Numerical computations of Maass cusp forms

Andrei Seymour-Howell

University of Bristol

May 4, 2021

Maass cusp forms

Let $\mathbb{H} = \{z = x + iy \mid y > 0\}$ denote the upper half-plane. We define the group $\Gamma = \mathrm{SL}(2, \mathbb{Z}) = \{\gamma \in \mathrm{GL}(2, \mathbb{Z}) | \det(\gamma) = 1\}$. This group acts on $\mathbb{H}$ by linear fractional transformations, i.e

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d} \qquad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \mathbb{H}.$$

Maass cusp forms

Let $\mathbb{H} = \{z = x + iy \mid y > 0\}$ denote the upper half-plane. We define the group $\Gamma = \mathrm{SL}(2, \mathbb{Z}) = \{\gamma \in \mathrm{GL}(2, \mathbb{Z}) | \det(\gamma) = 1\}$. This group acts on $\mathbb{H}$ by linear fractional transformations, i.e

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d} \qquad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \mathbb{H}.$$

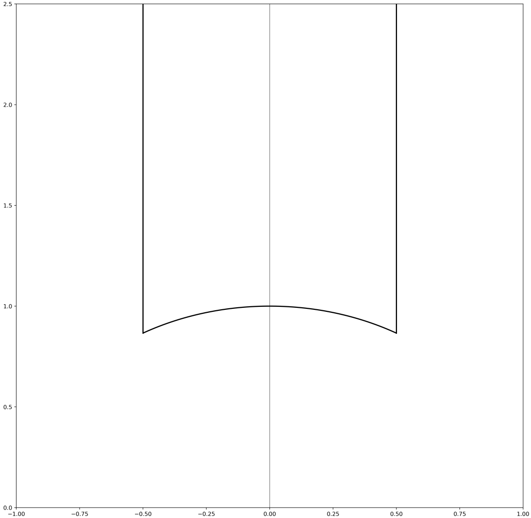
$\mathrm{SL}(2, \mathbb{Z})$ is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

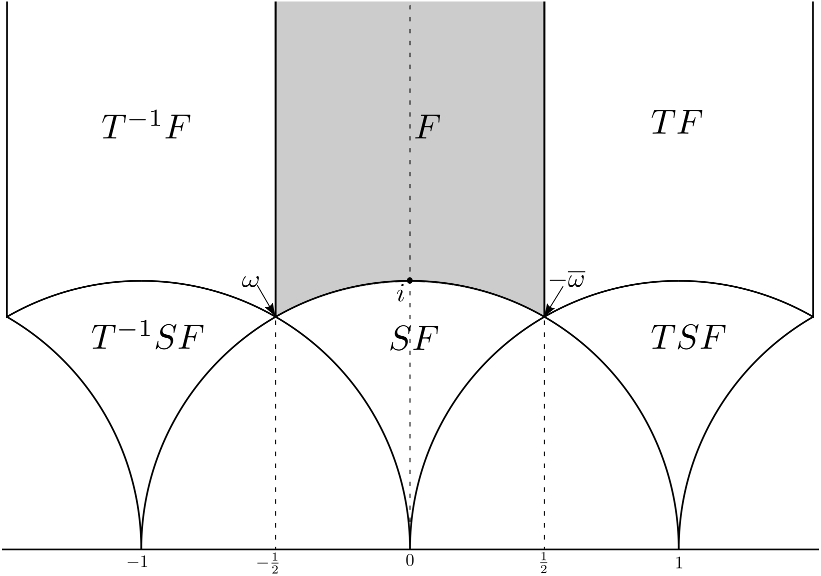
This gives the following fundamental domain for this action

$$F = \left\{ z \in \mathbb{H} : |z| \geq 1 \text{ and } |\mathrm{Re}(z)| \leq \frac{1}{2} \right\}.$$

Fundamental domain



Fundamental domain



Maass cusp forms

The modular surface $X = \Gamma \backslash \mathbb{H}$ is a finite volume non-compact surface with Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

We also have the measure

$$\frac{dx\, dy}{y^2}.$$

Maass cusp forms

We call a function $f : \mathbb{H} \to \mathbb{C}$ a **Maass cusp form** on Γ if

1. f is an eigenfunction of the Laplacian, $\Delta f = \lambda f, \lambda \geq 0$,
2. f is automorphic, $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
3. $f \in L^2(X)$, i.e f is square-integrable,
4. f vanishes at all of the cusps of X.

