mu-1} (1 - x)^{\nu-1}}{B[\mu, \nu]} \quad x \in (0, 1)
\]

where \( B[\mu, \nu] = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu + \nu)} \) is the Beta function and we write \( X \sim B(\mu, \nu) \). We say that the r.v \( X \) has beta prime distribution with parameters \( \mu, \nu > 0 \) if it has pdf of the form

(3.5)

\[
f(x, \mu, \nu) = \frac{x^{\mu-1} (1 + x)^{-\mu-\nu}}{B[\mu, \nu]} \quad x > 0
\]

and we write \( X \sim B'(\mu, \nu) \). It was noted in [12, II.4] that if \( X \sim B(\mu, \nu) \) then \( \frac{X}{1-X} \sim B'(\mu, \nu) \).

The distribution (3.5) can be further generalized to the so called generalized Beta prime distribution.
also known as the general Beta of the second kind distribution whose pdf is

\[
(3.6) \quad f(x, \mu, \nu, h) = \frac{\left(\frac{x}{h}\right)^{\mu-1} \left(1 + \frac{x}{h}\right)^{-\mu-\nu}}{h \cdot B[\mu, \nu]} \quad x > 0
\]

with \( h, \mu, \nu > 0 \). If \( X \) has generalized Beta prime distribution of the form \( (3.6) \) then we write \( X \sim GB2(\mu, \nu, h) \).

**Theorem 1.** Let \( Y^{t_{0}}_{t_{1}} = A_{E_{t_{1}}} - A_{E_{t_{0}}} \) where \( t_{0} > 0 \) be the aging process. Let \( B_{1}, B_{2}, \ldots, B_{k} \) be Borel sets such that \( 0 \notin B_{1} \). Let \( p_{r_{0}}(r) = f(r, 1 - \alpha, \alpha, t_{0}) \) be a generalized beta prime distribution as in \( (3.6) \). Then we have for \( 0 < t_{1} < t_{2} < \cdots < t_{k} \)

\[
(3.7) \quad P(Y^{t_{0}}_{t_{1}} \in B_{1}, Y^{t_{0}}_{t_{2}} \in B_{2}, \ldots, Y^{t_{0}}_{t_{k}} \in B_{k}) = \int_{0}^{t_{1}} P(Y_{t_{1}-r} \in B_{1}, Y_{t_{2}-r} \in B_{2}, \ldots, Y_{t_{k}-r} \in B_{k}) p_{r_{0}}(r) dr.
\]

**Proof.** For simplicity, we proof the result for \( k = 2 \), the proof for \( k > 2 \) is similar. We have

\[
(3.8) \quad P(Y^{t_{0}}_{t_{1}} \in B_{1}, Y^{t_{0}}_{t_{2}} \in B_{2}) = Q_{t_{0}}[1_{[0, \infty)}(y_{0}, r_{0}) \times Q_{t_{1}}[1_{[y_{1}, r_{1})}(y_{1}, r_{1}) Q_{t_{2}-t_{1}}[1_{[y_{2}, r_{2})}(y_{2}, r_{2}) (y_{0}, r_{0})] (0, 0).
\]

It is easy to see that by \( (3.4) \) the semi-group operator \( Q_{t} \) is translation invariant with respect to the space variable when \( r = 0 \), i.e., \( Q_{t}[f](y + a, 0) = Q[g](y, 0) \) where \( g(y, r) = f(y + a, r) \). Moreover,

\[
Q_{t}[f](y + a, r) = 1_{0 < t < r}f(y + a, r - t) + 1_{0 \leq r \leq t}Q_{t-r}[f](y + a, 0)
\]

\[
= 1_{0 \leq r < t}g(y, r - t) + 1_{0 \leq r \leq t}Q_{t-r}[g](y, 0)
\]

\[
= Q_{t}[g](y, r).
\]

Hence, \( Q_{t} \) is translation invariant with respect to the space variable. Consequently, since \( 0 \notin B_{1}, \) by \( (2.6) \) we have

\[
(3.9) \quad P(Y_{t_{1} - r_{0}} \in B_{1}, Y_{t_{2} - r_{0}} \in B_{2}).
\]

For ease of notation we write \( P(Y_{t_{1} - r_{0}} \in B_{1}, Y_{t_{2} - r_{0}} \in B_{2}) = f(r_{0}) \). Plug \( (3.9) \) in \( (3.8) \) and use \( (3.4) \) to obtain,

\[
(3.10) \quad P(Y^{t_{0}}_{t_{1}} \in B_{1}, Y^{t_{0}}_{t_{2}} \in B_{2}) = \int_{s_{0} \in [0, t_{0}]} \left( \int_{w' \in \mathbb{R}^{+}} g(s', u') du' \right)
\]

\[
\times \int_{w' \in [t_{0} - s', \infty)} \frac{c\alpha}{\Gamma(1-\alpha)} w^{-1-\alpha} dw ds'
\]

\[
\times \left[ 1_{0 \leq w' - (t_{0} - s') \leq t_{1}} \times f(w' - (t_{0} - s')) \right]
\]
\[
\begin{align*}
\int_{s' \in [0,t_0]} \int_{u' \in \mathbb{R}^+} g(s',u') \, du' \times \int_{u' \in [t_0-t',t_0-t'+s']} f(w'-(t_0-s')) \times \frac{c_\alpha}{\Gamma(1-\alpha)} w'^{-\alpha} \, du' \, ds'.
\end{align*}
\]

