FINITE DIMENSIONAL FOKKER-PLANCK EQUATIONS FOR UNCOUPLED CONTINUOUS TIME RANDOM WALK LIMITS

ABSTRACT. Continuous Time Random Walk(CTRW) is a model where particle's jumps in space are coupled with waiting times before each jump. A Continuous Time Random Walk Limit(CTRWL) is obtained by a limit procedure on a CTRW and can be used to model anomalous diffusion. The distribution $p\,(dx,t)$ of a CTRWL X_t satisfies a Fractional Fokker-Planck Equation(FFPE). Since CTRWLs are usually not Markovian, their one dimensional FFPE is not enough to completely define them. In this paper we find the FFPEs of the distribution of X_t at multiple times , i.e. the distribution of the random vector $(X_{t_1},...,X_{t_n})$ for $t_1 < ... < t_n$ for a large class of CTRWLs. This allows us to define CTRWLs by their finite dimensional FFPEs.

1. INTRODUCTION

CTRW models the movement of a particle in space, where the k'th jump J_k of the particle in space is preceded by the k'th waiting time W_k. We let $N_t = \sup\{k : T_k \le t\}$ where $T_k = \sum_{i=1}^{k} W_i$, if $T_1 > t$ then $N_t = 0$. N_t is just the number of jumps of the particle up to time t. Then

$$X'_t = \sum_{k=1}^{N_t} J_k,$$

is the CTRW associated with the time-space jumps $\{(J_k, W_k)\}_{k\in\mathbb{N}}$. Let us now assume that $\{J_k\}$ and $\{W_k\}$ are independent i.i.d sequences of random variables. In order to model the long time behavior of the CTRW we write $\{(J_k^c, W_k^c)\}_{k\in\mathbb{N}}$ for $c > 0$. Here the purpose of c is to render the trajectories of $\{(J_k^c, W_k^c)\}_{k\in\mathbb{N}}$ convergent weakly on a proper space. More precisely, we let $\mathcal{D}\left([0,\infty),\mathbb{R}^2\right)$ be the space of cádlág functions $f : [0,\infty) \to \mathbb{R}^2$ equipped with the Skorokhod J_1 topology. We assume that

$$(S_u^c, T_u^c) = \sum_{k=1}^{\lfloor cu \rfloor} (J_k^c, W_k^c) \Rightarrow (A_u, D_u) \qquad c \to \infty,$$

where $\Rightarrow$ denotes weak convergence of measures with respect to the J_1 topology. We further assume that the processes A_t and D_t are independent Lévy processes and that D_t is a strictly increasing subordinator. Denote by X_t^c the CTRW associated with $\{(J_k^c, W_k^c)\}_{k\in\mathbb{N}}$. We then have ([12, Theorem 3.6] and [11, Lemma 2.4.5])

$$X_t^c \Rightarrow X_t = A_{E_t} \qquad c \to \infty, \tag{1.1}$$

where $E_t = \inf\{s : D_s > t\}$ is the inverse of D_t and $\Rightarrow$ means weak convergence on $\mathcal{D}\left([0,\infty),\mathbb{R}\right)$ equipped with the J_1 topology. It is well known that X_t is usually not Markovian, a fact that makes the task of finding basic properties of X_t nontrivial. One such task is finding the finite dimensional distributions of the process X_t, i.e. $P\left(X_{t_1} \in dx_1, ..., X_{t_n} \in dx_n\right)$. In [8], Meerschaert and Straka used a semi-Markov approach to find the finite dimensional distributions for a large class of CTRWL. It turns out that the discrete regeneration times of X_t^c converge to a set of points where X_t is renewed. Once we know the next time of regeneration of X_t, we no longer need older observations in order to determine the future behavior of X_t. More mathematically, denote by $R_t = D_{E_t} - t$ the next time for regeneration of X_t then (X_t, R_t) is a Markov process. One can then use the transition probabilities of (X_t, R_t) along with the Chapman-Kolmogorov's equation in order to find $P(X_{t_1} \in dx_1, ..., X_{t_n} \in dx_n)$ for $t_1 < ... < t_n$ and $n \in \mathbb{N}$. This method was used in [4] in order to find the finite dimensional distribution of the aged process $X_t^{t_0} = X_t - X_{t_0}$. It is well known ([7, Section 4.5]) that the one dimensional distribution $p(dx, t) = P(X_t \in dx)$ satisfies a FFPE. Once again, as X_t is non Markovian the FFPE satisfied by $p(dx, t)$ is not enough to fully describe X_t(as it would if X_t were Markovian). Hence, a dual problem to finding the finite dimensional distributions is that of finding the finite dimensional FFPEs of the finite dimensional distributions of X_t. In this paper we obtain the finite dimensional FFPEs for a large class of CTRWL. The results generalize the well known one dimensional FFPE of CTRW([6]) as well as results in the finite dimensional case([1],[2]).