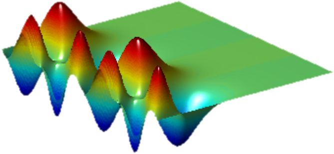
Maass cusp forms

We call a function $f : \mathbb{H} \to \mathbb{C}$ a **Maass cusp form** on Γ if

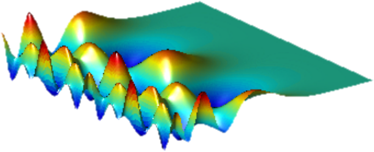
1. f is an eigenfunction of the Laplacian, $\Delta f = \lambda f, \lambda \geq 0$,
2. f is automorphic, $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
3. $f \in L^2(X)$, i.e f is square-integrable,
4. f vanishes at all of the cusps of X.

We will denote the space of Maass cusp forms on Γ with Laplace eigenvalue λ by $\mathcal{S}_\lambda$.
The set of functions that just satisfy points (2), (3) and (4) we shall denote as $L^2_{\text{cusp}}(X)$.

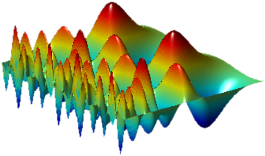
Pictures of Maass forms



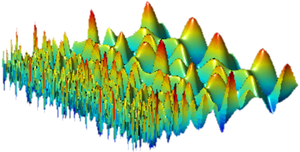
(a) $\lambda = 91.141345\ldots$



(b) $\lambda = 190.131547\ldots$



(c) $\lambda = 404.529171\ldots$



(d) $\lambda = 2468.702167\ldots$

Figure: Images of Maass forms from the LMFDB.

Hecke operators

For any $f \in \mathcal{S}_\lambda$ and any non-zero integer n, we define the **Hecke operator** T_n by

$$T_n f(z) = \frac{1}{\sqrt{|n|}} \sum_{\substack{ad=n \\ d>0}} \sum_{j=0}^{d-1} \begin{cases} f\left(\dfrac{az+j}{d}\right) & \text{if } n > 0, \\ f\left(\dfrac{a\bar{z}+j}{d}\right) & \text{if } n < 0. \end{cases}$$

This will map $\mathcal{S}_\lambda \to \mathcal{S}_\lambda$.

Hecke operators

For any $f \in \mathcal{S}_\lambda$ and any non-zero integer n, we define the **Hecke operator** T_n by

$$T_n f(z) = \frac{1}{\sqrt{|n|}} \sum_{\substack{ad=n \\ d>0}} \sum_{j=0}^{d-1} \begin{cases} f\left(\dfrac{az+j}{d}\right) & \text{if } n > 0, \\ f\left(\dfrac{a\bar{z}+j}{d}\right) & \text{if } n < 0. \end{cases}$$

This will map $\mathcal{S}_\lambda \to \mathcal{S}_\lambda$.
Now a famous result then tells us that there exists an orthogonal basis $\{f_j\}$ in $L^2_{\text{cusp}}(X)$ consisting of eigenfunctions to all Hecke operators T_n.

Hecke eigenvalues

A Maass cusp form f with Laplace eigenvalue $\lambda = \frac{1}{4} + R^2$ has a Fourier expansion (at ∞) of the form

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{iR}(2\pi |n| y) e(nx)$$

where $e(nx) = \exp(2\pi i n x)$ and $K_\nu(u)$ is the K-Bessel function.