By [31, Eq. 37.12] if \( D_t \) is a stable subordinator of index \( 0 < \alpha < 1 \) with \( E \left( e^{-uX_t} \right) = e^{-tcu^\alpha} \) and probability distribution \( P(D_t \in dx) = g(x,t) \, dx \) then its potential density is given by

\[
(3.11) \quad v(s) = \int_{u \in \mathbb{R}^+} g(s,u) \, du = \frac{1}{c\Gamma(\alpha)} s^{\alpha-1} \quad s > 0.
\]

Substitute \((3.11)\) in \((3.10)\) and apply the change of variables \( r = w' + s' - t_0 \) to obtain

\[
(3.12) \quad P\left( Y_{t_1}^{t_0} \in B_1, Y_{t_2}^{t_0} \in B_2 \right) = \int_{0}^{t_1} \int_{s' \in [0,t_0]} \frac{s'^{\alpha-1}}{c\Gamma(\alpha)} f(r) \frac{c_\alpha}{\Gamma(1-\alpha)} (r-s'+t_0)^{-1-\alpha} \, ds' \, dr.
\]

Now apply the change of variables \( v = s'(r-s'+t_0)^{-1} \) to compute the integral with respect to \( s' \) and to obtain

\[
P\left( Y_{t_1}^{t_0} \in B_1, Y_{t_2}^{t_0} \in B_2 \right) = \int_{0}^{t_1} f(r) \left( \frac{r}{t_0} \right)^{-\alpha} \left( 1 + \frac{r}{t_0} \right)^{-1} \, dr \]
\[
= \int_{0}^{t_1} P(Y_{t_1-r} \in B_1, Y_{t_2-r} \in B_2) p_{t_1}(r) \, dr.
\]

\(
\square
\)

Remark 2. It follows from Theorem 1 that

\[
(3.13) \quad P\left( Y_{t_1}^{t_0} = 0 \right) = \int_{0}^{t} p_{t_0}(r) \, dr + \int_{0}^{t} P(Y_{t-r} = 0) \cdot p_{t_0}(r) \, dr > 0.
\]

Therefore, the distribution of \( Y_{t_1}^{t_0} \) has an atom at the origin for every \( t \). More interesting is the fact that if \( P(A_t = 0) = 0 \) (this is true for all processes with pdf) then \( P(Y_{t_1}^{t_0} = 0) \) does not depend on the choice of the process \( A_t \). On the other hand it can be easily seen that for every \( t \) the process \( Y_{t_1}^{t_0} \) has density on \( \mathbb{R} \setminus \{0\} \) given by \( p_{t_0}(x,t) = \int_{0}^{t} p(x,t-r) \cdot p_{t_0}(r) \, dr \) whenever \( A_t \) has pdf \( p(x,t) \). Furthermore, note that the finite dimensional distributions of the process \( Y_{t_1}^{t_0} \) on Borel sets \( B_1, \ldots, B_k \) such that \( 0 \notin B_1 \), determine completely the finite dimensional distributions of the process \( Y_{t_0} \). We demonstrate this for \( k = 2 \); if \( B_2 \) is a Borel set then

\[
P\left( Y_{t_1}^{t_0} = 0, Y_{t_2}^{t_0} \in B_2 \right) = P\left( Y_{t_2}^{t_0} \in B_2 \right) - P\left( Y_{t_1}^{t_0} \in \mathbb{R} \setminus \{0\}, Y_{t_2}^{t_0} \in B_2 \right),
\]

which by \((3.13)\) determines the two dimensional distributions completely.
Remark 3. In [6], a result similar to Theorem 1 for the one dimensional distribution is obtained for CTRW for large $t_0$ and $t$. The proof in [6] sheds light on our result, as it was derived from showing that the distribution of the first epoch $\tau_1$ of the aging CTRW $X_{t_0}$ has beta prime distribution, i.e $\tau_1 \sim B^\prime (1-\alpha, \alpha, t_0)$. This can be shown by a result by Dynkin on renewal processes ([9, Theorem 8.6.3]). Interestingly, the distribution of the first epoch $\tau_1$ does not scale out as we move to the limit and obtain the process $Y_{t_0}$. Indeed, one can show (similarly to the proof of Theorem 1) that the distribution of the process $R_t$, the time left before the next regeneration at time $t$, is

\begin{equation}
(3.14) \quad f_{R_t} (r) = \frac{(\frac{r}{t})^{-\alpha} \left( 1 + \frac{r}{t} \right)^{-1}}{t \cdot B [\alpha, 1-\alpha]} \quad r > 0.
\end{equation}

