CONTINUOUS TIME RANDOM WALK AS A RANDOM WALK IN A RANDOM ENVIRONMENT

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Abstract. We show that for a weakly dense subset of the domain of attraction of a positive stable random variable of index $0 < \alpha < 1 (DOA(\alpha))$ the functional stable convergence is a time-changed renewal convergence of distribution of finite mean. Applied to Continuous Time Random Walk(CTRW) á la Montroll and Wiess we show that CTRW with renewal times in a weakly dense set of $DOA(\alpha)$ can be realized as random walk in a random environment. We find the quenched limit and give a bound on the error of the approximation.

1. Introduction

Let $\{W_i\}_{i=1}^{\infty}$ (abbrv. $\{W_i\}$) be a sequence of i.i.d positive r.vs s.t $\mathbb{P}(W_1 > t) \sim t^{-\alpha}$ for $0 < \alpha < 1$. Then it is well known that the process $D_t^n = n^{-\frac{1}{\alpha}} \sum_{i=1}^{[nt]} W_i$, converges weakly in the J_1 topology to a stable subordinator, that is

$$D^n \overset{J_1}{\Rightarrow} D, \tag{1.1}$$

where $\overset{J_1}{\Rightarrow}$ denotes weak convergence w.r.t to J_1-Skorohod topology. The fact that W_1 typically has big jumps carries over to the limit. This is in contrast to the SLLN of the Renewal Theorem that says that if $\{U_i\}$ is a sequence of i.i.d r.vs s.t $\mathbb{E}(U_1) = 1$ then $T^n = n^{-1} \sum_{i=1}^{[nt]} U_i$ converges in the Skorohod topology to the function $t \mapsto t$, i.e

$$T^n \overset{J_1}{\to} t, \tag{1.2}$$

where $\overset{J_1}{\to}$ denotes a.s convergence w.r.t J_1 topology. We wish to show here that these two apparently different convergences are closely related. That in fact, observing the convergence in (1.1) is essentially observing the convergence in (1.2) viewed through a sequence of random embedding of the positive real line into itself. One use of the convergence in (1.1) is in the model of Continuous Time Random Walks (CTRW) introduced in [11] by Montroll and Wiess. In the most simple setup $\{J_i\}$ and $\{W_i\}$ are two independent sequences of i.i.d r.vs. Define $(S_n, T_n) = (\sum_{i=1}^n J_i, \sum_{i=1}^n W_i)$, the (uncoupled) CTRW associated with space-time jumps $\{(J_i, W_i)\}_{i=1}^{\infty}$ (abbrv. (J_i, W_i)) is

$$X_t = \sum_{i=1}^{N_t} J_i,$$

where $N_t = \sup\{n : T_n \le t\}$. In order to model the microscopic behavior of a particle with long binding times to a substrate, one assumes that W_1 is heavy tailed, that is

$$\mathbb{P}(W_1 > t) \sim t^{-\alpha},$$

for some $0 < \alpha < 1$. The functional limit of X_t for large t was first considered in [10] in the mathematics literature although earlier in the physics literature ([3]). Limits for coupled CTRW were considered in [4], and in [9] that of CTRW with space-time jumps that are infinitely divisible. It was shown that

$$n^{-1}X_{tn^{\frac{2}{\alpha}}} \stackrel{J_1}{\Rightarrow} B_{E_t}, \tag{1.3}$$

where B_t is a Brownian motion and E_t is the inverse stable subordinator independent of B_t defined by

$$E_t = \inf\{s : D_s > t\}.$$

The process B_{E_t}, sometimes called the Fractional Kinetics process, is a sub-diffusion in the since that it is self-similar with exponent $\frac{\alpha}{2}$, i.e

$$B_{E_{tc}} \sim c^{\frac{\alpha}{2}} B_{E_t}.$$

Our results show that the invariance principle in (1.3) where the limit is a Bm subordinated to an independent inverse subordinator, is not merely a property of the limit but is the case for the CTRW itself, even when the CTRW is coupled, i.e. when the r.v W_i and J_i are dependent. In fact, we show this for a larger set of CTRWs, namely CTRW with waiting times with infinite mean with some restriction on their Laplace Transform. A simple case is when X_t is an uncoupled CTRW associated with the i.i.d space-time jumps (J_i, W_i), where $W_i \in DOA(\alpha)$. Then we show that for every $\epsilon > 0$ one can construct a probability space where one can find a sequence of i.i.d r.vs (J_i, U_i) where $\mathbb{E}(U_1) < \infty$ and an inverse subordinator (not necessarily stable) E_t, independent of $\{U_i\}$, s.t if Y_t is the CTRW associated with (J_i, U_i) then

$$\rho_{d_{J_1}}(Y_{E_t}, X_t) < \epsilon, \tag{1.4}$$

where $\rho_{d_{J_1}}$ is the Prohorov metric on probability distributions metrizing the weak topology of distributions on the Skorohod space $\mathbb{D}([0,\infty))$. This enables us to show that by enriching the filtration of a CTRW one may realize CTRW as an annealed process of a random walk in a random environment (RWRE). One of are two main results (Theorem 1) shows that there exists a set of distributions $\mathcal{A}$ which is weakly dense in $DOA(\alpha)$ for which CTRW is an annealed process of RWRE. The random environment is a random time change while the quenched process is a CTRW with finite mean waiting times (independent of the enviornment) time-changed by the random enviornment. The results also show that there exists a set of distributions $\mathcal{B} \subset \mathcal{A}$ which can be realized as another RWRE. This time the random environment is traps in time, that is, for each time $n \in \mathbb{Z}_+$ one randomizes i.i.d trappings τ_n from a heavy tailed distribution, the quenched process will then be a CTRW with waiting times $\{\tau_i U_i\}$, where $\mathbb{E}(U_1) < \infty$. We also show that under proper scaling of CTRW, the quenched process converges to an interesting diffusion time changed by the inverse of a stable subordinator. It shows that in the quenched limit the dynamics of the space-time jumps (J_i, U_i) are translated to that of the regenerative points of the enviornment. Our second main result (Theorem 2) deals with trying to bound the distance in (1.4) when we scale the process' Y_{E_t} and X_t . We give a polynomial bound Cn^{-c}, however, the proof gives way to finding a better c if one only finds a good way of matching the tail of a subordinator with that of W_1. Note that CTRW were considered in [1] as one instance of a RWRE on $\mathbb{Z}$ called *a Randomly Trapped Random Walk* (RTRW). However, there, the random environment is probability measures $\{\pi_z(dt)\}_{z\in\mathbb{Z}}$ on the the positive real line. Given such a random environment, one preforms a simple random walk on $\mathbb{Z}$ with waiting times $\{W_i^z\}_{i\in\mathbb{Z}_+,z\in\mathbb{Z}}$ s.t the sequence $\{W_i^z\}_{i\in\mathbb{Z}_+}$ of waiting times at site z is drawn independently from the the distribution π_z. Reaching the site $x \in \mathbb{Z}$ for the i'th time, the random walk waits W_i^x before moving on to the next site, i.e. traps are in space. In contrast, we show that CTRWs can, at some instances (e.g. stable distribution, Mittag-Leffler distribution), be realized as trap models where the traps are in time rather than space. Moreover, presented as a RTRW, CTRWs are essentially degenerate in the sense that the environment is deterministic, and therefore the limit is completely annealed. Here we show that by considering a larger filtration, the quenched limit retains its environment.

2. Preliminaries

Recall that a Bernstein function is a function $f : (0, \infty) \to \mathbb{R}$ that is infinitely differentiable, $f(s) \geq 0$ and $(-1)^{n-1} f^{(n)}(s) \geq 0$ for $n \geq 1$, where $f^{(n)}$ is the n'th derivative of f. A function f is a Bernstein function iff f is of the form

$$f(s) = a + bs + \int_0^\infty \left(1 - e^{-sy}\right) \mu(dy), \tag{2.1}$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ s.t $\int_0^\infty (1 \wedge y) \mu(dy) < \infty$, one can then identify a Bernstein function with the characteristics (a, b, μ). We shall be interested in the set

$$\mathfrak{B} := \{f : f \text{ is an unbounded Bernstein function of characteristics } (0, b, \mu)\}.$$

We denote the Laplace Transform(LT) of a positive measure μ on $(0, \infty)$ by $\mathcal{L}\mu(s) = \int_0^\infty e^{-st}\mu(dt)$. Let $\mathcal{CM}$ denote the space of completely monotone functions, i.e., $f \in \mathcal{CM}$ iff $f : (0, \infty) \to \mathbb{R}$ and $(-1)^n f^{(n)}(s) \geq 0$ for $n \geq 0$. Define

$$\hat{\mathfrak{L}} := \left\{f \in \mathcal{CM} : f\left(0^+\right) = 1\right\}.$$

Recall that $\hat{\mathfrak{L}}$ is just the set of Laplace Transforms of probability measures on the positive real line. For $\psi \in \mathfrak{B}$ we define the mapping $\hat{\Phi}_\psi : \hat{\mathfrak{L}} \to \hat{\mathfrak{L}}$ by

$$\hat{\Phi}_\psi(f)(s) = f(\psi(s)).$$

Note that the mapping is indeed into $\hat{\mathfrak{L}}$; if $f \in \mathcal{CM}$ and ψ is a Bernstein function then $f(\psi(s)) \in \mathcal{CM}$([13, Theorem 3.6]). We say the function $L : \mathbb{R}^+ \to \mathbb{R}$ is slowly varying at ∞ if

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1, \tag{2.2}$$

for every $\lambda \in \mathbb{R}^+$. In fact it is enough to show that (2.2) holds for every $\lambda \in \Lambda$, where $\Lambda \subset \mathbb{R}$ is of positive Lebesgue measure. Next, define

$$\hat{\mathfrak{L}}_\psi := \left\{f \in \hat{\mathfrak{L}} : f \sim 1 - \psi(s) L\left(s^{-1}\right)\right\},$$

where L is a slowly varying function, and where $f \sim 1-\psi(s)L\left(s^{-1}\right)$ means that

$$\lim_{s\to 0}\frac{|f(s)-1|}{\psi(s)L\left(s^{-1}\right)}=1.$$

We also use $X \sim f$, where X is a r.v and f is a distribution or a r.v, to say that X is distributed according to f, there should not be a confusion there. We denote by $\mathfrak{L}$ and $\mathfrak{L}_\psi$ the space of measures whose Laplace transform(LT) is in $\hat{\mathfrak{L}}$ and $\hat{\mathfrak{L}}_\psi$ respectively, that is, the LT $\mathcal{L}$ is a bijection between $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ and between $\mathfrak{L}_\psi$ and $\hat{\mathfrak{L}}_\psi$. We also define $\Phi_\psi:\mathfrak{L}\to\mathfrak{L}$ as $\Phi_\psi:=\mathcal{L}^{-1}\hat{\Phi}_\psi\mathcal{L}$. If X is a r.v with distribution f, we sometimes write $\Phi_\psi(X)$ instead of $\Phi_\psi(f)$. Finally, let ψ_1 and ψ_2 be two Bernstein functions in $\mathfrak{B}$, we define $\hat{\mathfrak{L}}_{\psi_1}^{\psi_2}=\hat{\Phi}_{\psi_2}\left(\hat{\mathfrak{L}}_{\psi_1}\right)$ and $\mathfrak{L}_{\psi_1}^{\psi_2}=\Phi_{\psi_2}\left(\mathfrak{L}_{\psi_1}\right)$.

Recall that a positive r.v X is said to be in the domain of attraction of a stable (totally asymmetric) r.v Y of index $0<\alpha<1$, i.e. $\mathbb{E}\left(e^{-sY}\right)=e^{-s^\alpha}$ (abbr. $X\in DOA(\alpha)$), if there exists a sequence of normalizing constants $a_n\to 0$ s.t

$$a_n\sum_{i=1}^{n}X_i\Rightarrow Y,$$

where $\{X_i\}$ are i.i.d copies of X and $\Rightarrow$ denotes weak convergence of measures. It is well known that $X\in DOA(\alpha)$ iff $\mathbb{P}(X>t)\sim L(t)\,t^{-\alpha}$, where $L(t)$ is a slowly varying function. It is also known that the sequence a_n is regularly varying, i.e,

$$\lim_{n\to\infty}\frac{a_{[\lambda n]}}{a_n}=\lambda^{-\alpha}\qquad \lambda>0,$$

and that

$$nL\left(a_n^{-1}t\right)\left(a_n^{-1}t\right)^{-\alpha}\to\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.3}$$

For convenience we let $a_1=1$. Our interest in Bernstein functions and the mappings Φ_ψ is in part due to the following fact: for $0<\alpha<1$ $\mathfrak{L}_{s^\alpha}=DOA(\alpha)$. Moreover,

$$\mathfrak{L}_s=\left\{\mu:\mu\in\mathfrak{L},\int_0^t y\mu(dy)\quad\text{is slowly varying}\right\}. \tag{2.4}$$