Hecke eigenvalues

A Maass cusp form f with Laplace eigenvalue $\lambda = \frac{1}{4} + R^2$ has a Fourier expansion (at ∞) of the form

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{iR}(2\pi |n| y) e(nx)$$

where $e(nx) = \exp(2\pi i n x)$ and $K_\nu(u)$ is the K-Bessel function. We call a Maass form f **even** if $a(-n) = a(n)$ or **odd** if $a(-n) = -a(n)$.

Hecke eigenvalues

A Maass cusp form f with Laplace eigenvalue $\lambda = \frac{1}{4} + R^2$ has a Fourier expansion (at ∞) of the form

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{iR}(2\pi |n| y) e(nx)$$

where $e(nx) = \exp(2\pi i n x)$ and $K_\nu(u)$ is the K-Bessel function.
We call a Maass form f **even** if $a(-n) = a(n)$ or **odd** if $a(-n) = -a(n)$.
If f is also a Hecke eigenfunction for all Hecke operators T_n, i.e $T_n f = \lambda(n) f$, then we can normalise such that $a(1) = 1$ and we have

$$a(n) = \lambda(n)$$
$$a(-n) = \varepsilon \lambda(n)$$

where ε is 1 is f is even and -1 if f is odd.

Hejhal's Algorithm

There are a few methods known for computing Maass forms, the most widely used is an algorithm due to Hejhal from the 1990's. The algorithm goes in the following steps

Hejhal's Algorithm

There are a few methods known for computing Maass forms, the most widely used is an algorithm due to Hejhal from the 1990's. The algorithm goes in the following steps

1. Truncate the Fourier series and treat it like a discrete Fourier series.

Hejhal's Algorithm

There are a few methods known for computing Maass forms, the most widely used is an algorithm due to Hejhal from the 1990's. The algorithm goes in the following steps

1. Truncate the Fourier series and treat it like a discrete Fourier series.
2. Do an inverse Fourier transform along a certain horocycle of points away from the fundamental domain.

Hejhal's Algorithm

There are a few methods known for computing Maass forms, the most widely used is an algorithm due to Hejhal from the 1990's. The algorithm goes in the following steps

1. Truncate the Fourier series and treat it like a discrete Fourier series.
2. Do an inverse Fourier transform along a certain horocycle of points away from the fundamental domain.
3. This will give an expression for the Fourier coefficients, however to make it a non-tautology, we use the automorphy of the Maass form to produce a linear system for the Fourier coefficients.

Hejhal's Algorithm

There are a few methods known for computing Maass forms, the most widely used is an algorithm due to Hejhal from the 1990's. The algorithm goes in the following steps

1. Truncate the Fourier series and treat it like a discrete Fourier series.
2. Do an inverse Fourier transform along a certain horocycle of points away from the fundamental domain.
3. This will give an expression for the Fourier coefficients, however to make it a non-tautology, we use the automorphy of the Maass form to produce a linear system for the Fourier coefficients.
4. We then use a non-linear search strategy to zoom in on an Laplace eigenvalue.

Hejhal's algorithm

Let f be a Maass cusp form with Laplace eigenvalue $\lambda = 1/4 + R^2$. To begin we truncate the Fourier series

$$f(z) = f(x+iy) = \sum_{0<|n|\leq M} a(n)\sqrt{y}K_{iR}(2\pi|n|y)e(nx) + [[\varepsilon]].$$

(We use $[[\varepsilon]]$ to denote a quantity with absolute value less than ε.)

Hejhal's algorithm

Let f be a Maass cusp form with Laplace eigenvalue $\lambda = 1/4 + R^2$. To begin we truncate the Fourier series

$$f(z) = f(x+iy) = \sum_{0<|n|\leq M} a(n)\sqrt{y}K_{iR}(2\pi|n|y)e(nx) + [[\varepsilon]].$$

(We use $[[\varepsilon]]$ to denote a quantity with absolute value less than ε.)

We can now view this a discrete Fourier transform in x. Thus we can do an inverse transform over some points so that we can get an expression for $a(n)$. However, we will need to be careful choosing these points so that we can achieve a non-trivial linear system.