Since it was noted in [25] that the process $Y_t$ starts afresh at time $H_t = D_{E_t} = t$ depending only on the position of $Y_t$, and by the fact that in our case the process $Y_t$ is homogeneous in space, it follows that once the process $Y_{t_0}$ leaves the state 0 it behaves like the process $Y_t$ from that point on. Now, condition the probability $P \left( Y_{t_0} \in B_1, Y_{t_0} \in B_2, ..., Y_{t_0} \in B_k \right)$ on the event $\{R_{t_0} = r\}$ and integrate with respect to $r$ to obtain (3.7). It should be clear now why $0 \notin B_1$ as we would like to make sure that the system is mobilized before time $t_1$.

Remark 4. Let $X_t$ be a renewal process with interarrival times $\{W_i\}$ whose tail distribution $1 - F(x) \in R(-\alpha)$ for $0 < \alpha < 1$, namely, there exists a slowly varying function $L(x)$ such that $1 - F(x) \sim x^{-\alpha} L(x)$ when $x \to \infty$. Define the arrival times $T_n = \sum_{i=1}^{n} W_i$ and let $S_t = t - T_{n_t}$, be the age process, the time spent at the current state. It was shown in [9, Theorem 8.6.3] that the distribution of $\frac{S_t}{\alpha}$ converges, as $t \to \infty$. The limit is the so called Generalized Beta of the first kind distribution $GB1(1-\alpha, \alpha, 1)$ whose pdf equals $f_{V_1}$, where

\begin{equation}
(3.15) \quad f_{V_1} (v) = \frac{(\frac{v}{t})^{-\alpha} \left( 1 - \frac{v}{t} \right)^{\alpha-1}}{tB[\alpha, 1-\alpha]} \quad 0 < v < t.
\end{equation}

In [25], the analogous process $V_t = t - D_{E_t-}$ was defined to track the time that has passed since the last regeneration of the process $Y_t$. It can be easily shown, along similar lines to the proof of Theorem 1, that the process $V_t$ has distribution $GB1(1-\alpha, \alpha, 1)$. Equations 3.14 and 3.15 explain the results of Jurlewicz et al in [14]. There it was proven ([14, Eq. 5.12]) that $D_{E_t}$ has pdf

\begin{equation}
(3.16) \quad g(r) = \frac{r^{-1}}{B[\alpha, 1-\alpha]} \left( \frac{t}{r-t} \right)^{\alpha} \quad r > t,
\end{equation}

and that $D_{E_t-}$ has pdf ([14, Eq. 5.9])

\begin{equation}
(3.17) \quad h(v) = \frac{v^{\alpha-1} (t-v)^{-\alpha}}{B[\alpha, 1-\alpha]} \quad 0 < v < t.
\end{equation}

Equation 3.16 and 3.17 can be obtained by 3.14 and 3.15 respectively, by translation and reflection.

4. Aging self similarity

Recall that a process $X_t$ is called self-similar if for every $a > 0$ there exists $b > 0$ such that the finite dimensional distributions of the time scaled process $X_{at}$ equals that of the process $bX_t$. It is well known ([31, Section 13]) that if $X_t$ is a Lévy process then it is self-similar if and only if $X_t$ is strictly stable, i.e for every $a > 0$ there exist $b > 0$ such that $E(e^{iuX_t})^a = E(e^{iuX_1})^a$. For self-similar non trivial processes that are stochastically continuous at $t = 0$, $b = aH$ ([10, Theorem 1.1.1]), where $H > 0$ if and only if $X_t = 0$ with probability one. $H$ is sometimes called the Hurst parameter. For example, for fractional Brownian motion $0 < H < 1$ while the Hurst parameter
of the stable subordinator of index $0 < \alpha \leq 2$ is $1/\alpha$. For self-similar processes with stationary increments and finite second moment the Hurst parameter (when it exists) determines long range dependence ([10, Section 3.2]). Throughout this section we consider the process $Y^t_{E_t} = A_{E_t+t_0} - A_{E_t}$ where $A_t$ is a strictly stable process whose Hurst parameter we denote by $1/\beta$ and $E_t$ is the inverse of a stable subordinator of index $\alpha$. We wish to find whether $Y^t_{E_t}$ has the property of self-similarity or a different property that resembles self-similarity to some extent. From Theorem 1 it is only reasonable that any self-similarity-like property of $Y^t_{E_t}$ should be strongly connected to the self-similar-like behavior of the process $Y_t$.