These are consequences of [5, Corollary 8.1.7 and Theorem 8.3.1].
Let $f:\mathbb{R}\to\mathbb{R}$ be a right continuous function with left limits. We denote

$$f_{t-}:=\lim_{\epsilon\to 0^+}f(t-\epsilon),$$

the left limit of f_t. If f is a left continuous function with right limits then

$$(f_t)^+:=\lim_{\epsilon\to 0^+}f(t+\epsilon),$$

the right limit of f_t. Note that whenever $g(t)$ is a continuous strictly increasing function and f is right continuous with left limits $\left(f_{g(t)-}\right)^+=f_{g(t)}$ (note that we first compute f_{t-} and then evaluate at $g(t)$). This may not be the case when there exists $\epsilon>0$ s.t $g(t-\epsilon)=g(t+\epsilon)$ and f is not continuous at $g(t)$. We say that X_t is a CTRW with space-time jumps $\{J_i,W_i\}$ or that X_t is a CTRW associated with the space-time jumps $\{J_i,W_i\}$, if

$$X_t=\sum_{n=1}^{\infty}J_n1(t)_{\{y:T_n\le y\}},$$

where $T_n = \sum_{i=1}^{n} W_i$. We use $\mathbb{D}[0,T]$ ($\mathbb{D}[0,\infty)$) to denote the subspace of $\mathbb{R}^{[0,T]}$ ($\mathbb{R}^{\mathbb{R}_+}$)for $T > 0$ of cï¿œdlï¿œg functions, and $\overset{J_1[0,T]}{\sim}$ to denote the equivalence of law of processes in the Skorohod J_1 topology on $\mathbb{D}[0,T]$. We shall use $\mathbb{D}$ and $\overset{J_1}{\sim}$ when we refer to $\mathbb{D}[0,\infty)$ and $\overset{J_1}{\sim}$. We use $X_t^n \overset{J_1}{\Rightarrow} X_t$ ($X_t^n \overset{J_1[0,T]}{\Rightarrow} X_t$) to say that the law of the process X_t^n converges weakly to that of X_t w.r.t the J_1 topology on $\mathbb{D}$ ($\mathbb{D}[0,T]$). Let d be a metric on the set V and let $\mathcal{P}(V)$ be the set of all probability measures on the Borel sets (with respect to d) of V. Recall that a sequence of probability measures $p_n \in \mathcal{P}(V)$ converges weakly to $p \in \mathcal{P}(V)$ if for every bounded continuous (with respect to d) function $h : V \to \mathbb{R}$ we have

$$\int h(x)\, p_n(dx) \to \int h(x)\, p(dx).$$

Recall further that the weak topology of $\mathcal{P}(V)$ is metrizable by the following metric

$$\rho_d(p_1,p_2) = \inf_{p_{1,2}} \inf\{\epsilon : p_{1,2}(|X-Y| > \epsilon) < \epsilon\},$$

where the infimum runs over all couplings of the r.vs X and Y whose distribution is given by p_1 and p_2 respectively. For two r.vs X and Y we sometimes write $\rho_d(X,Y)$, which should be understood as $\rho(p_X,p_Y)$, where p_X and p_Y are the distributions of X and Y respectively. Recall that the Skorohord J_1 topology on $\mathbb{D}[0,T]$ is metrizable in the following way; a sequence $f_t^n \in \mathbb{R}^{[0,T]}$ converges in the J_1 topology to $f_t \in \mathbb{R}^{[0,T]}$ if there exists a sequence of homeomorphisms $\lambda_t^n : [0,T] \to [0,T]$ s.t

$$\|f_{\lambda^n}^n - f\| \to 0 \quad \text{and} \quad \|\lambda_t^n - t\| \to 0,$$

as $n \to \infty$, where $\|\cdot\|$ is the sup norm, that is, for $f_t, g_t \in \mathbb{R}^{[0,T]}$

$$\|g - f\| = \sup_{t\in[0,T]} |g_t - f_t|.$$

Denote by Λ the set of all homeomorphismes form $[0,T]$ to itself. One way to metrize the J_1 topology is to use the following metric

$$d_{J_1}(f,g) = \inf_{\lambda\in\Lambda} \{\|g_\lambda - f\| \vee \|\lambda_t - t\|\}.$$

Let $\mathbb{D}_{\uparrow\uparrow}$ be the subset in $\mathbb{D}$ whose elements are strictly increasing. If $d_t \in \mathbb{D}$ and increasing, we define the generalized inverse of d_t to be

$$d_t^{-1} = \inf\{s : d_s > t\}.$$

Note that d_t^{-1} is continuous iff d_t is strictly increasing. Define the mapping $\mathcal{H} : \mathbb{D} \times \mathbb{D}_{\uparrow\uparrow} \to \mathbb{D}$ by

$$\mathcal{H}(f_t, d_t) = \left(f_{d_t^{-1}-}\right)^{+}.$$

The results in [14] show that $\mathcal{H}$ is continuous w.r.t the J_1 topology. In fact, we shall often make use of the following result by Straka and Henry ([14, Theorem 3.6]).

Lemma 1. *(Straka and Henry, 2011) Suppose we have a sequence of random space-time jumps $\{J_i^n, W_i^n\}$ and a sequence of random increasing step process N_t^n s.t*

$$\left(J_{N_t^n}^n, W_{N_t^n}^n\right) \overset{J_1}{\Rightarrow} (A_t, D_t),$$

where $D_t \in \mathbb{D}_{\uparrow\uparrow}$. *If* X_t^n *is the CTRW associated with* $\{J_i^n, W_i^n\}$. *Then*

$$X_t^n \overset{J_1}{\Rightarrow} \mathcal{H}(A_t, D_t). \tag{2.5}$$

As in this paper we are interested mostly in the temporal jumps of our CTRWs one may assume throughout that the spatial jumps $\{J_i^n\} \in \mathbb{R}^d$ for $d \in \mathbb{N}$ are i.i.d such that

$$\lim_{n\to\infty} \sum_{i=1}^{[nt]} J_i^n \overset{J_1}{\Rightarrow} B_t,$$

where B_t is a standard Bm in $\mathbb{R}^d$. We use the term *time-change* for a function f s.t $f(0) = 0$, f is increasing and continuous.

3. FROM RELATIVE STABILITY TO SUB-DIFFUSION

We begin with some technical lemmas that will be useful in understanding the mapping Φ_ψ.

Lemma 2. *Let* L *be a slowly varying function and* $\phi(s)$ *a positive function s.t for every* $\lambda > 0$ *there exist positive constants* $C_1(\lambda)$ *and* $C_2(\lambda)$ *s.t*

$$C_1(\lambda) \le \frac{\phi(\lambda s)}{\phi(s)} \le C_2(\lambda) \qquad \forall s > S(\lambda), \tag{3.1}$$

for some positive constant $S(\lambda)$ *that may depend on* λ. *Then* $L(\phi(s))$ *is again slowly varying.*

Proof. Indeed, by the Uniform Convergence Theorem (UCT) ([5, Theorem 1.2.1]) for slowly varying functions we know that

$$\lim_{s\to\infty} \frac{L(\lambda s)}{L(s)} = 1,$$

uniformly on any compact λ-set in $(0,\infty)$. Since by (3.1) there exists $\lambda' \in [C_1, C_2]$ s.t for every $s > S$

$$\frac{L(\phi(\lambda s))}{L(\phi(s))} = \frac{L(\lambda' \phi(s))}{L(\phi(s))},$$

taking the limit while using the uniform convergence we obtain the result. □

Lemma 3. *Let* $\psi_1, \psi_2 \in \mathfrak{B}$, *then*

$$\hat{\Phi}_{\psi_2}\left(\hat{\mathfrak{L}}_{\psi_1}\right) \subset \hat{\mathfrak{L}}_{\psi_1(\psi_2)}. \tag{3.2}$$

Proof. Suppose first that $f \in \hat{\mathfrak{L}}_{\psi_1}$. By definition $f(s) \sim 1 - \psi_1(s) L\left(\frac{1}{s}\right)$ when $s \to 0$ where L is a slowly varying function. It then follows that $\hat{\Phi}_{\psi_2} f(s) \sim 1 - \psi_1(\psi_2(s)) L\left(\frac{1}{\psi_2(s)}\right)$. Denote $L'\left(\frac{1}{s}\right) = L\left(\frac{1}{\psi_2(s)}\right)$. We must show that $L'(s) = L\left(\frac{1}{\psi_2(s^{-1})}\right)$ is slowly varying. By Lemma 2 it is enough to show that

$$C_1 \le \frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} \le C_2,$$

for some positive constants C_1 and C_2 that may depend on λ. First assume that ψ_2 has representation $(0,0,\mu)$. From [13, Lemma 3.4] we see that

$$\frac{e-1}{e}\lambda\frac{I_\mu(s)}{I_\mu(\lambda s)} \le \frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} \le \frac{e}{e-1}\lambda\frac{I_\mu(s)}{I_\mu(\lambda s)}, \tag{3.3}$$

for every $s>0$ where $I_\mu(s)=\int_0^s \mu(y,\infty)\,dy$. Suppose first that $\lambda \ge 1$ then by the fact that ψ_2 is increasing $\frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} \ge 1$, which shows that

$$1 \le \frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} \le \frac{e}{e-1}\lambda.$$

Similarly, if $\lambda<1$ we have

$$\frac{e-1}{e}\lambda \le \frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} \le 1.$$

Now, if $\psi_2(s)=bs+\psi'(s)$, where $\psi'(s)$ has representation $(0,0,\mu)$ and $b>0$, then

$$\begin{aligned}\frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} &= \frac{bs^{-1}+\psi'\left(s^{-1}\right)}{b\lambda^{-1}s^{-1}+\psi'\left(\lambda^{-1}s^{-1}\right)}\\ &= \frac{b+\psi'\left(s^{-1}\right)s}{b\lambda^{-1}+\psi'\left(\lambda^{-1}s^{-1}\right)s}.\end{aligned}$$

We see that for $\lambda \ge 1$,

$$\frac{b+\psi'\left(s^{-1}\right)s}{b\lambda^{-1}+\psi'\left(s^{-1}\right)s} \le \frac{b+\psi'\left(s^{-1}\right)s}{b\lambda^{-1}+\psi'\left(\lambda^{-1}s^{-1}\right)s} \le \frac{b+\psi'\left(s^{-1}\right)s}{b\lambda^{-1}+\frac{e-1}{e}\lambda\psi'\left(s^{-1}\right)s}.$$

Note that by integration by parts and monotone convergence we see that the limit

$$\begin{aligned}M &= \lim_{s\to\infty}\psi'\left(s^{-1}\right)s\\ &= \lim_{s\to\infty} s\int_0^\infty s^{-1}e^{-s^{-1}y}\mu(y,\infty)\,dy\\ &= \int_0^\infty \mu(y,\infty)\,dy\end{aligned}$$

exists and $M\in[0,\infty]$. It follows that for some large enough S, for every $s>S$ we have

$$C_1(\lambda) \le \frac{\psi_2\left(s^{-1}\right)}{\psi_2\left((\lambda s)^{-1}\right)} \le C_2(\lambda).$$

This shows that $\frac{1}{\psi_2(s^{-1})}$ satisfies (3.1), $L\left(\frac{1}{\psi_2(s^{-1})}\right)$ is slowly varying and that (3.2) holds. $\square$

We say that a measure μ is sub-homogeneous(super-homogeneous) if for every $\lambda > 0$ there exists a constant $C(\lambda)$ s.t $\mu(C(\lambda)x,\infty) \leq \lambda\mu(x,\infty)(\lambda\mu(x,\infty) \leq \mu(C(\lambda)x,\infty))$ for every $x > 0$. For example, if $\mu(dx)$ is a finite measure and $\mu(x,\infty) = x^{-\alpha}L(x)$ where $L(x)$ converges to a constant at infinity, then μ is sub-homogeneous. The following is a partial uniqueness result.

Lemma 4. *Let $\psi_1,\psi_2 \in \mathfrak{B}$, and assume that the measure μ_2 of ψ_2 is sub-homogeneous or super-homogeneous. Then $\Phi_{\psi_2}^{-1}\left(\mathfrak{L}_{\psi_1(\psi_2)}\right) = \mathfrak{L}_{\psi_1}$.*

Proof. We prove this for when μ_2 is sub-homogeneous as the proof for the super-homogeneous is similar. Let $f \in \hat{\mathfrak{L}}$ s.t $\hat{\Phi}_{\psi_2}f \in \hat{\mathfrak{L}}_{\psi_1(\psi_2)}$, or equivalently that $\hat{\Phi}_{\psi_2}f \sim 1-\psi_1(\psi_2(s))L\left(s^{-1}\right)$ as $s \to 0$ where $L(s)$ is slowly varying. It follows that $\hat{\Phi}_{\psi_2}^{-1}f \sim 1-\psi_1(s)L\left(\frac{1}{\psi_2^{-1}(s)}\right)$, and we must show that $L'(s) = L\left(\frac{1}{\psi_2^{-1}\left(\frac{1}{s}\right)}\right)$ is slowly varying. By the characterization of regularly varying function, in order to show that $L'(s)$ is slowly varying it is enough to show that

$$\frac{L'(\lambda s)}{L'(s)} \to 1,$$

for $\lambda \in \Lambda$ where Λ is a set of positive measure. Let $\lambda \in [1,\infty)$, it is then enough to show that

$$C_1' \leq \frac{\frac{1}{\psi_2^{-1}\left(\frac{1}{\lambda s}\right)}}{\frac{1}{\psi_2^{-1}\left(\frac{1}{s}\right)}} \leq C_2', \tag{3.4}$$

for some positive constants C_1', C_2' that may depend on λ. Since $\psi_2^{-1}(s)$ is increasing we see that

$$1 \leq \frac{\psi_2^{-1}\left(\frac{1}{s}\right)}{\psi_2^{-1}\left(\lambda^{-1}\frac{1}{s}\right)}.$$

It is now enough to show that $\psi_2^{-1}(t) \leq C_2'(k)\psi_2^{-1}(kt)$ for $0 < k \leq 1$ and positive $C_2'(k)$. Let $t = bs + \int_0^\infty (1-e^{-sy})\mu(dy)$ so that $\psi_2^{-1}(t) = s$. By the fact that μ is sub-homogeneous we see that

$$\begin{aligned} kt &= kbs + k\int_0^\infty \left(1-e^{-sy}\right)\mu_2(dy) \\ &\geq kbs + \int_0^\infty \left(1-e^{-sy}\right)\mu_2(C(k)\,dy) \\ &= kbs + \int_0^\infty \left(1-e^{-sC(k)y}\right)\mu_2(dy) \\ &\geq C'(k)\,bs + \int_0^\infty \left(1-e^{-sC'(k)y}\right)\mu_2(dy), \end{aligned}$$

where $C'(k) = \min\{C(k), k\}$. By the fact that ψ_2^{-1} is increasing we have $\psi_2^{-1}(kt) \geq sC'(k)$, or that $\psi_2^{-1}(t) \leq C'(k)^{-1}\psi_2^{-1}(kt)$. It follows that (3.4) is satisfied with

$C_1' = 1$ and $C_2' = C'\left(\lambda^{-1}\right)^{-1}$. Then $L'(s)$ is slowly varying and the result follows. The proof for the super-homogeneous case follows along similar lines while taking $\lambda \in (0,1]$. □

Combining Lemma 3 and Lemma 4 we obtain the following.

Proposition 1. *Let $\psi \in \mathfrak{B}$, then the set of distributions $\mathfrak{L}_s^{\psi}$ is contained in $\mathfrak{L}_{\psi}$. Moreover, if the Lï¿œvy measure of ψ is sub-homogeneous or super-homogeneous then $\Phi_{\psi}^{-1}\left(\mathfrak{L}_{\psi}\right) = \mathfrak{L}_s$.*

We now apply Proposition 1 to CTRW.