Hejhal's algorithm

Let $Y < Y_0 = \frac{\sqrt{3}}{2}$ and $Q > M$. We will perform an inverse transform over the following set of sampling points along a horocycle:

$$\left\{ z_m = x_m + iY \,\middle|\, x_m = \frac{1}{2Q}\left(m - \frac{1}{2}\right), 1 - Q \leq m \leq Q \right\}.$$

Hejhal's algorithm

Let $Y < Y_0 = \frac{\sqrt{3}}{2}$ and $Q > M$. We will perform an inverse transform over the following set of sampling points along a horocycle:

$$\left\{ z_m = x_m + iY \,\middle|\, x_m = \frac{1}{2Q}\left(m - \frac{1}{2}\right), 1 - Q \le m \le Q \right\}.$$

This gives us

$$a(n)\sqrt{Y}K_{iR}(2\pi|n|Y) = \frac{1}{2Q}\sum_{m=1-Q}^{Q} f(z_m)e(-nx_m) + [[\varepsilon]].$$

Hejhal's algorithm

Let $Y < Y_0 = \frac{\sqrt{3}}{2}$ and $Q > M$. We will perform an inverse transform over the following set of sampling points along a horocycle:

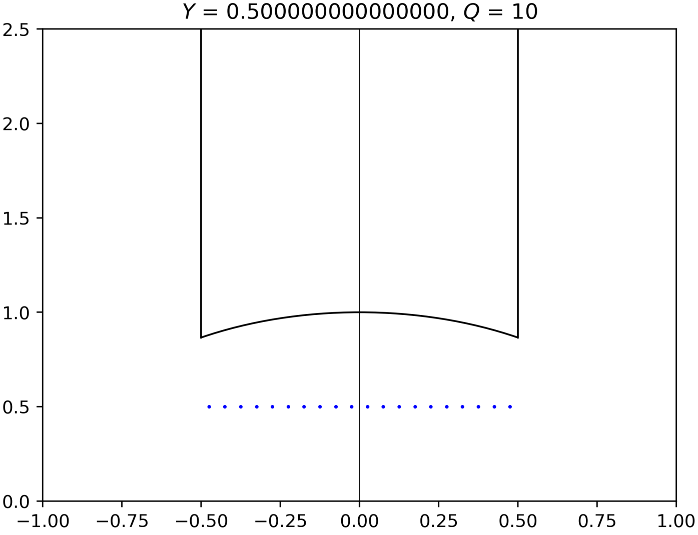
$$\left\{ z_m = x_m + iY \,\middle|\, x_m = \frac{1}{2Q}\left(m - \frac{1}{2}\right), 1 - Q \leq m \leq Q \right\}.$$

This gives us

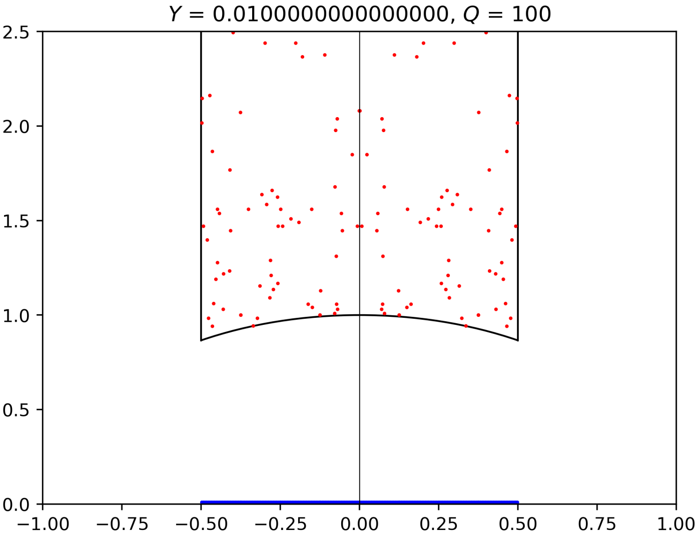
$$a(n)\sqrt{Y}K_{iR}(2\pi|n|Y) = \frac{1}{2Q}\sum_{m=1-Q}^{Q} f(z_m)e(-nx_m) + [[\varepsilon]].$$

Now to make this a non-trivial system we shall pullback the points z_m into the fundamental domain, that is find the matrices $P_m \in \Gamma$ such that $z_m^* = P_m z_m$, where $z_m^* = x_m^* + iy_m^*$ is in the fundamental domain. The automorphy of f gives us the non-trivial system since $f(z_m^*) = f(z_m)$.