2. Mathematical Background

2.1. **Notations.** A well known method of solving partial differential equations of distributions $p(dx_1, ..., dx_n; t_1, ..., t_n)$ on $\mathbb{R}^n$ is taking the Fourier Transform(FT) of the distribution with respect to the spatial variables and then the Laplace Transform(LT) with respect to the time variables. This is referred to as the Fourier Laplace Transform(FLT) of $p(dx_1, ..., dx_n; t_1, ..., t_n)$. More generally, for $m, n \in \mathbb{N}$ let $f(dx_1, ..., dx_m; t_1, ..., t_n)$ be a finite measure on $\mathbb{R}^m$ for a fixed $\mathbf{t} = (t_1, ..., t_n)$ where $0 < t_1 < \cdots < t_n$. Moreover, let $\int\limits_{\mathbf{x} \in A} f(dx_1, ..., dx_m; t_1, ..., t_n)$ be measurable as a function of $\mathbf{t}$ for each $A \subset \mathbb{R}^m$. We denote the FT of f by

$$\widetilde{f}(k_1, ..., k_m; t_1, ..., t_n) = \int\limits_{x_1 \in \mathbb{R}} \cdots \int\limits_{x_m \in \mathbb{R}} e^{-i \sum_{i=1}^m k_i x_i} f(dx_1, ..., dx_m; t_1, ..., t_n).$$

When f has density $f(x_1, ..., x_m; t_1, ..., t_n)$ we denote the LT of f by

$$\hat{f}(x_1, ..., x_m; s_1, ..., s_n) = \int\limits_{t_1=0}^{\infty} \cdots \int\limits_{t_n=0}^{\infty} e^{-\sum_{i=1}^n s_i t_i} f(x_1, ..., x_m; t_1, ..., t_n) \, dt_1 \cdots dt_n.$$

The FLT of f is

$$\bar{f}(dx_1, ..., dx_m; s_1, ..., s_n) = \int\limits_{t_1=0}^{\infty} \cdots \int\limits_{t_n=0}^{\infty} \int\limits_{x_1 \in \mathbb{R}} \cdots \int\limits_{x_n \in \mathbb{R}} e^{-\sum_{i=1}^n (ik_i x_i + s_i t_i)} f(dx_1, ..., dx_m; t_1, ..., t_n) \, dt_1 \cdots dt_n.$$

We also denote by $\tilde{f}$ the FT of f with respect to some of its spatial variables, therefore, $\widetilde{f}(dx_1, k_2; t_1, t_2)$ is the FT of f w.r.t x_2. Similarly, $\hat{f}(dx_1, dx_2; s_1, t_2)$ is the LT of f w.r.t t_1 and $\bar{f}(k_1, dx_2; s_1, t_2)$ is the FLT of f w.r.t x_1 and t_1. When using the hat symbol is cumbersome we also use $\hat{f} = \mathscr{L}(f)$.

2.2. **Caputo and Riemann-Liouville Fractional Derivatives.** The Riemann-Liouville(RL) fractional derivative of index $0 < \alpha < 1$ is given by

$$\mathbb{D}_t^{\alpha} f(t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-r)^{-\alpha} f(r) \, dr, \tag{2.1}$$

for a suitable function f defined on $\mathbb{R}_+$. When the variable with respect to which we take the derivative is obvious we drop the subscript and just write $\mathbb{D}^\alpha f(t)$. It can be easily verified that the LT of (2.1) is

$$\widehat{\mathbb{D}^\alpha f}(s) = s^\alpha \widehat{f}(s).$$

Hence, the RL derivative is a pseudo-differential operator of symbol s^α. Caputo's derivative is obtained by moving the derivative in (2.1) under the integral to obtain

$$\mathfrak{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \frac{\partial}{\partial r} f(r)\, dr. \tag{2.2}$$

The LT of (2.2) is

$$\widehat{\mathfrak{D}^\alpha f}(s) = s^\alpha \widehat{f}(s) - s^{\alpha-1} f\left(0^+\right).$$

We denote the classic derivative by $\frac{\partial}{\partial t} = \mathfrak{D}^1$, and note that $\mathbb{D}^1 = \mathfrak{D}^1$ iff $f(0^+) = 0$. For simplicity we drop the superscript and write $\frac{\partial}{\partial t} = \mathfrak{D}$(or $\frac{\partial}{\partial t} = \mathbb{D}$ when that is the case).

2.3. **On some pseudo-differential operators.** Here we investigate the pseudo-differential operators(PDO) acting on functions $f(\mathbf{t})$ that are differentiable on $C = \{\mathbf{t} : 0 < t_1 < t_2 < ... < t_n\}$ with support $\bar{C} = \{\mathbf{t} : 0 \le t_1 \le t_2 \le ... \le t_n\}$ with LT $\hat{f}$ and s.t $\lim_{t_n \to \infty} f(\mathbf{t}) = 0$. Let us begin by considering $\mathbb{D}_{\mathbf{t}} f(\mathbf{t}) = \sum_{i=1}^n \frac{\partial}{\partial t_i} f(\mathbf{t})$.