Proposition 2. *Let Y_t be a CTRW with i.i.d space-time jumps $\left(J_k, W_k^{\psi}\right)$ where $\left\{W_k^{\psi}\right\} \in \mathfrak{L}_s^{\psi}$ and $\psi(s) \in \mathfrak{B}$. Then there exists a CTRW X_t with i.i.d space-time jumps $\left(J_k', W_k^s\right)$ where $\left\{W_k^s\right\} \in \mathfrak{L}_s$ and an inverse of a subrodinator with symbol $\psi(s)$ E_t that is independent of $\left\{W_k^s\right\}$ s.t*

$$Y_t \overset{J_1}{\sim} \left(X_{E_t-}\right)^{+}. \tag{3.5}$$

Conversely, assume Y_t is a CTRW with waiting times $\left\{W_k^{\psi}\right\} \in \mathfrak{L}_{\psi}$ s.t $Y_t \overset{J_1}{\sim} \left(X_{E_t-}\right)^{+}$, where X_t is a CTRW with waiting times $\left\{W_k\right\}$ and E_t is the inverse of a subordinator of symbol $\psi(s)$ that is independent of $\left\{W_k\right\}$. Moreover, assume that $\psi(s) \in \mathfrak{B}$ has representation $(0, b, \mu)$ where μ is super-homogeneous or sub-homogeneous, then $W_1^{\psi} \in \mathfrak{L}_s^{\psi}$ and $W_1 \in \mathfrak{L}_s$.

Proof. We note that if T is a positive r.v then $\Phi_{\psi}(T) \sim D_T$ where D_t is a subordinator of symbol ψ independent of T. Indeed, by the independence of D_t and T we have $\mathbb{E}\left(e^{-sD_T}\right) = \mathbb{E}\left(e^{-\psi(s)T}\right) = \hat{\Phi}_{\psi}\left(\mathcal{L}(T)\right)$. Let $T_n^{\psi} = \sum_{k=1}^{n} W_k^{\psi}$ be the time of the n'th jump of Y_t. Since $W_1^{\psi} \in \mathfrak{L}_s^{\psi}$ there exists a distribution $f^s \in \mathfrak{L}_s$ s.t $W_1^{\psi} \sim \Phi_{\psi}(f^s)$. We now generate a sequence of i.i.d r.v's $\left\{W_k'^{\psi}\right\}$ on a common probability space s.t $W_1'^{\psi} \sim W_1^{\psi}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\left\{W_i^s\right\}$ be a sequence of i.i.d random variables in Ω s.t $\Phi_{\psi}\left(W_1^s\right) \sim W_1^{\psi}$, and let $T_n^s = \sum_{k=0}^{n} W_k^s$. Let D_t be a subordinator of symbol $\psi(s)$ in $(\Omega, \mathcal{F}, \mathbb{P})$ independent of $\left\{W_i^s\right\}$. Define $W_k'^{\psi} = D_{T_k^s} - D_{T_{k-1}^s}$, and note that $\left\{W_k'^{\psi}\right\}$ are i.i.d and $W_1'^{\psi} \sim W_1^{\psi}$. Indeed, by the fact that D_t is a strong Markov process, independent of T_n^s , with stationary increments we have

$$\begin{aligned} W_k'^{\psi} &= D_{T_k^s} - D_{T_{k-1}^s} \\ &\sim D_{T_k^s - T_{k-1}^s} \\ &\sim D_{W_k^s} \sim \Phi_{\psi}\left(W_k^s\right) \\ &\sim W_k^{\psi}. \end{aligned}$$

By the independence of increments of D_t, we see that $W_k'^{\psi}$ are also independent. Assume now that $\left\{J_k'\right\}$ are i.i.d r.v's in Ω s.t $\left(J_i', W_k'^{\psi}\right) \sim \left(J_i, W_k^{\psi}\right)$ and that X_t is the CTRW associated with the space-time jumps $\left(J_i', W_i^s\right)$. Let $T_n'^{\psi} = \sum_{k=1}^{n} W_k'^{\psi}$, and define the process

$$Y_t' = \sum_{n=1}^{\infty} J_n' 1(t)_{\left\{y: T_n'^{\psi} \leq y\right\}}.$$

Note that Y'_t is a CTRW with space-time jumps $\left(J'_i, W'^{\psi}_k\right)$ and therefore $Y'_t \overset{J_1}{\sim} Y_t$. Next we show (3.5). Since ψ is unbounded we see that D_t is strictly increasing and therefore E_t is continuous, and it follows that a.s for every $\omega \in \Omega$ we have

$$\begin{aligned} t \in \left\{y : D_{T_n^s}(\omega) < y\right\} &\iff t \in \left\{y : T_n^s(\omega) < E_y(\omega)\right\} \\ &\iff E_t(\omega) \in \left\{y : T_n^s(\omega) < y\right\}, \end{aligned}$$

and therefore

$$\begin{aligned} Y'_t &= \left(Y'_{t-}\right)^+ \\ &= \left(\sum_{n=1}^{\infty} J'_i 1(t)_{\left\{y : D_{T_n^s} < y\right\}}\right)^+ \\ &= \left(\sum_{n=1}^{\infty} J'_i 1(E_t)_{\left\{y : T_n^s < y\right\}}\right)^+ \\ &= \left(X_{E_t-}\right)^+. \end{aligned} \tag{3.6}$$

Now, suppose that Y_t is a CTRW associated with the waiting times $\left\{W_k^{\psi}\right\}$, where $W_1^{\psi} \in \mathfrak{L}_{\psi}$ and ψ has representation $(0, b, \mu)$ with μ being super-homogeneous or sub-homogeneous and that $Y_t \sim \left(X_{E_t-}\right)^+$, where X_t is a CTRW with space-time jumps (J_i, W_i) and E_t is the inverse-subordinator of symbol ψ independent of $\{W_i\}$. Let $T_n = \sum_{i=1}^n W_i$, going backwards in equation (3.6), we see that $W_1^{\psi} \sim \Phi_{\psi} W_1$. It is implied by Proposition 1 that $W_1 \in \mathfrak{L}_s$ and the result follows. □

Remark 1. In [8], the mapping Φ_{ψ} was used implicitly to obtain *fractional Poisson processes*. Let D_t be a subordinator of symbol ψ, E_t its inverse and let N_t be a Poisson process of intensity 1. Then it was shown in [8, Theorem 4.1] that N_{E_t} is a renewal process with waiting times $\{W_i\}$ s.t

$$\mathbb{P}(W_1 > t) = \mathbb{E}\left(e^{-\lambda E_t}\right).$$

Remark 2. Let us say a distribution $f(dx)$ is a *stable-mixture* if it is of the form

$$f(dx) = \int_0^{\infty} t^{-1/\alpha} g\left(t^{-\frac{1}{\alpha}} x\right) p(dt)\, dx,$$

where $g(x)$ is the density of a standard stable r.v of index $0 < \alpha < 1$ and $p(dt)$ is a measure whose first moment (maybe infinite) is slowly varying . In other words, $f(dx)$ is a stable-mixture if and only if $f \in \mathfrak{L}_s^{s^{\alpha}}$. It is obvious that $\mathfrak{L}_s^{s^{\alpha}} \subsetneq \mathfrak{L}_{s^{\alpha}}$. Firstly, distributions in $\mathfrak{L}_s^{s^{\alpha}}$ have densities which may not be the case for distributions in $\mathfrak{L}_{s^{\alpha}}$. Moreover, by (2.4) we see that whenever $\hat{f} \in \hat{\mathfrak{L}}_{\psi}$ s.t $\hat{f} \sim 1 - \psi(s) L\left(\frac{1}{s}\right)$ with L a slowly varying function s.t $\lim_{s\to\infty} L(s)$ is zero or does not exist, $f \notin \mathfrak{L}_s^{\psi}$. Indeed, [5, Corollary 8.1.7] states that if $L(t)$ is slowly varying then $\hat{f}(s) \sim 1 - sL\left(s^{-1}\right)$ is equivalent to $\int_0^t y df(y) \sim L(t)$ hence L must be increasing. A natural question is whether $\mathfrak{L}_s^{s^{\alpha}}$ is weakly dense in $\mathfrak{L}_{s^{\alpha}}$? Unfortunately we could not answer that. We could not even answer what appears to be a simpler version of that question, namely, if $0 < a < b$ and $A_a^b = \left\{\hat{f} \in \hat{\mathfrak{L}} : \hat{f}(s) \sim 1 - cs, a \le c \le b\right\}$, $B_a^b = \left\{\hat{f} \in \hat{\mathfrak{L}} : \hat{f}(s) \sim 1 - c\psi(s), a \le c \le b\right\}$ is $\Phi_{\psi(s)}\left(A_a^b\right)$ weakly dense in B_a^b?

Remark 3. In the case where A_t and E_t are independent, by the fact that Lï¿œvy process are stochastically continuous we see that the A_{E_t} and $(A_{E_t-})^+$ have the same law.

Remark 2 underlines the possibly limited range of measures in $\mathfrak{L}_s^{\psi}$ compared to $\mathfrak{L}_{\psi}$. In order to extend the set $\mathfrak{L}_s^{\psi}$ we may use $\Phi_{\psi'}$ where $\psi'(s) \in \mathfrak{B}$ s.t $\psi'(s) \sim \psi L'(s^{-1})$ where $L'(t)$ is slowly varying. As the product of two slowly varying functions is a slowly varying function we must have $\Phi_{\psi'}(\mathfrak{L}_s) \subset \mathfrak{L}_{\psi}$. Indeed, if $\hat{f}(s) \sim 1 - sL(s^{-1})$ where $L(s^{-1})$ is slowly varying then $\hat{f}(\psi'(s)) \sim 1 - \psi(s) L'(s^{-1}) L\left(\psi'(s)^{-1}\right)$ and $\hat{f}(\psi(s)) \in \mathfrak{L}_{\psi}$. Define the set

$$\mathfrak{B}_{\psi} := \left\{\psi'(s) \in \mathfrak{B} : \psi'(s) \sim \psi(s) L(s^{-1}), L \text{ is slowly varying}\right\},$$

and then define

$$\mathfrak{L}_s^{\bar{\psi}} := \cup_{\psi' \in \mathfrak{B}_{\psi}} \Phi_{\psi'}(\mathfrak{L}_s).$$

Note that the mapping $\Phi_{s^{\alpha}}$ reduces the "regularity" s around $s = 0$ for $\hat{f}(s) \in \hat{\mathfrak{L}}_s$ with the more coarse "regularity" s^{α}. In order to maintain general results we make the following assumption on $\psi(s)$.

Assumption 1. *We assume $\psi(s)$ satisfies*

$$\lim_{s \to 0^+} \frac{s}{\psi(s)} = 0. \tag{3.7}$$

Note that due to the relation between the regularity of the LT $\hat{f}$ around zero and the moments of the distribution f we see that if $f \in \mathfrak{L}_{\psi}$ where ψ satisfies (3.7) then the first moment of f is infinite. It turns out that the set of distribution $\mathfrak{L}_s^{\bar{\psi}}$ is indeed rich in $\mathfrak{L}_{\psi}$.

Lemma 5. *Let $\psi \in \mathfrak{B}$ that satisfies (3.7).Then the set of distributions $\mathfrak{L}_s^{\bar{\psi}}$ is weakly dense in $\mathfrak{L}_{\psi}$.*

Proof. Let $Y \in \mathfrak{L}_{\psi}$, that is, $\mathbb{E}\left(e^{-sY}\right) \sim 1 - \psi(s) L(s^{-1})$. Define $Y_n = Y 1_{[0,n]}$ and note that $Y_n \in \mathfrak{L}_s$. Next define

$$\psi_n(s) = s + \mu_n^{-1} \int_0^{\infty} \left(1 - e^{-sy}\right) f(dy), \tag{3.8}$$

where $\mu_n = \mathbb{E}(Y_n)$ and $f(dy)$ is the distribution of Y. Since ψ satisfies (3.7) we see that $\mu_n \to \infty$ by monotone convergence. It follows that

$$\psi_n(s) \to s, \tag{3.9}$$

for every $s > 0$. Moreover, denote by $\bar{f}(t) = \int_t^{\infty} f(dy)$ the tail of the distribution $f(dy)$. It is straightforward to verify that $\hat{\bar{f}}(s) := \int e^{-sy} \bar{f}(y)\, dy = \frac{1-\hat{f}(s)}{s}$ and therefore that $\hat{\bar{f}}(s) \sim s^{-1}\psi(s) L(s^{-1})$. Using integration by parts in (3.8) we see that for every n

$$\psi_n(s) \sim s + \mu_n^{-1} \psi(s) L(s^{-1}).$$

Let f_n be the distribution of Y_n. Since $\hat{f}_n(s) \sim 1 - \mu_n s$, it follows by (3.7) that

$$\hat{f}_n(\psi_n(s)) \sim 1 - \psi(s) L(s^{-1}), \tag{3.10}$$

in particular, $\Phi_{\psi_n} f_n \in \mathfrak{L}_{\psi}$. It is left to show that $\Phi_{\psi_n} f_n \to f$ as $n \to \infty$. But this follows easily form (3.9) and the fact that Y_n converges weakly to Y. □

Remark 4. The reason why we did not use $\psi_n = s + \mu_n^{-1}\psi(s)$ instead of the form in (3.8) is that the form in (3.8) has an advantage when $\psi(s) = s^{\alpha}L(s^{-1})$ where $L(t)$ is slowly varying. Indeed, by Karamata's Theorem we know that $\mathbb{E}\left(e^{-sY}\right) \sim 1 - s^{\alpha}L(s^{-1})\Gamma(1-\alpha)$ is equivalent to $\mathbb{P}(Y > t) \sim t^{-\alpha}L(t)$. It follows that for every n our approximation $\Phi_{\psi_n}Y_n$ of Y satisfies

$$\lim_{t\to\infty} \frac{\mathbb{P}(Y > t)}{\mathbb{P}(\Phi_{\psi_n}Y_n > t)} = 1. \tag{3.11}$$

Equation 3.11 will be utilized in the sequel in order to obtain quantitative results.

Proposition 3. *Let Y_t^n be the CTRW associated with the i.i.d space-time jumps $(J_i^n, a_n W_i)$ where $W_1 \in \mathfrak{L}_s^{\overline{s^{\alpha}}}$. Then there exists a CTRW X_t^n associated with the i.i.d space-time jumps $\left(J_i^n, n^{-1}U_i\right)$ s.t U_1 has finite mean, and a sequence of inverse-subordinators E_t^n independent of $\{U_i\}$ so that for every n*

$$Y_t^n \overset{J_1}{\sim} \left(X^n_{E_t^n-}\right)^+,$$

and s.t E_t^n converges in law w.r.t the J_1-topology to E_t, the inverse of a stable subordinator of index α.