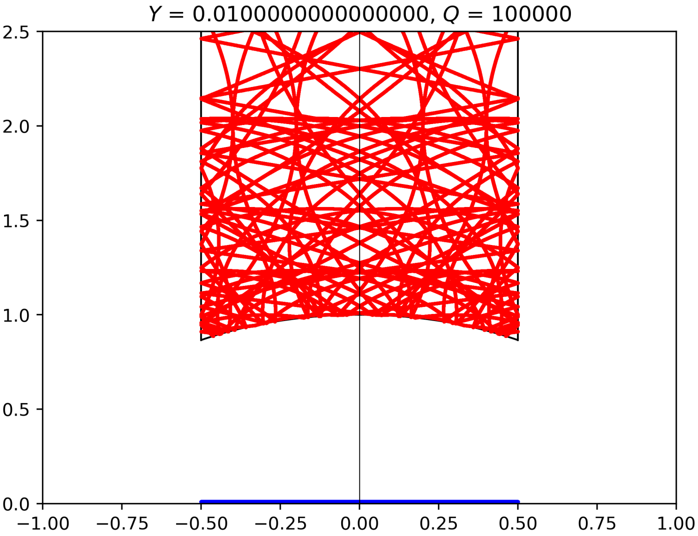
Picture of z_m



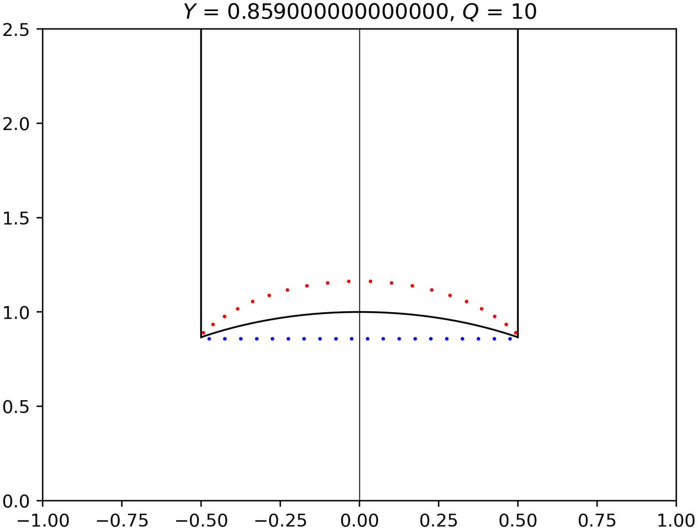
Picture of z_m



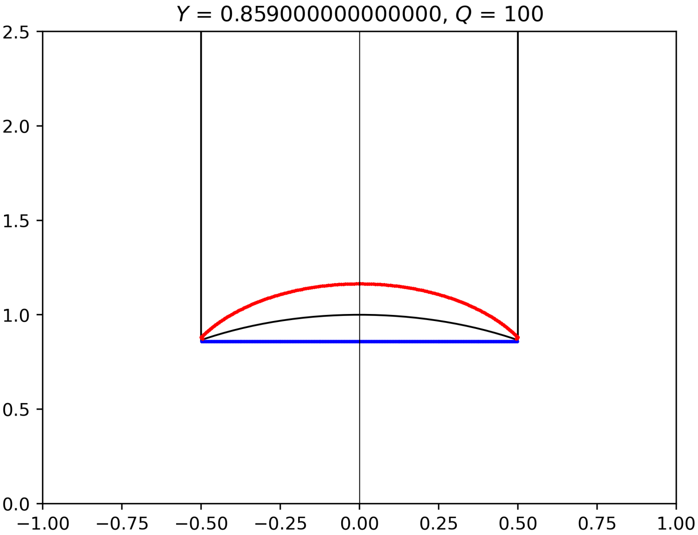
Picture of z_m



Picture of z_m



Picture of z_m



Hejhal's algorithm

This allows us to rewrite the system as

$$a(n)\sqrt{Y}K_{iR}(2\pi|n|Y) = \frac{1}{2Q}\sum_{m=1-Q}^{Q} f(z_m^*)e(-nx_m) + [[\varepsilon]]$$
$$= \sum_{0<|k|\leq M} a(k)V_{nk} + 2[[\varepsilon]]$$

where

$$V_{nk} = \frac{1}{2Q}\sum_{m=1-Q}^{Q} \sqrt{y_m^*}K_{iR}(2\pi|k|y_m^*)e(kx_m^* - nx_m).$$

Due to the non-trivial mixing of the points z_m and z_m^*, we get a non-trivial linear system for the Fourier coefficients.

Hejhal's algorithm

Restricting to $1 \leq |n| \leq M$, we can rewrite the linear system to get

$$0 = \sum_{0<|k|\leq M} a(k)\widetilde{V}_{nk} + 2[[\varepsilon]]$$

where $\widetilde{V}_{nk} = V_{nk} - \delta_{nk}\sqrt{Y}K_{iR}(2\pi|n|Y)$.

Hejhal's algorithm

Restricting to $1 \le |n| \le M$, we can rewrite the linear system to get

$$0 = \sum_{0<|k|\le M} a(k)\widetilde{V}_{nk} + 2[[\varepsilon]]$$

where $\widetilde{V}_{nk} = V_{nk} - \delta_{nk}\sqrt{Y}K_{iR}(2\pi|n|Y)$.
If we let V denote the $(2M \times 2M)$-matrix $\widetilde{V}_{nk}$ and let C denote the $2M$-vector of Fourier coefficients $a(n)$, we can write the linear system as

$$VC = 0.$$

Hejhal's algorithm

Restricting to $1 \leq |n| \leq M$, we can rewrite the linear system to get

$$0 = \sum_{0<|k|\leq M} a(k)\widetilde{V}_{nk} + 2[[\varepsilon]]$$

where $\widetilde{V}_{nk} = V_{nk} - \delta_{nk}\sqrt{Y}K_{iR}(2\pi|n|Y)$.
If we let V denote the $(2M \times 2M)$-matrix $\widetilde{V}_{nk}$ and let C denote the $2M$-vector of Fourier coefficients $a(n)$, we can write the linear system as