Lemma 1. *Let f be as above. Then the LT of $\mathbb{D}_{\mathbf{t}} f(\mathbf{t})$ is*

$$\widehat{\mathbb{D}_{\mathbf{t}} f(\mathbf{t})} = \left(\sum_{i=1}^n s_n\right) \widehat{f}(s_1, \dots s_n) - \lim_{x_1 \to 0^+} \hat{f}(x_1, s_2, ..., s_n). \tag{2.3}$$

Proof. In the following, we use $\check{a}_i$ to indicate that a_i is absent from where it normally should be. For $1 \le i \le n$ we have

(2.4)

$$\int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty}\cdots\int_{t_n=0}^{\infty} e^{-s_1t_1-s_2t_2\cdots-s_nt_n}\frac{\partial f(\mathbf{t})}{\partial t_i}dt_1dt_2\cdots dt_n$$

$$=\int_{t_1=0}^{\infty}\cdots\int_{t_i=0}^{\check{\infty}}\cdots\int_{t_n=0}^{\infty} e^{-s_1t_1\cdots-\check{s_it_i}\cdots-s_nt_n}\left[\int_{t_i=0}^{\infty} e^{-s_it_i}\frac{\partial f(\mathbf{t})}{\partial t_i}dt_i\right]dt_1\cdots \check{dt_i}\cdots dt_n$$

$$=\int_{t_1=0}^{\infty}\cdots\int_{t_i=0}^{\check{\infty}}\cdots\int_{t_n=0}^{\infty} e^{-s_1t_1\cdots-\check{s_it_i}\cdots-s_nt_n}\left[e^{-s_it_i}f(\mathbf{t})\Big|_{(t_1,\dots,t_{i-1},t_{i-1},t_{i+1},\dots,t_n)}^{(t_1,\dots,t_{i-1},t_{i+1},t_{i+1},\dots,t_n)}\right.$$

$$\left.+s_i\int_{t_i=0}^{\infty} e^{-s_it_i}f(t_1,t_2,\dots,t_n)\,dt_i\right]dt_1\cdots\check{dt_i}\cdots dt_n$$

$$=\int_{t_1=0}^{\infty}\cdots\int_{t_i=0}^{\check{\infty}}\cdots\int_{t_n=0}^{\infty} e^{-s_1t_1\cdots-\check{s_it_i}\cdots-s_nt_n}\left[e^{-s_it_{i+1}}f\left(t_1\dots,t_{i-1},\underbrace{t_{i+1}}_{\text{i'th coordinate}},t_{i+1}\dots,t_n\right)\right.$$

$$\left.-e^{-s_it_{i-1}}f\left(t_1\dots,t_{i-1},\underbrace{t_{i-1}}_{\text{i'th coordinate}},t_{i+1}\dots,t_n\right)+s_i\int_{t_i=0}^{\infty} e^{-s_it_i}f(t_1,t_2,\dots,t_n)\,dt_i\right]dt_1\cdots\check{dt_i}\cdots dt_n$$

(2.5)

$$=\int_{t_{i+1}=0}^{\infty} e^{-(s_i+s_{i+1})t_{i+1}}\hat{f}\left(s_1\dots,s_{i-1},\underbrace{t_{i+1}}_{\text{i'th coordinate}},t_{i+1},s_{i+2}\dots,s_n\right)dt_{i+1}$$

(2.6)

$$-\int_{t_{i-1}=0}^{\infty} e^{-(s_i+s_{i-1})t_{i-1}}\hat{f}\left(s_1\dots,t_{i-1},\underbrace{t_{i-1}}_{\text{i'th coordinate}},s_{i+1},s_{i+2}\dots,s_n\right)dt_{i-1}+s_i\hat{f}(s_1,s_2,\dots,s_n)$$

Note that since $\lim_{t_n\to\infty} f(t_1,\dots,t_n)=0$, summing over the variable i the first two terms in the last equation in 2.4 cancel out for every $i\neq 1$. For $i=1$ only the second term in the brackets cancels out and the result follows. □

$\mathbb{D}_{\mathbf{t}}$ is just the directional derivative along the vector $v=(1,\dots,1)$. Let Ψ_x be a PDO on $\mathbb{R}$ with symbol $\psi(k)$. Then $\psi(\sum_{i=1}^n k_i)$ is a symbol of the PDO $\Psi_{\mathbf{x}}$ where we use bold $\mathbf{x}$ subscript to emphasize the fact that $\Psi_{\mathbf{x}}$ is defined on functions(measures) on $\mathbb{R}^n$. One can think of $\Psi_{\mathbf{x}}$ as the directional version of Ψ_x with directional vector $v=(1,\dots,1)$.

Define the RL fractional derivative of index $0<\alpha<1$ of $f(\mathbf{t})$ to be

$$\mathbb{D}_{\mathbf{t}}^{\alpha}f=\left(\sum_{i=1}^n\frac{\partial}{\partial t_i}\right)\int_0^{t_1} f(t_1-r,t_2-r,\dots,t_n-r)\frac{r^{-\alpha}}{\Gamma(1-\alpha)}dr. \tag{2.7}$$

Equation (2.7) can be thought of as a fractional directional derivative.

Lemma 2. *The LT of* $\mathbb{D}^{\alpha} f$ *is* $\left(\sum_{i=1}^{n} s_n\right)^{\alpha} \widehat{f}(s_1, \ldots s_n)$.