Proof. Since $W_1 \in \mathfrak{L}_s^{\overline{s^{\alpha}}}$, there exists $\psi \in \mathfrak{B}_{s^{\alpha}}$ (we shall use the one in Lemma 5) s.t $W_1 \in \mathfrak{L}_s^{\psi}$. By Proposition 2 we see that

$$Y_t^1 = \left(X^1_{E_t^1-}\right)^+,$$

where E_t^1 is the inverse-subordinator of symbol $\psi_1 = sa_n b + \int_0^{\infty}\left(1 - e^{-sa_n y}\right)\mu(dy)$ and X_t^1 is a CTRW associated with i.i.d space-time jumps $\left(J_i^1, U_i\right)$ with $U_1 \in \mathfrak{L}_s$. By Proposition 2 it is enough to show that

$$a_n W_1 \sim \Phi_{\psi_n}\left(n^{-1}U_1\right), \tag{3.12}$$

where ψ_n is the symbol of a strictly increasing subordinator. Looking at the Laplace Transform of $a_n W_1$ we see that

$$\begin{aligned}\mathbb{E}\left(e^{-sa_nW_1}\right) &= \mathbb{E}\left[e^{-U_1\left(sa_nb+\int_0^{\infty}\left(1-e^{-sa_ny}\right)\mu(dy)\right)}\right]\\ &= \mathbb{E}\left[e^{-n^{-1}U_1\left(sa_nnb+\int_0^{\infty}\left(1-e^{-sy}\right)n\mu\left(a_n^{-1}dy\right)\right)}\right],\end{aligned} \tag{3.13}$$

which implies (3.12) with $\psi_n(s) = sna_n b + \int_0^{\infty}\left(1 - e^{-sy}\right)n\mu\left(a_n^{-1}dy\right)$. Letting E_t^n be the inverse of a strictly increasing subordinator of symbol ψ_n and invoking again Proposition 2 we see that

$$Y_t^n = \left(X^n_{E_t^n-}\right)^+.$$

We are left to show that E_t^n converges in law to E_t, the inverse of a stable subordinator of index α. To see that, first note that by the definition of ψ_1 and Karamata's Theorem we know that $\bar{\mu}_1(y) \sim L(y)y^{-\alpha}$. Let $h(y)$ be a smooth function with compact support $[a,b] \subset \mathbb{R}_+/\{0\}$, then by (2.3)

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\infty} h(y)\,n\mu_1\left(a_n^{-1}dy\right) &= \lim_{n\to\infty}\int_a^b \frac{\partial h(y)}{\partial y}n\bar{\mu}_1\left(a_n^{-1}y\right)dy\\ &= \int_a^b \frac{\partial h(y)}{\partial y}\frac{y^{-\alpha}}{\Gamma(1-\alpha)}dy,\end{aligned}$$

and form the fact that $a_n n \to 0$ (a_n is regularly varying with parameter $-\frac{1}{\alpha}$) we see that μ_n converges vaguely to $\mu(dy) = \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy$, the Lï¿œvy measure of a standard stable subordinator of index α. However, convergence of characteristics of Feller processes implies weak convergence of their law in the J_1 topology. In other words, if D_t^n is the subordinator whose symbol is ψ_n we see that $D_t^n \overset{J_1}{\Rightarrow} D_t$. Next we use Lemma 1 with $\left(n^{-1}, D_t^n\right)$ to obtain

$$E^n \overset{J_1}{\Rightarrow} E,$$

This completes the proof. □

Proposition 3 can be understood in the following way; let A_t be an increasing process and let $A(\omega, T)$ be the regenerative set of A_t in the interval $[0, T]$ (we may also consider $[0, \infty)$). That is,

$$A(\omega, T) = \{u \in [0, T] : A_{u-\epsilon}(\omega) < A_u(\omega) < A_{u+\epsilon}(\omega), \forall \epsilon > 0\}.$$

Note that the mapping Φ_ψ can be viewed as a mapping on processes. Let X_t be a process, then we define

$$\Phi_\psi(X_t) = X_{E_t},$$

where E_t is the inverse-subordinator of symbol ψ independent of X_t. Moreover, Φ_ψ can also be viewed as a mapping between regenerative set-valued random variables. That is, conditioned on $E(\omega, \infty)$, $\Phi_\psi(X_t)(\cdot, \infty)$ is a random regenerative set contained in $E(\omega, \infty)$. Lastly, note that conditioned on $E_t(\omega) = \xi$, Φ_ψ can be viewed as a function $\Phi_{\psi,\xi} : \mathfrak{L} \to \mathfrak{L}$. If $U \in \mathfrak{L}$ has distribution μ

$$\Phi_{\psi,\xi}(\mu) \sim \xi_U^{-1}.$$

$\Phi_{\psi,\xi}$ sends measures in $\mathfrak{L}$ to measures whose support is in the regenerative set of ξ. Let $\mu \in \mathfrak{L}$, and let f_t be a time-change, then we define the probability measure μ_f on Borel sets of $\mathbb{R}_+$ to be

$$\mu_f(A) = \mu\left(A_{f^{-1}}\right), \tag{3.14}$$

for every Borel set in $A \subset \mathbb{R}_+$, where for an increasing f A_f is the set

$$A_f = \{x \in \mathbb{R} : f(x) \in A\}.$$

We have $\Phi_{\psi,\xi}(\mu) = \mu_\xi(dx)$. If U_1 has finite mean then by the SLLN of Renewal Theory we know that with probability one the regenerative points of the CTRW T_t^n associated with the space-time jumps $\left(1, n^{-1}U_i\right)$ 'converge' to a set that is dense in $[0, T]$, namely

$$T(\omega, T) = \cup_n T^n(\omega, T).$$

Since $D_t(\omega)$ is right continuous we deduce that the mapping Φ_ψ is 'continuous' (if $x_n \in T(\omega, T)$ s.t $x_n > x$ and $x_n \to x$ then $D_{x_n} \to D_x$) and $E(\omega, T)$ is a perfect set (closed, with no isolated points). It follows that $\Phi_\psi(T(\omega, T))$ is dense in $E(\omega, T)$. In other words, as $n \to \infty$ the trajectory $E_t(\omega)$ is delineated by the regenerative points of T_t. This idea holds more generally. Let $f \in \mathfrak{L}$ and let $\{U_i\}$ be i.i.d r.vs with distribution f. Let us define $T_0 = 0$ and

$$T_n = \sum_{i=1}^{n} U_i.$$

We say that f is *relatively stable* ([5, 8.8])if there exist norming constants a_n s.t

$$a_n T_n \to 1,$$

where convergence is in probability. Next define the renewal process

$$N_t = \max\{k : T_k \le t\}.$$

Define the *residual lifetime* Z_t and the *aging* Y_t by

$$Y_t = t - T_{N_t}$$
$$Z_t = T_{N_t+1} - t.$$

Finally, we let $a_n > 0$ be any sequence s.t

$$1 - \hat{f}(a_n) \sim n^{-1}.$$

The following is known [5, Theorem 8.8.1].

Lemma 6. *Let $f \in \mathfrak{L}$ and let Y_t and Z_t be the aging and the residual lifetime processes associated with f. The following are equivalent:*

(1) $f \in \mathfrak{L}_s$.
(2) f is relatively stable.
(3) $\frac{Y_t}{t} \to 0$ in probability.
(4) $\frac{Z_t}{t} \to 0$ in probability.

Define $i_f^n = \sup\{i : T_i^n \le T\}$ and the set

$$A_{\delta,T}^n = \left\{\omega : \sup_{1\le i\le i_f^n} \left|T_i^n - T_{i-1}^n\right| < \delta\right\}. \tag{3.15}$$

The set $A_{\delta,T}^n$ is the event that one can not find two consecutive regenerative points whose distance is larger than δ. We shall need the next lemma.

Lemma 7. *Let $f \in \mathfrak{L}_s$ and $T_i^n = a_n \sum_{j=1}^{i-1} U_j$ where $\{U_i\}$ are i.i.d and $U_1 \sim f$. Then for every $\delta > 0$ $\mathbb{P}(A_\delta^n) \to 1$ as $n \to \infty$.*

Proof. Assume w.l.o.g that $T = 1$. Define the set $\{t_k\}_{k=0}^{2^m-1}$ where $t_k = 2^{-m}k$ where $m = \lceil \log_2 \delta^{-1}\rceil + 1$. If N_t^n is the renewal process of $\{T_i^n\}$ and $Z_t^n = T_{N_t^n+1} - t$ is its residual lifetime, then for every $\epsilon > 0$, by Lemma 6, we have for large enough n,

$$\mathbb{P}\left(Z_{t_k}^n > 2^{-m}\right) < 2^{-m}\epsilon \qquad \forall t_k, 1 \le k < 2^m.$$

It is left to note that $\left\{\cup_{1\le k<2^m}\{Z_{t_k} > {}^{\delta}/_{2}\}\right\}^c \subset A_{\delta,T}^n$. ☐

Proposition 4. *Assume*

$$(A_t^n, D_t^n) \overset{J_1}{\Rightarrow} (A_t, D_t,) \tag{3.16}$$

where D_t is a.s strictly increasing. Let $\{U_i\}$ be i.i.d r.vs independent of the sequence (A_t^n, D_t^n) where $U_1 \in \mathfrak{L}_s$. Let $T_i^n = a_n \sum_{j=1}^{i} U_j$ be the renewal epoch and let $\{X_t^n\}_{n=1}^\infty$ be the CTRWs associated with the space-time jumps

$$\{J_i^n, W_i^n\} = \left\{A_{T_i^n}^n - A_{T_{i-1}^n}^n, D_{T_i^n}^n - D_{T_{i-1}^n}^n\right\}.$$

Then

$$X_t^n \overset{J_1[0,T]}{\Rightarrow} (A_{E_t-})^+ \tag{3.17}$$

where E_t is the generalized inverse of D_t.

Proof. Let $\epsilon>0$. If E^n are the generalized inverses of D^n, Due to (3.16) we see that $E_T^n\Rightarrow E_T$ and therefore, one can find $\tilde{T}>0$ s.t

$$\sup_n \mathbb{P}\left(E_T^n>\tilde{T}\right)<\frac{\epsilon}{3}.$$

For every $\delta>0$, define the event $A_{\delta,\tilde{T}}^n$ as in (3.15). Consider the CTRW Y_t^n associated with the time-space jumps $\left(\left(J_i^n,W_i^n\right),a_nU_i\right)$ (note that $Y_t^n\in\mathbb{R}^d\times\mathbb{R}_+$). We now claim that for large enough n , we have

$$\rho_{d_{J_1[0,\tilde{T}]}}\left(\left(A_t^n,D_t^n\right),Y_t^n\right)<\epsilon. \tag{3.18}$$

Recall that if $f\in\mathbb{D}[0,\tilde{T}]$, then the modulus of continuity of f is given by

$$\omega_f^{\tilde{T}}(\delta)=\inf\Big\{\max_{1\le i\le m}\theta_f[t_{i-1},t_i):\exists m\ge1,\\ 0=t_0<t_1...<t_m=\tilde{T}\text{ s.t. }t_i-t_{i-1}>\delta\text{ for all }i\le m\Big\},$$

where

$$\theta_f[s,t)=\sup_{s\le u<w\le t}\left|f(u)-f(w)\right|.$$

Define

$$B_{\delta,\tilde{T}}^n=\left\{\omega^{\tilde{T}}_{\left(A_t^n,D_t^n\right)(\omega)}(\delta)<\epsilon\right\},$$

assumption (3.16), suggests that for every $\epsilon>0$ there exists $\delta>0$ s.t

$$\sup_n\mathbb{P}\left(B_{\delta,\tilde{T}}^n\right)>1-\frac{\epsilon}{3}. \tag{3.19}$$

Define the sequence $f^n\in\mathbb{D}_{\mathbb{R}^d\times\mathbb{R}^+}[0,\tilde{T}]$ by $f_t^n=\left(A_{T_i}^n,D_{T_i}^n\right)$ on $T_i\le t<T_{i+1}$ and let $\delta'<\frac{\delta}{2\tilde{T}}\min(\delta,\epsilon)$. We first condition on $A_{\delta',\tilde{T}}^n\cap B_{\delta,\tilde{T}}^n$, $(A_\cdot^n,D_\cdot^n)$, and $\left\{E_T^n>\tilde{T}\right\}$ i.e. we would like to show that

$$\mathbb{P}\left(d_{J_1}\left(\left(A_\cdot^n,D_\cdot^n\right),f_\cdot^n\right)>\epsilon|A_{\delta',\tilde{T}}^n\cap B_{\delta,\tilde{T}}^n,\left(A_\cdot^n,D_\cdot^n\right),\left\{E_T^n>\tilde{T}\right\}\right)=0. \tag{3.20}$$

Indeed, by (3.19), on $B_{\delta,\tilde{T}}^n$ one can find $0=t_0<t_1...<t_m=\tilde{T}$ s.t for every $n\ge1$, $t_i-t_{i-1}>\delta$ and $\theta_{(A_\cdot^n,D_\cdot^n)}[t_{i-1},t_i)<\epsilon$ for $1\le i\le m$. Let $T^n=\left\{T_i^n:T_i^n\le\tilde{T}\right\}$. On $A_{\delta',\tilde{T}}^n$ one can find the two points

$$T_i^{n,1}=\inf\left\{T^n\cap[t_i,t_{i+1})\right\}$$
$$T_i^{n,2}=\sup\left\{T^n\cap[t_i,t_{i+1})\right\},$$

s.t $t_i\le T_i^{n,1}<T_i^{n,2}<t_{i+1}$ for $0\le i\le m-1$. The distance between $T_i^{n,2}$ and $T_{i+1}^{n,1}$ is at most δ' and so one can find a homeomorphism $\lambda:[0,\tilde{T}]\to[0,\tilde{T}]$ s.t $\lambda\left(T_i^{n,1}\right)=t_i$ and s.t $\sup|\lambda(s)-s|\le m\delta'\le\frac{\tilde{T}}{\delta}\delta'<\epsilon$ (one simply maps the interval $[T_i^{n,2},T_{i+1}^{n,1})$ to $[T_i^{n,2},t_{i+1})$ which costs no more then δ' as $\left|T_i^{n,2}-T_{i+1}^{n,1}\right|<\delta'$) . Next note that by the definition of $\omega_f^{\tilde{T}}(\delta)$ and (3.19) we see that on $A_{\delta',\tilde{T}}^n\cap B_{\delta,\tilde{T}}^n$

$$\sup_{0\le s\le\tilde{T}}\left|f_{\lambda(s)}^n-\left(A_s^n,D_s^n\right)\right|<\epsilon,$$

$$\sup_{0\le s\le\tilde{T}}\left|\lambda(s)-s\right|<\epsilon.$$

Hence (3.20) holds. By Lemma 7, for large enough n

$$\mathbb{P}\left(A_{\delta}^{n}\right) > 1 - \frac{\epsilon}{3}.$$

Taking expectation in (3.20) while using independence we conclude that for large enough n

$$\mathbb{P}\left(d_{J_1[0,\tilde{T}]}\left(\left(A_{\cdot}^{n}, D_{\cdot}^{n}\right), f_{\cdot}^{n}\right) > \epsilon\right) < \epsilon,$$

or (3.18) which implies that

$$f_t^n \overset{J_1[0,\tilde{T}]}{\Rightarrow} \left(A_t, D_t\right).$$

From here we use Lemma 1 to obtain $X_t^n \overset{J_1[0,E_T]}{\Rightarrow} \left(A_{E_t-}\right)^+$. □

Remark 5. If U_1 in Proposition 4 has finite mean and $\left(A_t^n, D_t^n\right) \overset{J_1}{\to} \left(A_t, D_t,\right)$ a.s, then using the SLLN of Renewal Theory and same arguments as in Proposition 4 we see that conditioned on $\left\{A_t^n, D_t^n\right\}_{n=1}^{\infty}$ we have $X_t^n \overset{J_1}{\to} \left(A_{E_t-}\right)^+$ with probability 1.