The next corollary states that although the aging process $Y^t_{E_t}$ is not self-similar it exhibits a self-similar-like behavior. Intuitively it suggests that $Y^t_{E_t}$ behaves like a “younger” ($a > 1$) scaled version of itself.

**Corollary 1.** Let $Y^t_{E_t}$ be an aging process and let $B_i$ for $1 \leq i \leq k$ be Borel sets in $\mathbb{R}$. Then

$$
Y^t_{E_t}, Y^t_{E_t}, ..., Y^t_{E_t} \sim_d \left( a^\frac{\alpha}{\beta} Y_t, a^\frac{\alpha}{\beta} Y_t, ..., a^\frac{\alpha}{\beta} Y_t \right)
$$

**Proof.** For simplicity we only prove the result for $k = 2$ as the proof for $k > 2$ is similar. First assume that $B_1 \subseteq \mathbb{R}$ does not contain zero. By Theorem 1 we have

$$
P \left( Y^t_{E_t} \in B_1, Y^t_{E_t} \in B_2 \right) = \int_0^{t_1} P \left( Y_{a(t1-r)} \in B_1, Y_{a(t2-r)} \in B_2 \right) p_{t_0} (r) dr.
$$

Apply the change of variables $r' = \frac{t_0}{a}$ to obtain

$$
P \left( Y^t_{E_t} \in B_1, Y^t_{E_t} \in B_2 \right) = \int_0^{t_1} P \left( a^\frac{\alpha}{\beta} Y_{a(t1-r')} \in B_1, a^\frac{\alpha}{\beta} Y_{a(t2-r')} \in B_2 \right) \frac{\left( \frac{t_0}{a} \right)^{-\alpha}}{t_0 \cdot B_{\alpha, 1-\alpha}^{-1}} adr'.
$$

By [21, Corollary 4.1] $Y_t$ is self similar with Hurst parameter $\frac{\alpha}{\beta}$. Therefore we have

$$
P \left( Y^t_{E_t} \in B_1, Y^t_{E_t} \in B_2 \right) = \int_0^{t_1} P \left( a^\frac{\alpha}{\beta} Y_{a(t1-r')} \in B_1, a^\frac{\alpha}{\beta} Y_{a(t2-r')} \in B_2 \right) p_{t_0} (r) dr
$$

$$
= P \left( a^\frac{\alpha}{\beta} Y_{a(t1)} \in B_1, a^\frac{\alpha}{\beta} Y_{a(t2)} \in B_2 \right).
$$

Now, by Remark 2 it follows that (4.1) holds for any Borel sets $B_1, B_2 \subseteq \mathbb{R}$ and the result follows. □

5. **Asymptotic behavior and the Fractional Fokker-Planck equation**

An easy yet important consequence of Theorem 1 is the following.

**Corollary 2.** Let $B \subseteq \mathbb{R}$ be a Borel measurable subset such that $0 \notin B$ and $P \left( Y^t_{E_t} \in B \right) \neq 0$, then

$$
P \left( Y^t_{E_t} \in B \right) \sim C t_0^{-\alpha} \quad t_0 \to \infty
$$

where $C = \frac{\sin(\pi \alpha)}{\pi} \int_0^t P \left( Y_{t-r} \in B \right) r^{-\alpha} dr$. 

Proof. First note that by the continuity of \( P(Y_t \in B) \) (see (1)) \( C \neq 0 \iff P(Y_{t_0} \in B) \neq 0 \). By dominated convergence we then have,

\[
\lim_{t_0 \to \infty} \frac{P(Y_{t_0} \in B)}{C t_0^{\alpha-1}} = \lim_{t_0 \to \infty} \frac{t_0^{\gamma-1} \sin(\pi \alpha)}{\pi} \int_0^t P(Y_{t-r} \in B) r^{-\alpha} 1 + t_0^{-1} dr = 1. 
\]

Remark 5. When the process \( Y_t \) is a renewal process (that is the case for the FPP) the convergence of \( P(Y_{t_0} \in B) \) to zero is expected by the Renewal Theorem ([12, XI.1]) and the fact that the interarrival times have the Mittag-Leffler distribution with infinite expectation. Interestingly, it was shown by Erickson in [11, Theorem 1], that if \( Y_t \) is a renewal process with interarrival times \( W_n \) with \( F(t) = P(W_t \leq t) \) such that \( 1 - F(t) \in R(-\alpha) \) for \( 0 < \alpha < 1 \), i.e \( 1 - F(t) \sim t^{-\alpha} L(t) \) as \( t \to \infty \) where \( L(t) \) is a slowly varying function and \( F \) is not arithmetic, then

\[
E(Y_{t_0}) \sim \frac{\sin(\pi \alpha)}{\pi} \frac{t}{L(t_0)} t_0^{\alpha-1} \quad t_0 \to \infty.
\]