Proof. A simple computation shows that

$$\mathscr{L}\left(\int_0^{t_1} f(t_1 - r, t_2 - r, ..., t_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr\right) = \left(\sum_{i=1}^{n} s_i\right)^{\alpha-1} \hat{f}(s_1, \ldots s_n).$$

It follows by Lemma 1 that

$$\widehat{\mathbb{D}^{\alpha} f} = \mathscr{L}\left(\left(\sum_{i=1}^{n} \frac{\partial}{\partial t_i}\right) \int_0^{t_1} f(t_1 - r, t_2 - r, ..., t_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr\right) = \left(\sum_{i=1}^{n} s_n\right)^{\alpha} \widehat{f}(s_1, \ldots s_n).$$

□

2.4. **The Semi-Markov Approach.** Since the process $X_t = A_{E_t}$ is not Markovian, knowing its one dimensional distribution in not enough to construct its finite dimensional distributions. To circumvent this problem Meerschaert and Straka ([9]) constructed the Markov process (X_t, R_t), where $R_t = D_{E_t} - t$ is the time for regeneration of the process X_t. Let $Q_t(x', r'; dx, dr)$ be the transition probability of the process (X_t, R_t) and $0 < t_1 < t_2 < ... < t_n$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
&P\left(X_{t_1} \in dx_1, X_{t_2} \in dx_2, ..., X_{t_n} \in dx_n\right) \\
&= \int_{r_1=0}^{\infty} \int_{r_2=0}^{\infty} \cdots \int_{r_n=0}^{\infty} Q_{t_1}(0, 0; dx_1, dr_1) \\
&\times Q_{t_2 - t_1}(x_1, r_1; dx_2, dr_2) \cdots Q_{t_n - t_{n-1}}\left(x_{t_{n-1}}, r_{t_{n-1}}; dx_n, dr_n\right) \\
&= Q_{t_1}(0, 0; dx_1, dr_1) \circ Q_{t_2 - t_1}(x_1, r_1; dx_2, dr_2) \cdots Q_{t_n - t_{n-1}}\left(x_{t_{n-1}}, r_{t_{n-1}}; dx_n, dr_n\right) \circ .
\end{aligned}$$

Here, $Q_t(x', r'; dx, dr) \circ f(x, r) = \int_{r=0}^{\infty} f(x, r) Q_t(x', r'; dx, dr)$ and $Q_t(x', r'; dx, dr) \circ = \int_{r=0}^{\infty} Q_t(x', r'; dx, dr)$. In [9], the expression for Q_t is given for a large class of processes. Here, however, unless stated otherwise we consider processes of the form $X_t = A_{E_t}$, where A_t is a Lévy process and E_t is the inverse of a strictly increasing subordinator D_t that is independent of A_t. That is,

$$E_t = \inf\{s > 0 : D_s > t\}.$$

More precisely, the characteristic function of A_t and the Laplace transform of D_t are given respectively by

$$E\left(e^{iuA_t}\right) = exp\left[t\left(ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy1_{\{|y|<1\}}\right) K_1(dy)\right)\right] \tag{2.8}$$

$$E\left(e^{-sD_t}\right) = exp\left[t\left(\int_{\mathbb{R}_+} \left(e^{-sy} - 1\right) K_2(dy)\right)\right].$$

Here, $a \geq 0, b \in \mathbb{R}$. K_1 is a Lévy measure while K_2 is a measure whose support is $[0,\infty)$ and satisfies $\int (y \wedge 1) K_2(dy) < \infty$, $K_2(\{0\}) = 0$ and $\int K_2(dy) = \infty$. By (2.8) it can be easily verified that the infinitesimal generator $\mathcal{A}$ of the process (A_t, D_t) is

$$\mathcal{A}(f)(x,t) = b\frac{\partial}{\partial x} f(x,t) + \frac{a}{2}\frac{\partial^2}{\partial x^2} f(x,t) + \int_{\mathbb{R}^2} \left(f(x+y,t+w) - f(x,t) - y\frac{\partial f(x,t)}{\partial x} 1_{\{|(y,w)|<1\}} \right) K(dy,dw), \tag{2.9}$$

where K is again a Lévy measure. In [9], the case where the coefficients b and a as well as the measure K may be dependent on (x,t) is considered. However, when they do not(this is referred to as the homogeneous case), the transition probability Q_t is given by ([9, Equation. 4.4])

$$Q_t(x',r';dx,dr) = 1_{\{0<t<r'\}}\delta_0(dx-x')\delta_{r'-t}(dr) + 1_{\{0\leq r'\leq t\}} Q_{t-r'}(x',0;dx,dr)$$

$$Q_t(x',0;dx,dr) = \int_{y\in\mathbb{R}} \int_{w\in[0,t]} U^{x'}(dy,dw) K(dx-y, dr+t-w), \tag{2.10}$$

where $U^{x'}(dy,dw)$ is the occupation measure of (A_t, D_t), i.e

$$\int f(y,w) U^{x'}(dy,dw) = \mathbb{E}\left(\int_0^\infty f(A_u + x', D_u) du \right).$$

When the processes A_t and D_t are independent, it can be easily verified that

$$U^{x'}(dy,dw) = \int_0^\infty z(dy-x',u) g(dw,u) du, \tag{2.11}$$

where $z(dx,t) = P(A_t \in dx)$ and $g(dx,t) = P(D_t \in dx)$. Moreover, in the case of independence it was shown that ([3, Corollary 2.3])

$$K(dy,dw) = K_1(dy)\delta_0(dw) + \delta_0(dy) K_2(dw).$$

Hence, equations (2.10) translate into

$$\begin{aligned} Q_t(x',r';dx,dr) = {} & 1_{\{0\leq t<r'\}}\delta_0(dx-x')\delta_{r'-t}(dr) \\ & + 1_{\{0\leq r'\leq t\}} \int_{y\in\mathbb{R}} \int_{w\in[0,t-r']} \left(\int_0^\infty z(dy-x',u) g(dw,u) du \right) \\ & \times \left(\delta_0(dr+t-r'-w) K_1(dx-y) + \delta_0(dx-y) K_2(dr+t-r'-w) \right). \end{aligned} \tag{2.12}$$