As we have shown that CTRW with heavy tailed waiting times can be represented as CTRW with finite mean waiting times subordinated to a time-change, we see that CTRWs ï¿œ la Montroll and Weiss are essentially CTRWs in random environment. Among the well known Random Walks in Random Environment(RWRE) are the so-called trap models. The most basic of which is arguably the Bouchaud model. The most basic setup consists of a simple graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. We are also given the trapping environment

$$\boldsymbol{\tau} = \left\{\tau_x > 0 : x \in V\right\}.$$

On the graph G we preform a CTRW with exponential waiting times whose jump rate is given by

$$w_{xy} = \left\{\begin{array}{cc} \tau_x^{-1}, & (x,y) \in E, \\ 0, & \text{otherwise.} \end{array}\right. ,$$

and the generator is given by

$$Lf(x) = \sum_{y \sim x} w_{x,y}\left(f(y) - f(x)\right). \tag{3.21}$$

In words, the larger τ_x is, the deeper the trap at site x and the longer the CTRW stays at the site x. In order to obtain a non-trivial (simple random walk on G) limit we assume that $\{\tau_x\}$ are i.i.d and that $\tau_x \in \mathfrak{L}_{s^\alpha}$. In [6] Fontes et al studied the Bouchaud model where G is $\mathbb{Z}$ with nearest neighbor edges. The Markov process X_t associated with the generator (3.21) (conditioned on the environment $\boldsymbol{\tau}$) is called the *quenched* process. Taking expectation w.r.t the law of $\boldsymbol{\tau}$ we obtain the *annealed* process. Let a_n be the sequence defined in (2.3). One is interested in the limit (in distribution) of the Bouchaud model

$$n^{-1} X_{tna_n^{-1}} \to X_t. \tag{3.22}$$

It was proven in [6] that X_t is a Brownian motion time-changed by the generalized inverse of the local time of a standard Brownian motion integrated against a Poisson measure on $\mathbb{R} \times \mathbb{R}_+$ with intensity $\alpha t^{-\alpha-1} 1(t)_{(0,\infty)}\, dtdx$. This was referred to as *Singular Diffusion*. It turns out that the dimension of the lattice affects the limit in (3.22)(although the scaling is different). Indeed, it was proven in [2] that under proper scaling of the Bouchaud model on $\mathbb{Z}^d$ for $d > 1$ the limit is B_{E_t} , i.e. a Brownian motion time-changed by the inverse of a standard stable subordinator independent of B_t (this is referred to as *Fractional Kinetics*) . It is worth mentioning here that the scaling in dimension $d > 2$ is the same as that of the CTRW in the sense of Montroll and Wiess. The limit of the Bouchaud model for dimension larger than one is the same as that in the uncoupled Montroll and Wiess CTRW model. Proposition 3 suggests that the CTRW in the sense of Montroll and Wiess with waiting times in $\mathfrak{L}_{s^\alpha}$ has a representations as annealed process of possibly two different RWRE. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there exists a random continuous time-change (continuous increasing processes) E_t. We also have a CTRW $\mathring{X}_t \in \mathbb{Z}^d$ associated with the i.i.d space-time jumps (J_i, U_i) where U_1 has finite mean, $\{U_i\}$ is independent of E_t, and where the probability transition function $p_t\left((J_i, U_i) \in (dx, du)\right)$ may depend on time and the random enviornment. Unless J_i and U_i are independent $\mathring{X}_t$ need not be Markovian even if $U_1 \sim Exp\,(\lambda)$. Given a realization of the time change $E_t\,(\omega)$ (our random environment) we consider the process (the quenched process)

$$X_t^{\mathrm{I}} = \mathring{X}_{E_t(\omega)}. \tag{3.23}$$

We refer to X_t^{I} in (3.23) as the quenched process of RWRE of type I. We will say that $\tilde{X}_t^{\mathrm{I}}$ is an annealed process of RWRE of type I if there exists a CTRW $\mathring{X}_t$ s.t

$$\mathbb{P}\left(\tilde{X}_\cdot^{\mathrm{I}} \in dx.\right) = \int \mathbb{P}\left(\mathring{X}_{\xi.} \in dx.\right) P_E\,(d\xi.)\,,$$

where $P_E\,(d\xi.)$ is the law of our random environment E_t, that is

$$\int_A P_E\,(d\xi.) = \mathbb{P}\,(E \in A)\,,$$

with A a Borel set in the Borel sigma-algebra of $\mathbb{D}$. We are interested in the limit

$$n^{-1}\tilde{X}^{\mathrm{I}}_{tn^{\frac{2}{\alpha}}} \Rightarrow \bar{X}_t^{\mathrm{I}}. \tag{3.24}$$

Next we introduce another RWRE model which is somewhat of a temporal trap model. Let

$$\boldsymbol{\tau} = \{\tau_n > 0 : n \in \mathbb{Z}_+\}\,, \tag{3.25}$$

be our random temporal landscape. We also assume the existence of a family of probability transition functions

$$p_t\,(s, x; y) \qquad t > 0. \tag{3.26}$$

Let X_t be the CTRW who after the n'th jump ends up at site x and waits an exponential time s of mean τ_n, and then makes a spatial jump to one of its neighbors according to a distribution $p_{\tau_n}\,(s; y)$. In other words, the temporal landscape $\boldsymbol{\tau}$ affects both the temporal dynamics as well as the spatial. More precisely, assume we have a sequence of positive r.v $\{\tau_i\}$ and let $\{U_i\}$ be a sequence of i.i.d waiting times s.t $\mathbb{E}\,(U_1) = 1$ independent of $\{\tau_i\}$. We define $T_n = \sum_{i=1}^n \tau_i U_i$ to be the epochs of our random walk. Let $\{J_i\}$ be i.i.d r.vs in $\mathbb{Z}^d$ and $S_n = \sum_{i=1}^n J_i$ be a discrete random walk on $\mathbb{Z}^d$ s.t

$$\mathbb{P}\,(J_{n+1} = y | \tau_n = t, U_{i+1} = s) = p_t\,(s, x; y)\,.$$

Then, conditioning on $\{\tau_1 = t_1, \tau_2 = t_2, ...\}$ we define

$$X_t^{\mathrm{II}} = S_n \qquad T_n \le t < T_{n+1}.$$

We note that in general X_t^{II} is not a Markov process, however, if U_1 is exponentially distributed, (3.26) is independent of s and N_t counts the number of jumps of X_t^{II} until time t, then (X_t, N_t) is a Markov process with the generator

$$Lf(x,z)=\sum_{y\sim x}\tau_z^{-1}p_{\tau_z}(y)\left(f(y,z+1)-f(x,z)\right). \tag{3.27}$$

We shall refer to X_t^{II} as the quenched process of a RWRE of Type II. We define the annealed process of a RWRE of Type II similarly to that of type I. That is

$$\mathbb{P}\left(\tilde{X}_\cdot^{\mathrm{II}}\in dx_\cdot\right)=\int\mathbb{P}\left(X_\cdot^{\mathrm{II}}\in dx_\cdot\right)P_{\boldsymbol{\tau}}(d\boldsymbol{\tau}),$$

where $P_{\boldsymbol{\tau}}(d\boldsymbol{\tau})$ is a probability distribution on the Borel sigma-algebra with respect to the product topology on $\mathbb{R}_+^{\mathbb{N}}$ s.t for every cylinder set of the form $A=\mathbb{R}_+\cdots\times\mathbb{R}_+\times A_{n_1}\times A_{n_2}\cdots\times A_{n_m}\times\mathbb{R}_+\cdots$ with $A_{n_i}\subset\mathbb{R}_+$,

$$\int_A P_{\boldsymbol{\tau}}(d\boldsymbol{\tau})=\mathbb{P}\left(\tau_{n_1}\in A_{n_1},\tau_{n_1}\in A_{n_2},...,\tau_{n_m}\in A_{n_m}\right).$$

Here we shall be interested in the limit

$$n^{-1}\tilde{X}^{\mathrm{II}}_{tn^{\frac{2}{\alpha}}}\Rightarrow\bar{X}_t^{\mathrm{II}} \tag{3.28}$$

Let $\mathfrak{M}$ be the set of probability measures whose all moments are finite. Consider the sets

$$\mathcal{A}=\cup_{\psi\in\mathfrak{B}_{s^\alpha}}\Phi_\psi(\mathfrak{M}),$$
$$\mathcal{B}=\Phi_{s^\alpha}(\mathfrak{M}).$$

We have seen already in Lemma 5 that $\mathcal{A}$ is weakly dense in $DOA(\alpha)$. Since $\mathfrak{M}$ is dense in $\mathfrak{L}$ and Φ_ψ is weakly continuous for every $\psi\in\mathfrak{B}$, we conclude that $\mathcal{B}$ is weakly dense in $\mathfrak{L}_s^{s^\alpha}$.
In order to facilitate the exposition of our results me make the following technical assumption.

Assumption 2. *Assume $\{J_i,W_i\}\in\mathbb{R}^d\times\mathbb{R}_+$ are i.i.d space-time jumps. We assume that the conditional distribution $p(dx;w)=\mathbb{P}(J_1\in d\boldsymbol{x}|W_1=w)$ is weakly continuous in w, $\mathbb{E}(J_1|W_1=w)=\mathbf{0}$, $\boldsymbol{\sigma}^2(w)=\mathbb{E}\left(J_i^TJ_i|W_1=w\right)$ is a full rank $d\times d$ matrix for $\mathbb{P}(W_1\in dw)$ almost every w and*

$$\sup_w\left\|\boldsymbol{\sigma}^2(w)\right\|<\infty, \tag{3.29}$$

where $\|\cdot\|$ is any norm on the space of $d\times d$ matrices.

Define

$$\boldsymbol{\sigma}_\mu^2=\int_{\mathbb{R}_+}\boldsymbol{\sigma}^2(t)\,\mu(dt)\qquad\mu\in\mathfrak{L}.$$

Suppose $f_t\in\mathbb{D}$, we denote by f_t^s the time-shift of f, i.e.

$$f_t^s:=f_{t+s}\qquad t>0.$$

Recall the definition of $\mu_f(dx)$ in (3.14).

Theorem 1. *Let X_t be a CTRW associated with the i.i.d space-time jumps (J_i, W_i) satisfying Assumption 2 where $W_1 \in \mathcal{A}$ and $J_i \in \mathbb{Z}^d$. Then X_t is an annealed process of RWRE of type I. Moreover, if W_1 is also in $\mathcal{B}$ then X_t is also an annealed process of RWRE of type II. In both cases, the limits 3.24 and 3.28 exist and equal*

$$\bar{X}_t = B_{E_t}, \tag{3.30}$$

where E_t is the inverse of a stable subordinator, and conditioned on $E_\cdot = \xi$, B_t is a time-inhomogeneous diffusion whose generator is

$$L_t(f)(x) = \frac{1}{2}\nabla_x f^T \boldsymbol{\sigma}^2_{\mu_{(\xi^{-1})^t}} \nabla_x f, \tag{3.31}$$

where $\mu \in \mathfrak{L}_s$.

Proof. If $W_1 \in \mathcal{A}$, by Proposition 3 and the definition of $\mathcal{A}$, one can find $U_1 \in \mathfrak{M}$ and an inverse-subordinator E^1_t s.t

$$X_t \overset{J_1}{\sim} \left(\mathring{X}_{E^1_t-}\right)^+, \tag{3.32}$$

where $\mathring{X}_t$ is a CTRW with space-time jumps (J_i, U_i) with $\mathbb{E}(U_1) < \infty$. Considering (3.13) we see that we may assume that $\mathbb{E}(U_1) = 1$ as this would only change the convergence to a standard stable subordinator by a constant time change. This proves that X_t is an annealed process of RWRE of type I. Let X^n_t be the CTRW associated with the space-time jumps $\left(n^{-1}J_i, n^{-\frac{2}{\alpha}}W_i\right)$, then

$$X^n_t \overset{J_1}{\sim} n^{-1}X_{tn^{\frac{2}{\alpha}}},$$

and by Proposition 3 we may assume w.l.o.g that there exists a sequence of inverses of subordinators E^n_t s.t

$$E^n_t \overset{J_1}{\to} E_t \tag{3.33}$$

a.s. where E_t is the inverse of a stable subrodinator of index α. By Proposition 3 we have

$$X^n_t \overset{J_1}{\sim} \left(\mathring{X}^n_{E^n_t-}\right)^+,$$

where $\mathring{X}^n_t$ is the CTRW associated with $\left\{n^{-1}J_i, n^{-2}U_i\right\}$. We now wish to find $\mathbb{P}\left(\left(n^{-1}J_{i+1}, n^{-2}U_{i+1}\right) \in (dx, du) \,|E^n_\cdot = \xi^n_\cdot\right)$. Let $T_n = \sum_{i=1}^n U_i$, we have

$$\begin{aligned}
&\mathbb{P}\left(\left(n^{-1}J_{i+1}, n^{-\frac{2}{\alpha}}W_{i+1}\right) \in (dx, dw)\right) = \\
&\mathbb{P}\left(\left(n^{-1}J_{i+1}, D^n_{(n^{-2}U_{i+1}+n^{-2}T_i)} - D^n_{n^{-2}T_i}\right) \in (dx, dw)\right) = \\
&\int \mathbb{P}\left(n^{-1}J_{i+1} \in dx | n^{-2}T_i = t, n^{-2}U_{i+1} = u, D^n_\cdot = (\xi^n_\cdot)^{-1}\right) \\
&\times \mathbb{P}\left(n^{-2}U_{i+1} \in du\right) \mathbb{P}\left(n^{-2}T_i \in dt\right) \mathbb{P}\left(D^n_\cdot \in d\left(\xi^n_\cdot\right)^{-1}\right).
\end{aligned}$$