We now show how (5.2) can be obtained for the FPP by Corollary 2. First note that by similar arguments as in Corollary 2 we have

\[
E(Y_{t_0}) \sim t_0^{\alpha-1} \frac{\sin(\pi \alpha)}{\pi} \int_0^t E(Y_{t-r}) r^{-\alpha} dr \quad t_0 \to \infty.
\]

Let \( Y_t = N_t^\alpha \) be the fractional Poisson process with intensity \( \lambda = 1 \). By [8, Eq. 2.7], \( E(Y_{t_0}) \sim \frac{(t-r)^\alpha}{\Gamma(1+\alpha)} \) and so by (5.3) we have

\[
E(Y_{t_0}) \sim t_0^{\alpha-1} \frac{\sin(\pi \alpha)}{\pi} \int_0^t \frac{(t-r)^\alpha}{\Gamma(1+\alpha)} r^{-\alpha} dr 
\]

\[
= t_0^{\alpha-1} \frac{\sin(\pi \alpha)}{\pi} t \Gamma(1-\alpha).
\]

To see this, note that

\[
\int_0^t (t-r)^\alpha r^{-\alpha} dr = \Gamma(1-\alpha) \partial_t^\alpha \Gamma(\alpha+1)_{t \geq 0} = \Gamma(1-\alpha) \Gamma(\alpha+1) t, \quad \partial_t^\alpha
\]

is the Caputo derivative of index \( \alpha \) (5.8). This agrees with (5.2). Indeed, note that by [29] the asymptotic behavior of the Mittag-Leffler distribution pdf is \( f^\alpha(t) \sim \frac{t^{-1-\alpha}}{\Gamma(1-\alpha)} \) as \( t \to \infty \) (note that there is a typo there as \( \alpha \) should be in the numerator) and by the Karamata Tauberian Theorem ([9, Theorem 1.5.11]) we see that \( E^\alpha(-t^\alpha) = \int_t^\infty f(y) dy \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \) as \( t \to \infty \) so \( (L(t))^{-1} = \Gamma(1-\alpha) \).

While it is known that generally CTRWL lose their stationarity property for \( 0 < \alpha < 1 \) ([21, Corollary 4.3]), Theorem 1 suggests a way of measuring the stationarity of a process in the class \( S \). The FPP for example has no stationary increments for \( 0 < \alpha < 1 \), however, for \( \alpha = 1 \) we obtain
the Poisson process which is of course stationary as being a Levy process. We proceed with a useful lemma that states that the distribution of the processes in $S$ is continuous as a function of time.

**Lemma 1.** Let $Y_t \in S$ and $C \subset \mathbb{R}$ a Borel set, then the function $t \mapsto P(Y_t \in C)$ is continuous on $(0, \infty)$.

**Proof.** Since $Y_t = A_{E_t}$, by a simple conditioning argument ([24, Eq. (2.7)]) we have

$$P(Y_t \in C) = \int_0^\infty \int_0^x P(A_y \in C) h(y, t) dy,$$

where $h(x, t)$ is the pdf of the process $E_t$. Then

$$\limsup_{h \to 0} |P(Y_{t+h} \in C) - P(Y_t \in C)| = \limsup_{h \to 0} \int_0^\infty \int_0^\infty \int_0^\infty |P(A_y \in C) h(y, t+h) dy - P(A_y \in C) h(y, t) dy|$$

It was proved in ([21, Corollary 3.1]) that

$$h(x, t) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} g(tx^{-\frac{1}{\alpha}})$$

where $g(x)$ is the pdf of a stable r.v. Since $g(x)$ is smooth it follows that $h(x, t)$ is continuous on $t, x > 0$. Trivially we have

$$\lim_{h \to 0} \int_0^\infty h(y, t+h) dy = \int_0^\infty h(y, t) dy = 1.$$

Hence, a basic result in analysis [30, Chapter 7, Theorem 7] implies that

$$\lim_{h \to 0} \int_0^\infty |h(y, t+h) - h(y, t)| dy = 0,$$

and the result follows. □

The next result states that as $\alpha \to 1$ the process $Y_t$, in some sense, becomes more stationary.