However, since $\int K_2(dy) = \infty$, we see ([10, Theorem. 27.4]) that $g(dw,t)$ has no atoms. Therefore, (2.12) reduces to

$$\begin{aligned} Q_t(x',r';dx,dr) &= 1_{\{0\le t<r'\}}\delta_0(dx-x')\,\delta_{r'-t}(dr) \\ &+ 1_{\{0\le r'\le t\}} \int\limits_{w\in[0,t-r']} \left(\int_0^\infty z(dx-x',u)\,g(w,u)\,du \right) \\ &\times K_2(dr+t-r'-w)\,dw. \end{aligned} \tag{2.13}$$

3. Fokker-Planck Equations

Throughout this section, we let A_t be a Lévy process such that $E\left(e^{ikA_t}\right) = e^{t\psi(k)}$, its probability density is given by $z(dx,t) = P(A_t \in dx)$. E_t is the inverse of a subordinator D_t such that $E\left(e^{-sD_t}\right) = e^{t\phi(s)}$, its probability density is $h(dx,t) = P(E_t \in dx)$. We denote by Ψ and Φ the pseudo-differential operators of the symbols $\psi(-k)$ and $-\phi(s)$ respectively. We also denote the transition probability function of the Markov process (X_t,R_t) by Q_t and that of (E_t,R_t) by H_t. Next note that the occupation measure of (t,E_t) is just $U^{x'}(dx,dw) = g(dw,x-x')\,dx$, and similarly to (2.13) we have

$$\begin{aligned} H_t(x',r';dx,dr) &= 1_{\{0\le t<r'\}}\delta_0(dx-x')\,\delta_{r'-t}(dr) \\ &+ 1_{\{0\le r'\le t\}} \int\limits_{w\in[0,t-r']} g(dw,x-x')\,dx \times K_2(dr+t-r'-w)\,dr. \end{aligned}$$

Theorem 1. *Suppose the measure K_2 has a continuous density k_2. Let $h(dx_1,...,dx_n;t_1,...,t_n)$ be the finite dimensional distribution of E_t where $t_1<t_2<...<t_n$, i.e*

$$h(dx_1,...,dx_n;t_1,...,t_n) = P\left(E_{t_1}\in dx_1,...,E_{t_n}\in dx_n\right).$$

Then

$$\Phi_{\mathbf{t}} h(dx_1,...,dx_n;t_1,...,t_n) = -\mathbb{D}_{\mathbf{x}} h(dx_1,...,dx_n;t_1,...,t_n). \tag{3.1}$$

Proof. Let us take LT with respect to the spatial variables and with respect to the time variables, this will be abbreviated by LLT. Before taking the LLT of $h(dx_1,...,dx_n;t_1,...,t_n)$ we note that since $H_t(x',r';dx,dr)$ is translation invariant with respect to the spatial variable we have

$$\begin{aligned} & h(dx_1,...,dx_n;t_1,...,t_n) \\ &= H_{t_1}(0,0;x_1,dr_1)\circ H_{t_2-t_1}(0,r_1;x_2-x_1,dr_2)\cdots H_{t_n-t_{n-1}}(0,r_{n-1};x_n-x_{n-1},dr_n)\circ. \end{aligned} \tag{3.2}$$

Taking the LLT of (3.2), by a simple change of variables we see that

$$\begin{aligned} & \overline{h}(\lambda_1,\dots\lambda_n;s_1,\dots s_n) \\ &= \int\limits_{t_1=0}^{\infty}\int\limits_{x_1=0}^{\infty} e^{-\left(\sum_{i=1}^n s_i\right)t_1-\left(\sum_{i=1}^n \lambda_i\right)x_1} H_{t_1}(0,0;dx_1,dr_1)\circ dt_1 \\ & \cdots H_{s_n+s_{n-1}}(0,r_1;k_n+k_{n-1},dr_{n-1})\circ H_{s_n}(0,r_{n-1};k_n,dr_n)\circ. \end{aligned} \tag{3.3}$$

Now, let us look at

$$\begin{aligned}
&\int_{t_1=0}^{\infty}\int_{x_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1-\left(\sum_{i=1}^{n}\lambda_i\right)x_1} H_{t_1}\left(0,0;dx_1,dr_1\right)dt_1 \\
&= \int_{t_1=0}^{\infty}\int_{x_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1-\left(\sum_{i=1}^{n}\lambda_i\right)x_1} \int_{w\in[0,t_1]} g\left(dw,x_1\right)dx_1 \\
&\times k_2\left(r_1+t_1-w\right)dr_1 \\
&= \int_{x_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n}\lambda_i\right)x_1} \int_{w\in[0,\infty]} g\left(dw,x_1\right)dx_1 \\
&\times \int_{t_1=w}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1} k_2\left(r_1+t_1-w\right)dr_1 \\
&= \int_{x_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n}\lambda_i\right)x_1} \int_{w\in[0,\infty]} g\left(dw,x_1\right)dx_1 e^{-\left(\sum_{i=1}^{n} s_i\right)w} \\
&\times \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1} k_2\left(r_1+t_1\right)dr_1 \\
&= \frac{1}{\sum_{i=1}^{n}\lambda_i-\phi\left(\sum_{i=1}^{n} s_i\right)} \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1} k_2\left(r_1+t_1\right)dr_1. \qquad (3.4)
\end{aligned}$$