We conclude that

$$\mathbb{P}\left(n^{-1}J_{i+1} \in dx | n^{-2}T_i = t, n^{-2}U_{i+1} = u, D^n_\cdot = (\xi^n_\cdot)^{-1}\right) = p\left(ndx; (\xi^n)^{-1}_{n^2(t+u)}\right).$$

Let Y_t^n be the Markov process $\left(\left(n^{-1}S_{N_t^n}, n^{-2}T_{N_t^n}\right)\right)$ conditioned on $\left\{D_\cdot = \left(\xi_\cdot^n\right)^{-1}\right\}$, where $S_n = \sum_{i=1}^n J_i$ and N_t^n is a homogeneous Poisson process with intensity n^2. Y_t^n is a Markov process with generator

$$L^n\left(f\right)\left(x,t\right) = n^2 \int p\left(dy; \left(\xi^n\right)^{-1}_{(t+u)}\right) \mathbb{P}\left(U_1 \in du\right) \left(f\left(x + yn^{-1}, t + un^{-2}\right) - f\left(x,t\right)\right),$$

for every $f \in C^{2,1}\left(\mathbb{R}^d \times \mathbb{R}_+\right)$. Let $\xi^{-1} = \lim_{n\to\infty}\left(\xi^n\right)^{-1}$ where the limit is in J_1-topology. If we denote $\mu\left(du\right) = \mathbb{P}\left(U \in du\right)$, it is not hard to see that $\mu_{\left(\left(\xi^n\right)^{-1}\right)^t} \to \mu_{\left(\xi^{-1}\right)^t}$ for every $t \geq 0$ where convergence is in the weak topology of measures in $\mathfrak{L}$ and where $\mu_{\left(\xi^{-1}\right)^t}$ is as in (3.14). By Assumption 2 it is also not hard to verify that

$$L^n\left(f\right)\left(x,t\right) \to L\left(f\right)\left(x,t\right),$$

where

$$L\left(f\right)\left(x,t\right) = \frac{1}{2}\nabla_x f^T \boldsymbol{\sigma}^2_{\mu_{\left(\xi^{-1}\right)^t}} \nabla_x f + \frac{\partial}{\partial t} f \tag{3.34}$$

with $\nabla_x f^T = \left(\frac{\partial}{\partial x_1}f, ..., \frac{\partial}{\partial x_d}f\right)$. (3.29) ensures that (3.34) is indeed the generator of a Markov process on $\mathbb{D}$ (see [7, Theorem 5.4.2]). It follows that $Y_t^n \overset{J_1}{\Rightarrow} Y_t$ where Y_t is a Markov process whose generator is given by (3.34). By Lemma 1 we see that

$$\mathring{X}_t^n \overset{J_1}{\Rightarrow} B_t,$$

where B_t is a diffusion with the generator in (3.31). Finally we conclude that

$$\mathring{X}^n_{E_t^n} \overset{J_1}{\Rightarrow} \left(B_{\xi-}\right)^+.$$

Since the generator in (3.31) is a local operator we conclude that $t \mapsto B_t$ is continuous a.s, and that (3.30) holds. Next we assume that $W_1 \in \mathfrak{L}_s^{s^\alpha}$. Note that this suggests that $E_t^n = E_t$ for every $n \geq 1$ and that

$$\begin{aligned} W_i &\sim D_{\left(U_i + T_{i-1}\right)} - D_{T_{i-1}} \\ &\sim U_i^{\frac{1}{\alpha}} D_1. \end{aligned} \tag{3.35}$$

The mapping $U \mapsto U^{\frac{1}{\alpha}}$ maps the set $\mathfrak{M}$ onto $\mathfrak{M}$. It follows that X_t is the CTRW associated with the space-time jumps $\left\{J_i, U_i^{\frac{1}{\alpha}}\tau_i\right\}$ with $\boldsymbol{\tau} = \{\tau_i\}$ where $\tau_1 \sim D_1$. This shows that X_t is an annealed process of RWRE of type II with waiting times $U_i' = \left(\mathbb{E}\left(U_i^{\frac{1}{\alpha}}\right)\right)^{-1} U_i^{\frac{1}{\alpha}}$ and random environment $\boldsymbol{\tau}' = \left\{\mathbb{E}\left(U_i^{\frac{1}{\alpha}}\right)\tau_i\right\}$. Assume for simplicity that $\mathbb{E}\left(U_i^{\frac{1}{\alpha}}\right) = 1$. Using (3.35) and the calculations for the RWRE of type I we conclude that the quenched limit of the RWRE of type II is B_ξ. □

4. BOUND ON THE ERROR

In this section we give a polynomial bound on the distance between the law of a given uncoupled CTRW Y_t and the law of a time changed CTRW $\mathring{X}^n_{E_t^n}$ on the space $\mathbb{D}[0,T]$. The proof is constructive and therefore provides us with the space-time jumps of $\mathring{X}_t$ as well as with the inverse-subordinators E_t^n. The bound relies on the following lemma.

Lemma 8. *Let $X \in \mathfrak{L}_{s^\alpha}$ be a r.v. with tail $\bar{f}(t) = \mathbb{P}(X > t)$. There exists a r.v $Y \in \mathfrak{L}_s^{\overline{s^\alpha}}$ and a coupling $\mathbb{P}_{couple}$ of X and Y s.t*

$$\mathbb{P}_{couple}(|X - Y| > t) = o\left(\bar{f}(t)\right)$$

Proof. Suppose X is a r.v in $\mathfrak{L}_{s^\alpha}$. It follows that there exists a function $L(t)$ which is positive and slowly varying s.t $\mathbb{P}(X \geq t) = L(t)\, t^{-\alpha}$. By Lemma 5 and (3.10) we see that there exists $Y \in \mathfrak{L}_s^{\overline{s^\alpha}}$ s.t $\mathbb{P}(Y \geq t) = L(t)\, t^{-\alpha} + g(t)$, where $g(t) = o(L(t)\, t^{-\alpha})$. We denote $F_1(t) = \mathbb{P}(Y \geq t)$, $F_2(t) = \mathbb{P}(X \geq t)$ and $I_j = [j, j+1)$ for $j \in \mathbb{Z}_+$. We begin by coupling X and Y in any way on I_j, note that the mass that can be coupled on I_j is $\min\{F_1(I_j), F_2(I_j)\}$ where $F_i(I_j) = F_i(j) - F_i(j+1)$ for $i \in \{1, 2\}$, and the mass that is excessive and could not be coupled is $|F_1(I_j) - F_2(I_j)| = |g(j) - g(j+1)|$. Note also that the sign of $g(j) - g(j+1)$ determines whether $F_1(I_j) > F_2(I_j)$ $(g(j) - g(j+1) > 0)$, $F_1(I_j) < F_2(I_j)$ $(g(j) - g(j+1) < 0)$ and $F_1(I_j) = F_2(I_j)$ $(g(j) - g(j+1) = 0)$. Next we couple the excessive mass of X and Y on each interval of the form $[2^n, 2^{n+1})$ in the following way: let $\{I_{i_k}\}_{k=1}^{m_q}$ and $\{I_{j_k}\}_{k=1}^{m_s}$ be the sets of intervals whose excessive mass from the partial coupling before is negative and non-negative respectively. More precisely, let $I_j \subset [2^n, 2^{n+1})$ then $I_j \in \{I_{i_k}\}_{k=1}^{m_q}$ $(I_j \in \{I_{j_k}\}_{k=1}^{m_s})$ iff $g(j) - g(j+1) < 0$ $(g(j) - g(j+1) \geq 0)$. So $\left\{\cup_{k=1}^{m_q} I_{i_k}\right\} \cup \left\{\cup_{k=1}^{m_s} I_{j_k}\right\} = [2^n, 2^{n+1})$. Imagine that each I_{i_k} is a customer with negative mass q_k and each $I_{j_k}$is a server with positive mass s_k. Customers enter the queue according to their original order in $[2^n, 2^{n+1})$, that is , $I_{i_{k_1}}$ is in front of $I_{i_{k_2}}$ iff $i_{k_1} < i_{k_2}$. The customer I_{i_k}leaves the queue only after he was served by m servers whose total mass is at least q_k. Server I_{j_k} leaves the line as soon as he has served all its mass. For example, if in the interval $[4, 8)$ we have the following

$$-\overset{I_4}{0.2}, -\overset{I_5}{0.4}, \overset{I_6}{0.1}, \overset{I_7}{0.7}. \tag{4.1}$$

In this case, $I_{i_1} = [4, 5), I_{i_2} = [5, 6)$ and $I_{j_1} = [6, 7), I_{j_2} = [7, 8)$. then the coupling will be

$$\begin{aligned} -0.4, -0.2&|0.1, 0.7 \\ -0.4, -0.1&|0.7 \\ -0.4&|0.6 \\ &0.2, \end{aligned}$$

and so $g(4) - g(8) = 0.2$, which is the excessive mass of $F_1([4, 8))$ over $F_2([4, 8))$ that can not be coupled in the interval $[4, 8)$. We say that the interval $I_{i_k}$is i-bad if the last server $I_{j_{k'}}$ that served him is such that $|i_k - j_{k'}| > i$. For example, in (4.1) the customer I_4 was served by both I_6 and I_7 and since $7 - 4 = 3 < 4$ it is not 4-bad but is 2-bad. Note that if $I_j \in [2^n, 2^{n+1})$ then I_j is i-bad iff one of the following conditions is satisfied

$$\begin{aligned} F_1([2^n, j - i)) &\geq F_2([2^n, j + 1)) \\ F_1([2^n, j + i + 1)) &< F_2([2^n, j + 1)). \end{aligned}$$

Define

$$\epsilon_i^1 = \sup_{j\geq i}\sup_{1\leq\lambda\leq 2}\left|\frac{L(j\lambda)}{L(j)} - 1\right| \tag{4.2}$$

$$\epsilon_i^2 = \sup_{t\geq i}\left|\frac{g(t)}{L(t)t^{-\alpha}}\right|. \tag{4.3}$$

Note that by the UCT and the definition of $g(t)$ $\epsilon_i^1, \epsilon_i^2 \overset{i\to\infty}{\to} 0$. Fix a positive integer i. Note that potential i- bad intervals I_j should be looked for for $j \geq 2^{[\log i]+1}$, where throughout the proof we use $\log x = \log_2 x$. Let us now check the two conditions. Let $t = 2^{[\log j]}$, then condition one is

$$\begin{aligned} F_1(t) - F_1(j-i) &\geq F_2(t) - F_2(j+1)\\ F_2(t) - F_2(j-i) + g(t) - g(j-i) &\geq F_2(t) - F_2(j+1) \end{aligned}$$

$$F_2(j+1) - F_2(j-i) \geq g(j-i) - g(t). \tag{4.4}$$

Note that by (4.3) it is enough to look for j's that satisfy

$$L(j+1)(j+1)^{-\alpha} - L(j-i)(j-i)^{-\alpha} \geq -2\epsilon_t^2 L(t)t^{-\alpha}.$$

If $L_{max} = \sup_{t\leq y\leq 2t}|L(y)|$, by (4.2) we can look for j's that satisfy

$$L_{max}\left((j+1)^{-\alpha} - \left(1-\epsilon_t^1\right)(j-i)^{-\alpha}\right) \geq -2\epsilon_t^2 t^{-\alpha}L_{max}.$$

Using the convexity of $t\mapsto t^{-\alpha}$ we may consider

$$-\alpha(j+1)^{-\alpha-1}(i+1) + \epsilon_t^1(j-i)^{-\alpha} \geq -2\epsilon_t^2 t^{-\alpha},$$

or

$$j \geq t^{\frac{\alpha}{1+\alpha}}((i+1)\alpha)^{\frac{1}{\alpha+1}}\left(2\epsilon_t^2+\epsilon_t^1\right)^{-\frac{1}{1+\alpha}} - 1.$$

Note that t is at least $2^{[\log(i)]}$ and so

$$j \geq ((i+1)\alpha)^{\frac{1}{\alpha+1}}2^{[\log(i)]\frac{\alpha}{1+\alpha}}\left(2\epsilon_t^2+\epsilon_t^1\right)^{-\frac{1}{1+\alpha}} - 1. \tag{4.5}$$

It follows that for a fixed i , i-bad j's who satisfy the first condition should be looked for above a number that increases super-linearly with i. Similarly, for the second condition we obtain the following condition

$$j > (i\alpha)^{\frac{1}{\alpha+1}}2^{[\log(i)]\frac{\alpha}{1+\alpha}}\left(2\epsilon_t^2+\epsilon_t^1\right)^{-\frac{1}{1+\alpha}} - 1. \tag{4.6}$$

Let us denote by ic_t ($t = 2^{[\log i]}$) the r.h.s of (4.6). It follows that one cannot find i-bad j's between i and ic_t where the latter increases super-linearly in i. Let $W = |X-Y|$ be the absolute difference between X and Y in our coupled space $(\Omega_{couple}, \mathcal{F}_{couple}, \mathbb{P}_{couple})$. If I_j is not i-bad and was coupled in the second stage, then $\{X\in I_j\}\subset\{W\leq i\}$ for $i>1$. Since on each interval of the form $[2^n, 2^{n+1})$, for $n\geq\log(ic_t)$, we coupled the r.v in such a way that it has no i-bad intervals, the only mass that may affect the event $\{W>i\}$ is $\left|g(2^n) - g\left(2^{n+1}\right)\right|$. It follows that

$$\mathbb{P}_{couple}(W>i) \leq \sum_{k=[\log(i)]}^{[\log(ic_t)]-1}\left|g\left(2^k\right) - g\left(2^{k+1}\right)\right| + L\left(\frac{ic_t}{4}\right)\left(\frac{ic_t}{4}\right)^{-\alpha} + \left|g\left(\frac{ic_t}{4}\right)\right|. \tag{4.7}$$

We claim now that

$$\sum_{k=[\log(i)]}^{[\log(ic_t)]-1}\left|g\left(2^k\right)-g\left(2^{k+1}\right)\right|=o\left(L\left(i\right)i^{-\alpha}\right). \tag{4.8}$$

To see that we note that by (4.3) and the UCT, for any $C>\frac{1+2^{\alpha}}{1-2^{\alpha}}$and large enough i we have

$$\begin{aligned}\left|g\left(2^k\right)-g\left(2^{k+1}\right)\right| &\le \epsilon_t^2 2^{-\alpha k}L\left(2^k\right)\left(1+2^{-\alpha}\frac{L\left(2^{k+1}\right)}{L\left(2^k\right)}\right)\\ &\le C\epsilon_t^2 2^{-\alpha k}L\left(2^k\right)\left(1-2^{-\alpha}\frac{L\left(2^{k+1}\right)}{L\left(2^k\right)}\right),\end{aligned}$$

It follows that for large enough i we have

$$\begin{aligned}\sum_{k=[\log(i)]}^{[\log(ic_t)]-1}\left|g\left(2^k\right)-g\left(2^{k+1}\right)\right| &\le \epsilon_i^2 C\sum_{k=[\log(i)]}^{[\log(ic_t)]-1}F_2\left(2^k\right)-F_2\left(2^{k+1}\right)\\ &\le \epsilon_i^2 C\left(L\left(2^{[\log(i)]}\right)2^{-[\log(i)]\alpha}-L\left(ic_t\right)2^{-ic_t\alpha}\right),\end{aligned}$$

and (4.8) is implied. It follows form (4.7) and (4.8) that

$$\mathbb{P}_{couple}\left(W>i\right)=o\left(L\left(i\right)i^{-\alpha}\right), \tag{4.9}$$

and it is straightforward to see that (4.9) holds when $i\in\mathbb{R}_+$. □

In the following result we limit ourselves to case where the waiting times of Y_t^1 is such that $\mathbb{P}\left(W_i>t\right)\sim ct^{-\alpha}$, where c is some positive constant. This assumption is important for the result.