**Proposition 1.** Let $Y_t \in S$, then for every $t, t_0 > 0$

$$Y_{t_0}^{t_0} = Y_{t+t_0} - Y_{t_0} \overset{d}{\to} Y_t \quad \alpha \to 1.$$

**Proof.** In [33, eq. 3.1.19] it was shown that

$$U(a, b, s) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-sx} x^{a-1} (1+x)^{b-a-1} dx,$$

where $U(a, b, s)$ is a hypergeometric function that solves the confluent hypergeometric equation, also known as Kummer’s equation

$$s \frac{\partial^2 U}{\partial^2 s} + (b-s) \frac{\partial U}{\partial s} - aU = 0.$$
By (5.4) and a simple change of variables we find that the Laplace transform of the generalized Beta prime distribution is given by

\[
\hat{p}_{t_0}(s) = \frac{U(1-\alpha, 1-\alpha, st_0)}{\Gamma(\alpha)} \quad s > 0.
\]

Using the identity \(U(1-\alpha, 1-\alpha, x) = e^x \Gamma(\alpha, x)\) where \(\Gamma(\alpha, x)\) is the incomplete gamma function defined by \(\Gamma(\alpha, x) = \int_x^{\infty} t^{\alpha-1} e^{-t} dt\), we can write (5.6) in a more familiar notation

\[
\hat{p}_{t_0}(s) = \frac{e^{st_0} \Gamma(\alpha, st_0)}{\Gamma(\alpha)}.
\]

Now, by dominated convergence

\[
\lim_{\alpha \to 1} \hat{p}_{t_0}(s) = \lim_{\alpha \to 1} \frac{e^{st_0} \Gamma(\alpha, st_0)}{\Gamma(\alpha)} = \frac{e^{st_0} e^{-st_0}}{1} = 1.
\]

Therefore, by [15, Theorem 4.3] we have \(p_{t_0} \overset{w}{\to} \delta\) as \(\alpha \to 1\) where \(\overset{w}{\to}\) denotes weak convergence of probability measures and \(\delta\) is the Dirac delta measure. For a Borel set \(B\) such that \(0 \notin B\) define

\[
f(r) = \begin{cases} \quad P(Y_{t-r} \in B), & 0 \leq r \leq t \\ 0 & t < r \end{cases},
\]

and note that \(P(Y_0 \in dx) = \delta_0(dx)\) and therefore \(P(Y_{t-r} \in B) = 0\) at \(r = t\). Consequently, Lemma 1 suggests that \(f(r)\) is continuous. By the fact that

\[
P(Y_{t_0}^{t_0} \in B) = \int_0^t f(r) p_{t_0}(r) dr
\]

we also have \(P(Y_{t_0}^{t_0} = 0) \to P(Y_t = 0)\) and the proof is complete.

\[
\text{Remark 6. It was shown in [33, eq. 4.1.12] that } U(a, b, s) \sim Cs^{-a} \text{ as } s \to \infty. \text{ It follows that }
\]

\[
\hat{p}_{t_0}(s) \sim C(s_{t_0})^{\alpha-1} \quad t_0 \to \infty
\]

and therefore \(\hat{p}_{t_0}(s) \to 0\) as \(t_0 \to \infty\). Hence, \(p_{t_0} \overset{v}{\to} 0\) as \(t_0 \to \infty\) where \(\overset{v}{\to}\) denotes vague convergence of distributions, and \(P(Y_{t_0}^{t_0} \in dx) \overset{v}{\to} \delta_0(dx)\), another proof for the fact that \(P(Y_{t_0}^{t_0} \in B) \to 0\) as \(t_0 \to \infty\) for \(B\) such that \(0 \notin B\). It is not hard to verify that \(\hat{p}_{t_0} \to 0\) as \(\alpha \to 0\). Intuitively, this is expected since a small \(\alpha\) suggests long waiting times between jumps and that \(Y_t\) is very subdiffusive.

Let \(p(dx, t)\) be a stochastic kernel, that is, for every \(t > 0\) \(p(dx, t)\) is a probability measure on \(\sigma(\mathbb{R})\) and for each Borel set \(B \subseteq \mathbb{R}\) \(p(B, \cdot)\) is measurable. Denote the Fourier transform of \(p(dx, t)\) by \(\tilde{p}(k, t) = \int_\mathbb{R} e^{-ikx} p(dx, t)\), and the Fourier-Laplace transform (FLT) by \(\tilde{\rho}(k, s) =

\[ f_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-st - ikx} p(dx, t) \, dt. \] Recall the definition of the Caputo \( 0 < \alpha < 1 \) fractional derivative of a function \( f(t) \),

\[
\partial^\alpha_t f = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \frac{\partial f(r)}{\partial r} \, dr.
\]

(5.8)

For \( 0 < \alpha < 1 \) the Laplace transform of \( \partial^\alpha_t f \) is ([22, p. 39])

\[
\hat{\partial^\alpha_t f} = s^\alpha \hat{f} - s^{\alpha-1} f(0^+).
\]

A closely related operator is the Riemann Liouville derivative \( D^\alpha_t \) for \( 0 < \alpha < 1 \), which is defined by

\[
D^\alpha_t f = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-r)^{-\alpha} f(r) \, dr.
\]

(5.9)

The LT of (5.9) can be shown to be \( \hat{D^\alpha_t f} = s^\alpha \hat{f} \). It follows that

\[
\partial^\alpha_t f = D^\alpha_t f - f(0^+) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.
\]

(5.10)