Next note that,

(3.5)

$$\begin{aligned}
&\lim_{x_1\to 0^+}\overline{h}\left(x_1,\lambda_2,\ldots\lambda_n;s_1,\ldots s_n\right) \\
&= \lim_{x_1\to 0^+}\int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1-\left(\sum_{i=2}^{n}\lambda_i\right)x_1} \int_{w\in[0,t_1]} g\left(dw,x_1\right)\times\int_{r_1=0}^{\infty} k_2\left(r_1+t_1-w\right)dr_1 \\
&\times H_{\sum_{i=2}^{n} s_i}\left(0,r_1;\sum_{i=2}^{n}\lambda_i,dr_2\right)\circ\cdots H_{s_n+s_{n-1}}\left(0,r_{n-2};\lambda_n+\lambda_{n-1},dr_{n-1}\right)\circ H_{s_n}\left(0,r_{n-1};\lambda_n,dr_n\right)\circ . \\
&= \int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^{n} s_i\right)t_1}\int_{r_1=0}^{\infty} k_2\left(r_1+t_1\right)dr_1 \\
&\times H_{\sum_{i=2}^{n} s_i}\left(0,r_1;\sum_{i=2}^{n}\lambda_i,dr_2\right)\cdots H_{s_n+s_{n-1}}\left(0,r_{n-2};\lambda_n+\lambda_{n-1},dr_2\right)\circ H_{s_n}\left(0,r_{n-1};\lambda_n,dr_n\right)\circ,
\end{aligned}$$

since $g(dw, x_1)$ converges weakly to $\delta_0(dw)$ as $x_1 \to 0^+$ and k_2 is continuous. Finally, plugging (3.4) in (3.3), using (3.5) and rearranging terms we arrive at

(3.6)
$$-\phi\left(\sum_{i=1}^{n} s_i\right)\overline{h}\left(\lambda_1,\dots\lambda_n;s_1,\dots s_n\right)=-\left(\sum_{i=1}^{n}\lambda_i\right)\overline{h}\left(\lambda_1,\dots\lambda_n;s_1,\dots s_n\right)+\overline{h}\left(0^+,\lambda_2,\dots\lambda_n;s_1,\dots s_n\right).$$

Taking the inverse LLT of (3.6) and using Lemma 1 we obtain (3.1). □

Theorem 1 paves the way for the finite dimensional FFPEs of the process X_t.

Corollary 1. *Let $p(dx_1,...,dx_n;t_1,...,t_n) = P(X_{t_1}\in dx_1,\dots,X_{t_n}\in dx_n)$. Then*

$$\Phi_{\mathbf{t}}p\left(dx_1,...,dx_n;t_1,...,t_n\right)=\Psi_{\mathbf{x}}p\left(dx_1,...,dx_n;t_1,...,t_n\right)$$
$$+\int_{u_2=0}^{\infty}\cdots\int_{u_n=u_n-1}^{\infty}\delta_0\left(dx_1\right)z\left(dx_2,...,dx_n;u_2,...,u_n\right)h\left(0^+,du_2,\dots,du_n;t_1,\dots,t_n\right) \tag{3.7}$$

Proof. By the independence of A_t and D_t

(3.8)
$$\begin{aligned}
&p\left(dx_1,...,dx_n;t_1,...,t_n\right)\\
&=\int_{u_1=0}^{\infty}\int_{u_2=u_1}^{\infty}\cdots\int_{u_n=u_{n-1}}^{\infty}z\left(dx_1,...,dx_n;u_1,...,u_n\right)h\left(du_1,\dots,du_n;t_1,\dots,t_n\right)\\
&=\int_{u_1=0}^{\infty}\cdots\int_{u_n=u_{n-1}}^{\infty}z\left(dx_1,u_1\right)z\left(dx_2-x_1,u_2-u_1\right)\cdots z\left(dx_n-x_{n-1},u_n-u_{n-1}\right)h\left(du_1,\dots,du_n;t_1,\dots,t_n\right).
\end{aligned}$$

Taking the FLT of $p(dx_1,...,dx_n;t_1,...,t_n)$ and applying the change of variables $u_2' = u_2 - u_1$ we obtain

$$\overline{p}\left(k_1,...,k_n;s_1,...,s_n\right)$$
$$=\int_{u_1=0}^{\infty}\int_{t_1=0}^{\infty}\int_{x_1\in\mathbb{R}}e^{-\left(\sum_{i=1}^{n}s_i\right)t_1-\left(i\sum_{i=1}^{n}k_i\right)x_1}z\left(dx_1,u_1\right)H_{t_1}\left(0,0;du_1,dr_1\right)\circ dt_1dx_1$$

(3.9)
$$\times\int_{u_2=0}^{\infty}\cdots\int_{u_n=u_{n-1}}^{\infty}\widetilde{z}\left(\sum_{i=2}^{n}k_i,u_2\right)\cdots\widetilde{z}\left(k_n,u_n-u_{n-1}\right)H_{\sum_{i=2}^{n}s_i}\left(0,r_1;du_2-u_1,dr_2\right)\circ\cdots H_{s_n}\left(0,r_{n-1};du_n,dr_n\right)\circ$$