Theorem 2. *Let Y_t^n be the CTRW associated with the i.i.d space-time jumps $\left\{n^{-\frac{1}{\alpha}}W_i,n^{-\frac{1}{2}}J_i\right\}$ where $\mathbb{P}\left(W_1>t\right)=\left[\Gamma\left(1-\alpha\right)\right]^{-1}t^{-\alpha}+g_2\left(t\right)$ and $g_2\left(t\right)=O\left(t^{-\beta}\right)$ for $\beta>\alpha$ and $J_1\in\mathbb{R}$ has variance 1 and zero mean. Then there exists a CTRW $\mathring{X}_t^n$ associated with the i.i.d space-time jumps $\left(n^{-\frac{1}{2}}J_i,n^{-1}\mathring{U}_i\right)$ s.t $\mathring{U}_1$ has finite mean, a sequence of inverse subordinators E_t^n so that for every $c<\xi_0$,*

$$\rho_{J_1}\left(Y_t^n,\mathring{X}_{E_t^n}^n\right)<Cn^{-c},$$

where $\xi_0=\min\left\{\frac{\alpha}{7\alpha+4},\frac{\beta-\alpha}{3\beta+\alpha+4}\right\}$.

Proof.

Step 1 First consider for every n the sequence $\{a_nW_i\}_{i=1}^{\infty}$. If $\mathbb{P}\left(W_1>t\right)=L\left(t\right)t^{-\alpha}$, by Lemma 5 we can approximate W_1 by a distribution $U\in\overline{\mathfrak{L}_s^{s^{\alpha}}}$ s.t $\mathbb{P}\left(U>t\right)\sim L\left(t\right)t^{-\alpha}$. Let $\{U_i\}$ be a sequence of i.i.d r.v's s.t $U_1\sim U$. Define X_t^n to be the CTRW associated with the space time jumps $\left\{n^{-\frac{1}{2}}J_i,n^{-\frac{1}{\alpha}}U_i\right\}$ We wish to construct a set $A\in\Omega_{couple}$ of probability larger than $1-\epsilon$ on which we can bound the distance (d_{J_1}) between two trajectories of the processes X_t^n and Y_t^n. In order to use Lemma 8 we must limit our discussion to finite number of jumps by time T. We shall use the fact that for every coupling $p_{X,Y}$ of some r.vs X and Y , if $p_X\left(X\in A\right)<\frac{\epsilon}{4}$ and $p_Y\left(Y\in B\right)<\frac{\epsilon}{4}$ then

$p_{X,Y}\left(|X-Y|\,1_{\{X\in A\}\cup\{Y\in B\}}>\epsilon\right)<\frac{\epsilon}{2}$, for any coupling $p_{X,Y}$ of p_X and p_Y. And so, if we show that

$$p_{X,Y}\left(|X-Y|\,1_{\{X\in A^c\}\cap\{Y\in B^c\}}>\epsilon\right)<\frac{\epsilon}{2},$$

we see that

$$p_{X,Y}\left(|X-Y|>\epsilon\right)=$$
$$p_{X,Y}\left(|X-Y|\,1_{\{X\in A\}\cup\{Y\in B\}}+|X-Y|\,1_{\{X\in A^c\}\cap\{Y\in B^c\}}>\epsilon\right)<\epsilon.$$

Suppose there exists a sequence $M_1(n)\to\infty$ s.t for large enough n

$$\mathbb{P}\left(\sum_{i=1}^{M_1(n)}a_nW_i\le T\right)\le\frac{\epsilon}{8},$$
$$\mathbb{P}\left(\sum_{i=1}^{M_1(n)}a_nU_i\le T\right)\le\frac{\epsilon}{8}.$$

Next assume there exists a sequence $M_2(n)\to\infty$ s.t for large enough n

$$\mathbb{P}\left(\sum_{i=1}^{M_2(n)}a_nW_i\le\frac{\epsilon}{2}\right)<\frac{\epsilon}{8}, \tag{4.10}$$
$$\mathbb{P}\left(\sum_{i=1}^{M_2(n)}a_nU_i\le\frac{\epsilon}{2}\right)<\frac{\epsilon}{8}. \tag{4.11}$$

Moreover, assume that for large enough n

$$\mathbb{P}\left(\overline{S}_{\{0,1,\ldots,M_2(n)\}}>\frac{\epsilon}{4}\right)<\frac{\epsilon}{8},$$

where for a set $A\subset\mathbb{Z}_+$, with $j_i=\inf A$,

$$\overline{S}_A=\sup_{i\in A}\left|\sum_{j=j_i}^{i}n^{-\frac{1}{2}}J_i\right|.$$

Define the random sets

$$B_n^Y[a,b]=\left\{j:\sum_{i=1}^{j}a_nW_i:\in[a,b]\right\}$$
$$B_n^X[a,b]=\left\{j:\sum_{i=1}^{j}a_nU_i:\in[a,b]\right\},$$

that is, $B_n^Y[a,b]$ is the set of the indices of the jumps that occurred in the time interval $[a,b]$. Also define $i_0^Y=\inf B_n^Y[a,b]$ and $i_0^X=\inf B_n^X[a,b]$. By Lemma 8 we know that we can construct a probability space $(\Omega_{couple},\mathcal{F}_{couple},\mathbb{P}_{couple})$ on which one can find the sequence $\{W_i\}$ and $\{U_i\}$ s.t for large enough n we have

$$\mathbb{P}_{couple}\left(\sum_{i=1}^{M_1(n)}a_n|W_i-U_i|>\frac{\epsilon}{2}\right)<\frac{\epsilon}{4}. \tag{4.12}$$

It is implied that for N_0 large enough, for every $n > N_0$, one can find a set $A_n \in \mathcal{F}_{couple}$ s.t $\mathbb{P}_{couple}(A_n) > 1 - \epsilon$ and conditioned on A_n we have

$$\mathbb{P}_{couple}\left(\left|B_n^Y[0,T]\right| > M_1(n) \,|A_n\right) = 0$$
$$\mathbb{P}_{couple}\left(\left|B_n^Y[T-\frac{\epsilon}{2},T]\right| > M_2(n) \,|A_n\right) = 0$$
$$\mathbb{P}_{couple}\left(\overline{S}_{B_n^Y[T-\frac{\epsilon}{2},T]} > \frac{\epsilon}{4}|A_n\right) = 0$$
$$\mathbb{P}_{couple}\left(\sum_{i=1}^{M_1(n)} a_n\,|W_i - U_i| > \frac{\epsilon}{2}|A_n\right) = 0, \tag{4.13}$$

where the first three equations in (4.13) are true for the sets $B_n^X[0,T]$ and $B_n^X[T-r^{-1},T]$ as well.

Step 2 Let $M \in \mathbb{Z}_+$ and $d_{l_1}^M : (\mathbb{R}\times\mathbb{R}_+)^M \to \mathbb{R}_+$ be the metric on vectors of real numbers defined by

$$d_{l_1}^M\left(\left\{a_n^1, a_n^2\right\}, \left\{b_n^1, b_n^2\right\}\right) = \sum_{n=1}^{M}\left|a_i^1 - b_i^1\right| + \left|a_i^2 - b_i^2\right|.$$

Consider the set $\mathcal{A} = \left\{(J_i, W_i) \in (\mathbb{R}\times\mathbb{R}_+)^M : \sum_{i=1}^M W_i \le T\right\}$ equipped with $d_{l_1}^M$,i.e.

$$d_{l_1}^M\left(\left(J_i^1, W_i^1\right), \left(J_i^2, W_i^2\right)\right) = \sum_{i=1}^{M}\left|J_i^1 - J_i^2\right| + \left|W_i^1 - W_i^2\right|.$$

Define the mapping $\mathcal{T} : \mathcal{A} \to \mathbb{D}[0,T]$ by

$$(J_i, W_i)_{i=1}^M \mapsto f_t = \sum_{i=1}^{M} J_i 1_{\left\{\sum_{j=1}^i W_j \le t\right\}}.$$

We claim that $\mathcal{T} : \left(\mathcal{A}, d_{l_1}^M\right) \to (\mathbb{D}[0,T], d_{J_1})$ is a contraction. To see that, let $\left(J_i^1, W_i^1\right), \left(J_i^2, W_i^2\right) \in (\mathbb{R}\times\mathbb{R}_+)^M$, and define

$$\lambda_t = \begin{cases} t\frac{W_1^2}{W_1^1} & 0 \le t < W_1^1 \\ \left(t - W_1^1\right)\frac{W_2^2}{W_2^1} + W_1^2 & W_1^1 \le t < W_1^1 + W_2^1 \\ \vdots & \vdots \\ \left(t - \sum_{i=1}^{M-1} W_i^1\right)\frac{W_M^2}{W_M^1} + \sum_{i=1}^{M-1} W_1^2 & \sum_{i=1}^{M-1} W_i^1 \le t \le T \end{cases}.$$

Note that

$$\left\|\mathcal{T}\left[\left(J_i^1, W_i^1\right)\right](\lambda_t) - \mathcal{T}\left[\left(J_i^2, W_i^2\right)\right](t)\right\| \le \sup_i\left|J_i^1 - J_i^2\right| \le \sum_{i=1}^{M}\left|J_i^1 - J_i^2\right|,$$

since the regeneration points of $\mathcal{T}\left[\left(J_i^2, W_i^2\right)\right](t)$ and $\mathcal{T}\left[\left(J_i^1, W_i^1\right)\right](\lambda_t)$ are the same. Next note that since λ_t is piece-wise linear

$$\|\lambda_t - t\| \le \sup_{t_i}|\lambda_{t_i} - t_i|,$$

where $t_i \in \left\{\sum_{j=1}^{i} W_j^1 : 1 \le i \le M\right\}$. Or equivalently,

$$\|\lambda_t - t\| = \sup_{1\le i\le M} \left|\sum_{j=1}^{i} W_j^2 - \sum_{j=1}^{i} W_j^1\right| \le \sum_{i=1}^{M} \left|W_i^1 - W_i^2\right|.$$

It follows that

$$\begin{aligned} & d_{J_1}\left(\mathcal{T}\left[\left(J_i^1, W_i^1\right)\right](t) - \mathcal{T}\left[\left(J_i^2, W_i^2\right)\right](t)\right) \\ & \le \|\lambda_t - t\| \wedge \left\|\mathcal{T}\left[\left(J_i^1, W_i^1\right)\right](\lambda_t) - \mathcal{T}\left[\left(J_i^2, W_i^2\right)\right](t)\right\| \\ & \le d_{l_1}\left(\left(J_i^1, W_i^1\right), \left(J_i^2, W_i^2\right)\right), \end{aligned} \tag{4.14}$$

so that $\mathcal{T}$ is indeed a contraction. Next, let x_t^n and y_t^n be two realizations of X_t^n and Y_t^n respectively on the set A^n. Suppose w.l.o.g that x_t^n has at least the same number of jumps by time $T - \frac{\epsilon}{2}$ as y_t^n, that is

$$\left|B_n^Y[0, T - \frac{\epsilon}{2}]\right| \le \left|B_n^X[0, T - \frac{\epsilon}{2}]\right|.$$

By (4.13) we have

$$\sum_{i=1}^{M_1(n)} a_n \left|W_i - U_i\right| < \frac{\epsilon}{2},$$

which implies that one can find $J_{diff} := \left|B_n^X[0, T - \frac{\epsilon}{2}]\right| - \left|B_n^Y[0, T - \frac{\epsilon}{2}]\right|$ jumps of y_t^n in the interval $[T - \frac{\epsilon}{2}, T]$. Let $\tilde{x}_t \in \mathbb{D}[0, T]$ s.t

$$\tilde{x}_t = x_t - \sum_{i \in B^X[T - \frac{\epsilon}{2}, t]} n^{-\frac{1}{2}} J_i,$$

where if $t < T - \frac{\epsilon}{2}$ the summation vanishes. In words, $\tilde{x}_t$ equals to x_t up to time $T - \frac{\epsilon}{2}$ and equals $x_t\left(T - \frac{\epsilon}{2}\right)$ on the interval $[T - \frac{\epsilon}{2}, T]$. Next we define the time $T_{diff} = \inf\left\{t : \left|B_n^Y[T - \frac{\epsilon}{2}, t]\right| \ge J_{diff}\right\}$ and

$$\tilde{y}_t = y_{(T_{diff} \wedge t)}.$$

$\tilde{y}_t$ stands for the function that equals y_t up to the point where it has jumped the same number of jumps as $\tilde{x}_t$. Next note that on A^n,

$$\begin{aligned} d_{J_1}(x_t, y_t) & \le d_{J_1}(\tilde{x}_t, \tilde{y}_t) + d_{J_1}(x_t - \tilde{x}_t, y_t - \tilde{y}_t) \\ & \le d_{l_1}^M\left(\left(J_i^n(\omega), a_n W_i(\omega)\right), \left(J_i^n(\omega), a_n U_i(\omega)\right)\right) \\ & \quad + \overline{S}_{B_n^Y[T - \frac{\epsilon}{2}, T]} + \overline{S}_{B_n^X[T - \frac{\epsilon}{2}, T]} \\ & < \epsilon, \end{aligned} \tag{4.15}$$

where $\omega \in \Omega_{couple}$ is such that

$$\begin{aligned} \mathcal{T}\left(J_i^n(\omega), a_n W_i(\omega)\right) &= x_t \\ \mathcal{T}\left(J_i^n(\omega), a_n U_i(\omega)\right) &= y_t \\ \left|B_n^X[0, T - \frac{\epsilon}{2}]\right| &= M. \end{aligned}$$