The following is a short summary of results in [3]. Let \( V^\omega = L^1_\omega(\mathbb{R} \times \mathbb{R}_+) \) be the space of real valued measurable functions on \( \mathbb{R} \times \mathbb{R}_+ \) such that

\[
\| f \|_\omega = \int_0^\infty e^{-\omega t} \int_{\mathbb{R}^d} |f(x, t)| \, dx \, dt < \infty,
\]

for some \( \omega > 0 \). \( V^\omega \) is a Banach space w.r to \( \| \cdot \|_\omega \). If \( (A_t, D_t) \) is a Lévy process where \( D_t \) is a subordinator and s.t \( E(e^{-ikA_t - sD_t}) = e^{s\eta(-k, s)} \), then the distribution of \( (A_t, D_t) \) gives way to a semi group of operators whose infinitesimal generator \( L \) satisfies \( \hat{L} f = \eta(-k, s) \hat{f}(k, s) \) (for \( \omega \leq s \)). In fact, \( f \) is in the domain of \( L \), \( D(L) \), iff \( \hat{\eta}(k, s) = \eta(-k, s) \hat{f}(k, s) \) where \( \hat{\eta}(k, s) \) is the FLT of some \( g \in V^\omega \). If the first and second order spatial weak derivatives as well as the first order time weak derivative of \( f \) is in \( V^\omega \) then \( f \in D(L) \). Let \( p^\alpha(dx, t) \) be the probability measure of the process \( Y^\alpha_t \), i.e. \( p^\alpha(dx, t) = P(Y^\alpha_t \in dx) \). Suppose \( A_t \) has the symbol \( \eta(k) \) and the infinitesimal generator \( L \), and \( D_t \) is an independent standard stable subordinator. We then have \( \eta(-k, s) = -s^\alpha + \psi(-k) \), and \( L = -D^\alpha_t + L \) (since \( f \in V^\omega \), \( Lf \) should be understood as \( f(\cdot, t) \in D(A) \) for every \( t > 0 \)). Note that by [4, Theorem 2.2] smooth functions on \( \mathbb{R} \) are contained in \( D(L) \). The FLT of \( p^\alpha(dx, t) \) is well known([22, Eq. 4.43]) and given by

\[
\overline{p^\alpha}(k, s) = \frac{s^{\alpha-1}}{-\eta(-k, s)} = \frac{s^{\alpha-1}}{s^\alpha - \psi(-k)},
\]

(5.11)

which in turn implies that

\[
\partial^\alpha_t p^\alpha(dx, t) = Lp^\alpha(dx, t)
\]

\[
p^\alpha(dx, 0) = \delta_0(dx).
\]

(5.12)

Equation (5.12) describes the dynamics of \( p^\alpha(dx, t) \) and therefore is called the Fractional Fokker Planck Equation (FFPE) of \( p^\alpha(dx, t) \). Suppose that the process \( Y^\alpha_t \) starts from the random point \( X_0 \) with density \( p(x) \in C^\infty_c(\mathbb{R}) \), that is, smooth with compact support and that \( X_0 \) is independent
of $Y_{t_0}^a$. The distribution of $Y_{t_0} + X_0$ is $C(x, t) = \int_{\mathbb{R}} p(x - y) p_{t_0}^a(dy, t)$ which is again smooth. The next theorem obtains the governing equation of $C(x, t)$.

**Theorem 2.** Let $Y_t = A_{E_t}$ have probability measure $p^0(dx, t)$ whose FLT is given by (5.11) for $0 < \alpha < 1$. Let $L$ be the generator of $A_t$. Then we have

$$\partial_t^\alpha C(x, t) = L \left( C(x, t) - p(x) \int_t^\infty p_{t_0}^a(r) \, dr \right)$$

$$C(x, 0) = p(x).$$

**Proof.** Let

$$p(x, t) = C(x, t) - p(x) \int_t^\infty p_{t_0}^a(r) \, dr,$$

and note that the FLT of (5.14) is

$$\tilde{p}(k, s) = C(k, s) - \tilde{p}(k) \left( \frac{1}{s} - \frac{1}{s} \tilde{p}_{t_0}^a(s) \right).$$

By Remark 2 we have

$$p(x, t) = \int_{\mathbb{R}} p(x - y) \int_0^t p^0(dy, t - r) p_{t_0}^a(r) \, dr.$$

By a general version of Fubini’s Theorem [2, Theorem 2.6.4] we have

$$\int_{\mathbb{R}} e^{-ikx} \int_0^t p^0(dx, t - r) p_{t_0}^a(r) \, dr = \int_0^t \tilde{p}^0(k, t - r) p_{t_0}^a(r) \, dr.$$

Take the LT of both sides of equation (5.17) to obtain

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-st - ikx} \int_0^t p^0(dx, t - r) p_{t_0}^a(r) \, dr = \int_0^t \tilde{p}^0(k, t - r) p_{t_0}^a(r) \, dr.$$