Let us look at

$$\begin{aligned}
&\int_{u_1=0}^{\infty}\int_{t_1=0}^{\infty}\int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1-\left(i\sum_{i=1}^n k_i\right)x_1} z\left(dx_1,u_1\right) H_{t_1}\left(0,0;du_1,dr_1\right)dt_1\\
&=\int_{u_1=0}^{\infty}\int_{t_1=0}^{\infty}\int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1-\left(i\sum_{i=1}^n k_i\right)x_1} z\left(dx_1,u_1\right)\\
&\times\int_{w\in[0,t_1]} g\left(dw,u_1\right)du_1 k_2\left(r_1+t_1-w\right)dr_1dt_1\\
&=\int_{u_1=0}^{\infty}\int_{w=0}^{\infty}\int_{x_1\in\mathbb{R}} e^{-i\left(\sum_{i=1}^n k_i\right)x_1-\left(\sum_{i=1}^n s_i\right)w} z\left(dx_1,u_1\right)g\left(dw,u_1\right)du_1\\
&\times\int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^n s_i\right)t_1} k_2\left(r_1+t_1\right)dr_1dt_1
\end{aligned}$$

$$=\frac{1}{-\psi\left(-\sum_{i=1}^n k_i\right)-\phi\left(\sum_{i=1}^n s_i\right)}\int_{t_1=0}^{\infty} e^{-\left(\sum_{i=1}^n s_i\right)t_1}k_2\left(r_1+t_1\right)dr_1. \tag{3.10}$$

Plugging (3.10) in (3.9) and using (3.5) we have

$$\begin{aligned}
\overline{p}\left(k_1,...,k_n;s_1,...,s_n\right)&=\frac{1}{-\psi\left(-\sum_{i=1}^n k_i\right)-\phi\left(\sum_{i=1}^n s_i\right)}\\
&\times\int_{u_2=0}^{\infty}\cdots\int_{u_n=u_{n-1}}^{\infty}\widetilde{z}\left(\sum_{i=2}^n k_i,u_2\right)\cdots\widetilde{z}\left(k_n,u_n-u_{n-1}\right)\hat{h}\left(0^+,du_2,\ldots,du_n;s_1,\ldots,s_n\right).
\end{aligned}$$

Rearranging and taking the inverse FLT we arrive at (3.7). □

Working along similar lines to the proof of Theorem 1 one can also obtain the finite dimensional FFPEs of the process $X_t=A_{E_t}$ where E_t is the inverse of a strictly increasing subordinator D_t and (A_t,D_t) is a Lévy process, i.e. the processes A_t and D_t are not necessarily independent. More precisely, suppose $E\left(e^{ikA_t-sD_t}\right)=e^{t\xi(k,s)}$ and that $\xi\left(k,s\right)=ibk-\frac{1}{2}ak^2+\int_{\mathbb{R}}\left(e^{iky-sw}-1-iky1_{\{|(y,w)|<1\}}\right)K\left(dy,dw\right)$ and that Ξ is the operator whose symbol is $-\xi\left(-k,s\right)$.

Corollary 2. *Let (A_t,D_t) be a Lévy process s.t $E\left(e^{ikA_t-sD_t}\right)=e^{t\xi(k,s)}$. Let E_t be the inverse of the strictly increasing subordinator D_t and let $p\left(dx_1,...,dx_n;t_1,...,t_n\right)=P\left(X_{t_1}\in dx_1,\ldots,X_{t_n}\in dx_n\right)$. Then*

(3.11)

$$\Xi_{\mathbf{x},\mathbf{t}}p\left(dx_1,...,dx_n;t_1,...,t_n\right)=\int_{r_1=0}^{\infty}K\left(dx_1,dr_1+t_1\right)$$

$$\times Q_{t_2-t_1}\left(x_1,r_1;dx_2,dr_2\right)\circ\cdots Q_{t_n-t_{n-1}}\left(x_{n-1},r_{n-1};dx_n,dr_n\right)\circ. \tag{3.12}$$

Proof. Using (2.10) we see that Q_t is again translation invariant with respect to the spatial variable. Note that here

$$U^{x'}(dy, dw) = \int_0^\infty v(dy, dw; u)\, du,$$

where $v(dy, dw; u) = P(A_u \in dy, D_u \in dw)$. Using the same ideas as in the proof of Theorem 1 we obtain

$$\begin{aligned}
\overline{p}(k_1, ..., k_n; s_1, ..., s_n) &= \int_{t_1=0}^\infty \int_{x_1 \in \mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1 - \left(\sum_{i=1}^n k_i\right)x_1} \int_{u=0}^\infty v(dy, dw; u)\, du \int_{r_1=0}^\infty \int_{y\in\mathbb{R}} \int_{w=0}^{t_1} K(dx_1 - y, dr_1 + t_1 - w) \\
&\times Q_{\sum_{i=2}^n s_i}\left(0, dr_1; \sum_{i=2}^n k_i, dr_2\right) \circ \cdots Q_{s_n}(0, r_{n-1}; k_n, dr_n) \circ \\
&= \int_{y\in\mathbb{R}} \int_{w=0}^\infty e^{-\left(\sum_{i=1}^n s_i\right)w - \left(\sum_{i=1}^n k_i\right)y} \int_{u=0}^\infty v(dy, dw; u)\, du \int_{r_1=0}^\infty \int_{t_1=0}^\infty \int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1 - \left(\sum_{i=1}^n k_i\right)x_1} \\
&\times K(dx_1, dr_1 + t_1)\, Q_{\sum_{i=2}^n s_i}\left(0, dr_1; \sum_{i=2}^n k_i, dr_2\right) \circ \cdots Q_{s_n}(0, r_{n-1}; k_n, dr_n) \circ \\
&= \frac{1}{-\xi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right)} \int_{r_1=0}^\infty \int_{t_1=0}^\infty \int_{x_1\in\mathbb{R}} e^{-\left(\sum_{i=1}^n s_i\right)t_1 - \left(\sum_{i=1}^n k_i\right)x_1} K(dx_1, dr_1 + t_1) \\
&\times Q_{\sum_{i=2}^n s_i}\left(0, dr_1; \sum_{i=2}^n k_i, dr_2\right) \circ \cdots Q_{s_n}(0, r_{n-1}; k_n, dr_n) \circ .
\end{aligned}$$