Inequality (4.15) follows from (4.13) and (4.14). We have showed that there exists a coupling s.t

$$\mathbb{P}_{couple}\left(d_{J_1}\left(X_t^n, Y_t^n\right) > \epsilon\right) < \epsilon,$$

or that

$$\rho_{J_1}\left(Y_t^n, X_t^n\right) < \epsilon.$$

Step 3 Let W be a r.v s.t $W \sim W_1$. In order to approximate W by elements in $\mathfrak{L}_s^{s^\alpha}$ we follow the recipe in Lemma 5. We first introduce $\mathcal{W}_m = W 1_{[0,m]}$ and

$$\mu_i^m = \mathbb{E}\left(\left(\mathcal{W}_m\right)^i\right),$$

the i'th moment of $\mathcal{W}_m$. We proceed to defining the symbol

$$\psi(s) = -s - \left(\mu_1^m\right)^{-1} \int_0^\infty \left(e^{-sy} - 1\right) \frac{\alpha y^{-\alpha-1}}{\Gamma(1-\alpha)} dy.$$

Note that $\psi(s)$ is the symbol of the subordinator $t + D_{t/\mu_1^m}$, where D_t is the standard stable subordinator of index $0 < \alpha < 1$ whose LT is $\mathbb{E}\left(e^{-sD_t}\right) = e^{-ts^\alpha}$. Note that we somewhat deviate form the recipe in Lemma 5, where we would use the symbol

$$\psi'(s) = -s - \left(\mu_1^m\right)^{-1} \int_0^\infty \left(e^{-sy} - 1\right) f(dy), \tag{4.16}$$

where $f(dy)$ is the distribution of W. However, since the purpose of the $f(dy)$ in Equation (4.16) is to obtain a regularity of $s^\alpha L\left(s^{-1}\right)$ around zero for $\psi(s)$, it is clear that in this case a stable subordinator would do the job. An expression for the tail of a stable subordinator at time $t > 0$ can be found in [15, Eq. 2.4.3] to be(with some algebraic manipulations)

$$F_t^D(x) = \sum_{n=1}^\infty (-1)^{n-1} \frac{x^{-\alpha n} t^n}{\Gamma(1-\alpha n)\, n!} \qquad x > 0, t > 0.$$

Let $h_m(dy) = \Phi_\psi(f_m)$, where $f_m(dy) = \mathbb{P}(\mathcal{W}_m \in dy)$. We have for $x > m$

$$\begin{aligned} \bar{h}_m(x) &= \mathbb{P}\left(\mathcal{W}_m + D_{\mathcal{W}_m/\mu_1^m} > x\right) \\ &= \int_0^\infty F_{y/\mu_1^m}^D(x-y) f_m(dy). \end{aligned}$$

Moreover, we see that for $x > m$

$$\int_0^\infty F_{y/\mu_1^m}^D(x) f_m(dy) \le \bar{h}_m(x) \le \int_0^\infty F_{y/\mu_1^m}^D(x-m) f_m(dy),$$

which, by the analyticity of F_t^D and the compact support of $f_m(dy)$ shows that

$$\begin{aligned} &\frac{x^{-\alpha}}{\Gamma(1-\alpha)} - \frac{x^{-2\alpha}}{\Gamma(1-2\alpha)} \frac{\mu_2^m}{\left(\mu_1^m\right)^2} + O\left(x^{-3\alpha}\right) \le \bar{h}_m(x) \\ &\le \frac{(x-m)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(x-m)^{-2\alpha}}{\Gamma(1-2\alpha)} \frac{\mu_2^m}{\left(\mu_1^m\right)^2} + O\left(x^{-3\alpha}\right), \end{aligned} \tag{4.17}$$

or that

$$\bar{h}_m(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + g_1(x),$$

where $g_1 \sim x^{-2\alpha}$. Let $U^m = \Phi_\psi(\mathcal{W}_m)$, so that $\mathbb{P}(U^m > x) = \bar{h}_m(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + g_1(x)$. Now, since $\mathbb{P}(W \geq x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + g_2(x)$ where $g_2(x) = O(x^{-\beta})$ then $\bar{h}(x) = \mathbb{P}(W \geq x) + g_3(x)$ where

$$g_3(x) = O(x^{-\gamma}), \tag{4.18}$$

with $\gamma = \min\{2\alpha, \beta\}$. We assume our probability space has two sequences of i.i.d r.vs $\{U_i\}$ and $\{W_i\}$ where $U_1 \sim U^m$ and $W_1 \sim W$. Applying the coupling in Lemma 8 with $Y = U^m$ and $X = W$ and substituting $g_3(x)$ in place of $g(x)$, making the same calculation down to (4.5) (note that here $L(t) = \frac{1}{\Gamma(1-\alpha)}$) we see that i-bad I_j's can be found for $j > Ci^{\frac{\gamma+1}{\alpha+1}}$. Using this in (4.7), denoting $Z = |X - Y|$, we see that the summation on the r.h.s contributes to $\mathbb{P}_{couple}(Z > t)$ at most $O(t^{-\gamma})$, the second term contributes $O\left(t^{-\frac{\alpha(\gamma+1)}{\alpha+1}}\right)$ whereas the last term gives not more than $O\left(t^{-\gamma\left(\frac{\gamma+1}{\alpha+1}\right)}\right)$. We conclude that

$$\mathbb{P}_{couple}(Z > t) = O\left(t^{-\xi_0}\right),$$

where $\xi_0 = \frac{\alpha(\gamma+1)}{\alpha+1}$. Let $c(\xi) = \frac{\xi-\alpha}{3\xi+\alpha}$ and note that $c(\xi)$ is strictly increasing on $[0, \infty)$. Fix $\xi \in [0, \xi_0)$ and write $c := c(\xi)$. Let $c' = \frac{3c}{\alpha}$, $M_1(n) = c' n \log n$, $M_2(n) = c' n^{1-\alpha c'} \log n$ in (4.13),by Chernoff's bound we have

$$\mathbb{P}_{couple}\left(\sum_{i=1}^{M_1(n)} n^{-\frac{1}{\alpha}} W_i \leq T\right) \leq e^{sT}\left(1 - n^{-1}s^\alpha + o\left(n^{-1}s^\alpha\right)\right)^{c' n \log n} \tag{4.19}$$

$$\mathbb{P}_{couple}\left(\sum_{i=1}^{M_2(n)} n^{-\frac{1}{\alpha}} W_i \leq n^{-c'}\right) \leq e^{s}\left(1 - n^{c'\alpha-1}s^\alpha + o\left(n^{c'\alpha-1}s^\alpha\right)\right)^{c' n^{1-\alpha c'} \log n}. \tag{4.20}$$

taking $s = 1$ in (4.19) and in (4.20) we see that for large enough n we have

$$\mathbb{P}_{couple}\left(\sum_{i=1}^{M_1(n)} n^{-\frac{1}{\alpha}} W_i \leq T\right) \leq Cn^{-c'}$$

$$\mathbb{P}_{couple}\left(\sum_{i=1}^{M_2(n)} n^{-\frac{1}{\alpha}} W_i \leq n^{-c'}\right) \leq Cn^{-c'}.$$

If $\{Z_i\}$ are i.i.d r.vs s.t $Z_1 \sim Z$, by [12, Eq. 1.1] (with $t = 1$) we have

$$\mathbb{P}_{couple}\left(n^{-\frac{1}{\alpha}} \sum_{i=1}^{M_1(n)} Z_i > n^{-c'}\right) \leq M_1(n)\, \mathbb{P}_{couple}\left(Z > n^{\frac{1}{\alpha}-c'}\right) + \exp\left(1 - \frac{A(n)}{n^{\left(\frac{1}{\alpha}-c'\right)}} - \log\left(\frac{n^{\left(\frac{1}{\alpha}-c'\right)}}{A(n)}\right)\right),$$

whenever

$$\frac{n^{\left(\frac{1}{\alpha}-c'\right)}}{A(n)} > 1, \tag{4.21}$$

where

$$A(n) = c' n \log n \int_0^{n^{\frac{1}{\alpha}-c'}} y \mathbb{P}_{couple}(Z \in dy).$$

To see that (4.21) indeed holds, use the fact that $\mathbb{P}_{couple}(Z > y) = O\left(y^{-\xi_0}\right)$ and therefore that $A(n) = O\left(cn \log(n)\, n^{\left(\frac{1}{\alpha}-c'\right)(1-\xi_0)}\right)$, and

$$\begin{aligned} -c &= \left(c' - \frac{1}{\alpha}\right)\xi + 1, \\ &> \left(c' - \frac{1}{\alpha}\right)\xi_0 + 1, \end{aligned} \tag{4.22}$$

to conclude that for large enough n

$$\frac{n^{\xi\left(\frac{1}{\alpha}-c'\right)}}{A(n)} \sim \frac{1}{[cn \log n]\, n^{-\xi_0\left(\frac{1}{\alpha}-c'\right)}} > 1,$$

and $\frac{A(n)}{n^{\xi\left(\frac{1}{\alpha}-c'\right)}} = o\left(n^{-c}\right)$. Using Markov's inequality we obtain for $\xi < \xi_0$

$$\begin{aligned} \mathbb{P}_{couple}\left(n^{-\frac{1}{\alpha}} \sum_{i=1}^{M_1(n)} Z > n^{-c'}\right) &\le n^{\left(c'-\frac{1}{\alpha}\right)\xi} n c' \log n \mathbb{E}\left(Z^{\xi}\right) \\ &\quad + o\left(n^{-c}\right). \end{aligned} \tag{4.23}$$

By (4.22) we see that (4.23) is bounded by $Cn^{-c}\log n$. By Doob's inequality we see that

$$\begin{aligned} \mathbb{P}_{couple}\left(\overline{S}_{\{1,\ldots,M_2(n)\}} > n^{-c}\right) &\le M_2(n)\, n^{2c} n^{-1} \mathbb{E}\left(J_1^2\right) \\ &\le n^{-\alpha c' + 2c} c \log n \mathbb{E}\left(J_1^2\right) \\ &= n^{-c} c \log n \mathbb{E}\left(J_1^2\right). \end{aligned}$$

Hence, we have verified the all the inequalities in (4.13) with the bound Cn^{-c} for $c < c(\xi_0)$. Following the arguments in Step 2 we see that if X_t^n is the CTRW associated with the space-time jumps $\left\{n^{-\frac{1}{2}} J_i, n^{-\frac{1}{\alpha}} U_i\right\}$ then

$$\rho_{J_1}\left(Y_t^n, X_t^n\right) < Cn^{-c}.$$

Finally, we note that since $U_1 \in \mathfrak{L}_s^{s^\alpha}$, by Proposition 3 we may represent X_t^n as $\mathring{X}_{E_t^n}^n$ where $\mathring{X}_t^n$ is the CTRW associated with the space-time jumps $\left\{n^{-\frac{1}{2}} J_i, n^{-1} \mathring{U}_i\right\}$, where $\mathring{U}_i \sim U^m$ (and so has finite mean) and E_t^n is a sequence of inverse-subordinators of the subordinators $D_t^n = t + D_{t/\mu_1^m}$ where $\mathbb{E}(U^m) = \mu_1^m$.

□

Remark 6. Note that the choice of the skew of the tail of W_1 need not be $[\Gamma(1-\alpha)]^{-1}$, i.e. one can take W s.t $\mathbb{P}(W_1 > t) \sim Ct^{-\alpha}$ for any $C > 0$. We then approximate W_1 by r.v's in $\mathfrak{L}_s^{C\Gamma(1-\alpha)s^\alpha}$.

Working along the same lines of Theorem 2 it is not hard to see that if Y_t^n is as in Theorem 2 but with i.i.d spatial jumps $\left\{n^{-1}J_i\right\}$ where $\mathbb{E}\left(J_1\right)=1$(and therefore $Y_t^n \overset{J_1}{\Rightarrow} E_t$) then there exists a time-changed CTRW $\mathring{X}^n_{E_t^n}$ s.t

$$\rho_{J_1}\left(Y_t^n, \mathring{X}^n_{E_t^n}\right) < Cn^{-c}, \tag{4.24}$$

for any $c<\frac{\xi_0-\alpha}{\alpha(\xi_0+1)}$. Note that one can not expect for a rate of convergence in (4.24) better then $O\left(n^{-\frac{1}{\alpha}}\right)$ as the scaling is of $n^{-\frac{1}{\alpha}}$(unless $U_1\sim W_1$). Nevertheless, is it possible to get arbitrarily close to $O\left(n^{-\frac{1}{\alpha}}\right)$? Suppose we somehow manage to find a subordinator D'_t s.t

$$\mathbb{P}\left(D'_{\mathcal{W}_m}>t\right)-\mathbb{P}\left(W_1>t\right)=O\left(t^{-\gamma}\right),$$

for $\gamma>2\alpha$. Then as $\gamma\to\infty$ we see that (4.24) holds for every $c<\frac{1}{\alpha}$. Unfortunately, we could not find a way to improve γ beyond 2α. Another point worth mentioning is that the constant controlling g_3 in (4.18) grows as we better approximate W by $\mathcal{W}_m$. This can be seen from the term $\mu_2^m/(\mu_1^m)^2$ in (4.17). Indeed, by the regular variation of the tail of W we see that $\mu_2^m/(\mu_1^m)^2\sim m^\alpha$ as $m\to\infty$. An interesting question in that regard is whether there exists a better choice of $\mathcal{W}_m$(possibly where $\mathcal{W}_m$ has infinite slowly varying mean) so that this undesirable phenomenon be avoided?

4.1. **Example: Pareto Distribution.** We would like to consider an example in which we use Theorem 2. Let Y_t^n be the CTRW associated with the i.i.d space-time jumps $\left(n^{-\frac{1}{2}}J_i, n^{-\frac{1}{\alpha}}W_i\right)$, where J_1 has finite second moment and zero mean and W_1 has the so-called Pareto distribution $f\left(dy\right)$, i.e.

$$\bar{f}\left(t\right)=\mathbb{P}\left(W_1\geq t\right)=\begin{cases}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} & t>\Gamma\left(1-\alpha\right)^{-1/\alpha}\\ 1 & t\leq\Gamma\left(1-\alpha\right)^{-1/\alpha}\end{cases},$$

where $0<\alpha<1$. Using Theorem 2, we have $\beta=\infty$ and therefore $\xi_0=\frac{\alpha}{7\alpha+4}$. It follows that

$$\rho_{J_1}\left(Y_t^n, \mathring{X}^n_{E_t^n}\right) < Cn^{-c},$$

with $c<\frac{\alpha}{7\alpha+4}$, where $\mathring{X}^n_{E_t^n}$ is the process constructed in Theorem 2.

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