It follows that

$$\tilde{p}(k, s) = \tilde{p}^0(k, s) \tilde{p}_{t_0}^a(s) \tilde{p}(k)$$

Since by (5.11) $\tilde{p}^0(k, s) = \frac{s^{\alpha - 1}}{s^\alpha - \psi(-k)}$ we have

$$\tilde{p}(k, s) = \frac{s^{\alpha - 1}}{s^\alpha - \psi(-k)} \tilde{p}_{t_0}^a(s) \tilde{p}(k).$$

Substitute (5.15) in (5.18) to obtain

$$\tilde{C}(k, s) s^\alpha - \tilde{C}(k, s) \psi(-k) - \tilde{p}(k) \left( \frac{1}{s} - \frac{1}{s} \tilde{p}_{t_0}^a \right) s^\alpha + \tilde{p}(k) \left( \frac{1}{s} - \frac{1}{s} \tilde{p}_{t_0}^a \right) \psi(-k) = s^{\alpha - 1} \tilde{p}_{t_0}^a(s) \tilde{p}(k),$$
which can be rearranged to obtain

\begin{equation}
\mathcal{C}(k,s)(s^\alpha - \psi(-k)) = \bar{\rho}(k) s^{\alpha-1} - \left(\frac{1}{s} - \frac{1}{s\hat{\rho}_{0}}\right) \bar{\rho}(k) \psi(-k).
\end{equation}

By the preceding discussion the right hand side of (5.19) inverts to a function in $V^\omega$, taking the IFLT of both sides of the equation. Since

\begin{equation}
\mathcal{C}(5.19)
\end{equation}

which can be rearranged to obtain

\begin{equation}
\tilde{\psi}(x,t) = \frac{t}{\Gamma(1-\alpha)} \int_0^t \frac{p_0(r)}{r} dr.
\end{equation}

Noting that $C(x,0^+) = p(x)$ one can rewrite (5.20) by using (5.10) to arrive at (5.13).

**Remark 7.** Although Equation (5.13) is not an abstract Cauchy problem, one may adopt the concept of a mild solution from [27, Chapter 4] and use it in our case. Let $f \in L^1(\mathbb{R})$, we say that a function

\begin{equation}
C(x,t) = f * p^0(dx,t) = \int_{\mathbb{R}} f(x-y)p^0(dy,t)
\end{equation}

is a mild solution of

\begin{equation}
\partial_t^\alpha C(x,t) = L C(x,t) - f(x) \int_t^\infty p_0(r) dr.
\end{equation}

if there exists a sequence $\phi_n \in D(L)$ s.t $\phi_n \overset{L^1}{\to} f$ (this implies that $\phi_n * p^0(dx,t) \overset{L^1}{\to} f * p^0(dx,t)$ uniformly in $t$ on bounded sets as $p^0(dx,t)$ is a contraction for every $t$). From Theorem 2 and the fact that $C^\infty_0(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we conclude that every $f \in L^1(\mathbb{R})$ is a mild solution of (5.21). We then write for simplicity

\begin{equation}
\partial_t^\alpha p^0(dx,t) = L \left( p^0(dx,t) - \delta_0(dx) \int_t^\infty p_0(r) dr \right)
\end{equation}

\begin{equation}
p^0(dx,0) = \delta_0(dx).
\end{equation}

Since (5.22) describes the dynamics of the probability kernel $p^0(dx,t)$ we call it its FFPE.

**Remark 8.** Theorem 2 shows that the dynamics of $p^0(dx,t)$ are the same as those of $p^0(dx,t)$ on $\mathbb{R}/\{0\} \times [0,\infty)$. There is a nice intuitive interpretation to equation (5.22) when $A_t$ is a stable process. Equation (5.13) can be explained as the behavior of a plume of particles by arguments of conservation of mass and Fick’s law ([22, Remark 2.3] and [37, Section 16.1]). However, note that the portion of the mass of particles that does not diffuse away from point $x = 0$ at time $t$ (and therefore does not contribute to the change in $p^0(dx,t)$ over time) is $\int_t^\infty p_0(r) dr$ by Remark 2 and the fact that stable processes have pdf. This accounts for the difference between (5.22) and (5.12).

**Remark 9.** In [5], a deterministic system was modeled by a CTRW and its aging properties were studied. There, the FFPE was given for the unnormalized distribution (5.14) of the aging process when $t_0$ and $t$ are large. To see that the results agree, simply plug $\psi(-k) = -\frac{k^2}{2A}$ in (5.18) and take the IFLT of both sides of the equation. Since $\bar{\rho}(k,s) s^\alpha$ is the FLT of the fractional Riemann-Liouville $\alpha$ derivative we obtain [5, eq. 18].
References