Rearrange and invert to obtain (3.11). □

4. Examples

Theorem 1 as well as Corollary 1 and Corollary 2 should be compared with their one-dimensional counterparts to gain a better understanding of the dynamics of the processes whose distributions govern those equations. We start with a specific case of the one dimensional analogue of Theorem 1.

Example 1. Let D_t be a standard stable subordinator of index $0 < \alpha < 1$, i.e. $E\left(e^{-sD_t}\right) = e^{t(-s^\alpha)}$. Its inverse E_t has a distribution $h(x,t)$ which satisfies ([8, Equation 5.5])

$$\mathbb{D}_t^\alpha h(x,t) = -\mathbb{D}_x h(x,t),$$

on $x, t > 0$. Since here $\phi(s) = -s^\alpha$, we see that $\Phi_t = \mathbb{D}_t^\alpha$.

Next we look at the one dimensional analogue of Corollary 1.

Example 2. Again we let D_t be a standard stable subordinator of index $0 < \alpha < 1$, and A_t be a Lévy process s.t $E\left(e^{ikA_t}\right) = e^{t\psi(k)}$. Then the distribution $p(dx,t)$ of A_{E_t} satisfies ([8, Equation 5.6])

$$\mathbb{D}_t^\alpha p(dx,t) = \Psi_x p(dx,t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\delta_0(dx). \tag{4.1}$$

To see why (3.7) can be thought of as a generalization of (4.1) note that $h\left(0^{+},t\right)=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ ([8, Equation 4.3]) and rewrite (4.1) as

$$\mathbb{D}_{t}^{\alpha}p\left(x,t\right)=\Psi_{x}p\left(x,t\right)+\delta_{0}\left(dx\right)h\left(0^{+},t\right).$$

Our last example concerns the one dimensional analogue of Corollary 2.

Example 3. Let (A_t, D_t) be a Lévy process as in Corollary 2. Then its one dimensional distribution $p\left(dx,t\right)$ satisfies

$$\Xi_{x,t}p\left(dx,t\right)=\int\limits_{r=0}^{\infty}K\left(dx,dr+t\right). \tag{4.2}$$

This was shown in [5, Theorem 4.1].

Remark 1. In [1, Equation 5.9] Buale and Friedrich essentially obtained Equation 3.1 for the case where D_t is a standard stable subordinator. In [2, Equation 14] Bauale and Friedrich obtained Equation 3.7(uncoupled case) for the two dimensional case.

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References

[1] Adrian Baule and Rudolf Friedrich. Joint probability distributions for a class of non-Markovian processes. *Physical Review E*, 71(2):026101, 2005.
[2] Adrian Baule and Rudolf Friedrich. A fractional diffusion equation for two-point probability distributions of a continuous-time random walk. *EPL (Europhysics Letters)*, 77(1):10002, 2007.
[3] Peter Becker-Kern, Mark M Meerschaert, and Hans-Peter Scheffler. Limit theorems for coupled continuous time random walks. *Annals of Probability*, pages 730–756, 2004.
[4] Ofer Busani. Aging of uncoupled continuous time random walk limits. *arXiv preprint arXiv:1402.3965*, 2015.
[5] Agnieszka Jurlewicz, P Kern, Mark M Meerschaert, and H-P Scheffler. Fractional governing equations for coupled random walks. *Computers & Mathematics with Applications*, 64(10):3021–3036, 2012.
[6] Mark M Meerschaert, Hans-Peter Scheffler, et al. Limit theorems for continuous-time random walks with infinite mean waiting times. *Journal of applied probability*, 41(3):623–638, 2004.
[7] Mark M Meerschaert and Alla Sikorskii. *Stochastic models for fractional calculus*, volume 43. Walter de Gruyter, 2011.
[8] Mark M Meerschaert and Peter Straka. Inverse stable subordinators. *Mathematical modelling of natural phenomena*, 8(02):1–16, 2013.
[9] Mark M Meerschaert, Peter Straka, et al. Semi-Markov approach to continuous time random walk limit processes. *The Annals of Probability*, 42(4):1699–1723, 2014.
[10] Ken-Iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68. Cambridge university press, 1999.
[11] Peter Straka. *Continuous Time Random Walk Limit Processes - Stochastic Models for Anomalous Diffusion*. Phd dissertation, University of New South Wales, 2011.
[12] Peter Straka and Bruce Ian Henry. Lagging and leading coupled continuous time random walks, renewal times and their joint limits. *Stochastic Processes and their Applications*, 121(2):324–336, 2011.