$$VC = 0.$$

Note that this solution space of this system for a true eigenvalue R is a 1-dimensional space, so in order to get a unique solution we use the normalisation $a(1) = 1$.

Hejhal's algorithm - Finding R

This algorithm works for any R and will yield a homogeneous linear system $V(R, Y)C = 0$. For a true eigenvalue R, this linear system should be independent of the choice of Y. We see in practice, that if R is far away from a true eigenvalue R, then the resulting coefficients in C change drastically as Y changes.

Hejhal's algorithm - Finding R

This algorithm works for any R and will yield a homogeneous linear system $V(R, Y)C = 0$. For a true eigenvalue R, this linear system should be independent of the choice of Y. We see in practice, that if R is far away from a true eigenvalue R, then the resulting coefficients in C change drastically as Y changes.

Thus to find the eigenvalues in a given range $[R_1, R_2]$, we divide this interval into smaller intervals and test the R values with 2 different Y values and measure the difference of the coefficients. We can then repeatedly do this to zoom into an eigenvalue R. This essentially amounts to minimising some *cost function* $\text{cost}(R)$ that is large when R is far away from a true eigenvalue and small when R is close to a true eigenvalue.

List of eigenvalues

R_1	9.53369526135 ...
R_2	12.1730083247 ...
R_3	13.7797513519 ...
R_4	14.3585095183 ...
R_5	16.1380731715 ...
R_6	16.6442592019 ...
R_7	17.7385633811 ...
R_8	18.1809178345 ...
R_9	19.4234814708 ...
R_{10}	19.4847138547 ...

Table: List of first 10 eigenvalues R on $\mathrm{SL}(2, \mathbb{Z})$. Data from the LMFDB.

Remarks

- This method is heuristic, so we require another method to certify if the data produced is correct.

Remarks

- This method is heuristic, so we require another method to certify if the data produced is correct.
- Despite being heuristic, in practice this method is very stable and can produce results to high precision.

Remarks

- This method is heuristic, so we require another method to certify if the data produced is correct.
- Despite being heuristic, in practice this method is very stable and can produce results to high precision.
- An alternative/additional way to test for a true eigenvalue is to test the multiplicativity of the coefficients, for example measure how well $a(2)a(3) = a(6)$.

Remarks

- This method is heuristic, so we require another method to certify if the data produced is correct.
- Despite being heuristic, in practice this method is very stable and can produce results to high precision.
- An alternative/additional way to test for a true eigenvalue is to test the multiplicativity of the coefficients, for example measure how well $a(2)a(3) = a(6)$.
- This method has been generalised to congruence subgroups (with characters) of $\mathrm{SL}(2, \mathbb{Z})$ by Strömberg in 2006.

Remarks

- This method is heuristic, so we require another method to certify if the data produced is correct.
- Despite being heuristic, in practice this method is very stable and can produce results to high precision.
- An alternative/additional way to test for a true eigenvalue is to test the multiplicativity of the coefficients, for example measure how well $a(2)a(3) = a(6)$.
- This method has been generalised to congruence subgroups (with characters) of $\mathrm{SL}(2, \mathbb{Z})$ by Strömberg in 2006.
- There are nearly 15000 examples of these Maass cusp forms computed and stored on the LMFDB.

Remarks

- This method is heuristic, so we require another method to certify if the data produced is correct.
- Despite being heuristic, in practice this method is very stable and can produce results to high precision.
- An alternative/additional way to test for a true eigenvalue is to test the multiplicativity of the coefficients, for example measure how well $a(2)a(3) = a(6)$.
- This method has been generalised to congruence subgroups (with characters) of $\mathrm{SL}(2, \mathbb{Z})$ by Strömberg in 2006.
- There are nearly 15000 examples of these Maass cusp forms computed and stored on the LMFDB.
- There is also a phase 2 to this algorithm which allows us to compute more Fourier coefficients once we have a good approximation to R and its first few Fourier coefficients.

Verification methods

- In 2006, Booker, Strömbergsson and Venkatesh proved that it is possible to certify whether a candidate Maass form is "close" to a true Maass form. Roughly, suppose you have a computed eigenvalue $\tilde{\lambda} = \frac{1}{4} + \tilde{R}^2$ and the coefficients of a suspected Maass form $\tilde{f}$. Then they showed that if $\tilde{f}$ is "almost automorphic", then $\tilde{f}$ is "close" to a true Maass cusp form f. They only showed this for level 1, i.e $\mathrm{SL}(2, \mathbb{Z})$ and computed and verified the first few Maass cusp forms to a hundred digits.

Verification methods

- In 2006, Booker, Strömbergsson and Venkatesh proved that it is possible to certify whether a candidate Maass form is "close" to a true Maass form. Roughly, suppose you have a computed eigenvalue $\tilde{\lambda} = \frac{1}{4} + \tilde{R}^2$ and the coefficients of a suspected Maass form $\tilde{f}$. Then they showed that if $\tilde{f}$ is "almost automorphic", then $\tilde{f}$ is "close" to a true Maass cusp form f. They only showed this for level 1, i.e SL$(2, \mathbb{Z})$ and computed and verified the first few Maass cusp forms to a hundred digits.
- For congruence subgroups of SL$(2, \mathbb{Z})$ current work is being done to verify the Laplace eigenvalues using that method that relies on an explicit version of the Selberg trace formula.

Thanks for listening!