Numerical Computations of the Riemann Zeta Function

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Some Complex Analysis

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Proposition - Analytic continuation

Let D, D' be domains with $D \subseteq D'$ and let $f : D \to \mathbb{C}$ be holomorphic. Then there exists (under certain conditions) an unique analytic extension $F : D' \to \mathbb{C}$ such that $F = f$ on D. We call F the **analytic continuation** of f to D'.

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Euler also managed to find some explicit values as well, the most famous being

$$\zeta(2) = \frac{\pi^2}{6}.$$

Facts about ζ

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- If $\rho \in S$ is a zero of $\zeta(s)$, then so is $\overline{\rho}, 1-\rho, 1-\overline{\rho} \in S$.

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Hadamard and de la Vallée Poussin in 1896 managed to use Riemann’s strategy to prove the Prime Number Theorem independently.

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I have no idea.

Where did Riemann come up with this and did he have some evidence to support this?
Yes! He computed the first few zeros using, what we now call, the **Riemann-Siegel formula**.

Riemann Siegel formula

Idea: Use functional equation and symmetry around $s = 1/2 + it$ to get an expression for $\zeta(1/2 + it)$. Then rotate this value(i.e multiply by complex exponential) so that it is now real. Then expand formula and use fancy maths to get a finite sum with a small error.

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Definitions

We define the **Riemann Siegel theta function** by

$$\theta(t) \approx \frac{t}{2} \log\left(\frac{t}{2\pi}\right) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \dots .$$

Then define the **Z-function** for $t \in \mathbb{R}$ by

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right).$$

This is a real-valued function. We have that the zeros of $Z(t)$ coincide with the zeros of $\zeta(1/2 + it)$ since $|Z(t)| = |\zeta(1/2 + it)|$.

Riemann Siegel formula

Final result

Let $N = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor$. Then we have

$$Z(t) = 2\sum_{n=1}^{N} n^{-1/2} \cos(t \log(n) - \theta(t)) + R(t)$$

where $R(t)$ is some error that can be improved.

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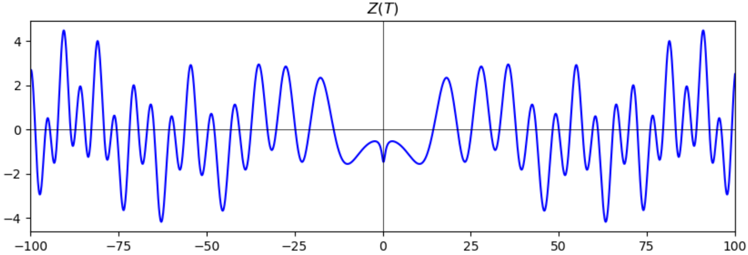

First few zeros

Now $Z(t)$ is just a real valued function and we are just looking for roots of this, so we can just employ your favourite root finding algorithm (I used secant for my computations) in steps along the real line and look for sign changes. In doing so, one can easily compute the first few zeros.

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n	t
1	$14.134725\ldots$
2	$21.022040\ldots$
3	$25.010858\ldots$
4	$30.424876\ldots$
5	$32.935062\ldots$
6	$37.586178\ldots$

A little History - Hand calculations

The first few computations were all done by hand and actually used a different method of computation called **Euler-Maclaurin summation**, which is actually slower than the method that Riemann used. However Riemann's method was not known to the world until Siegel rediscovered them 70 years after Riemann used them!

	Year	Range of t	Number of zeros
Riemann	1859	$t < 26$	3
Gram	1903	$t < 65$	15
Backlund	1914	$t < 200$	79
Hutchinson	1925	$t < 300$	138
Titchmarsh, Comrie	1935-1936	$t < 1468$	1041

A little History - Computers

In April 1949 the Manchester Mark I (one of the early electronic computers) became operational (woo!) and so began the new era of modern computation. Alan Turing, who was a professor at the University of Manchester at the time, used this machine to compute some more zeros.

	Year	Number of zeros
Turing	1950	1104
Lehmer	1956	25,000
Rosser et al.	1968	3,500,000
Brent et al.	1982	200,000,000
1988: Odlyzko-Schönhage algorithm published		
van der Lune	2001	10,000,000,000
Gourdon	2004	10,000,000,000,000

Turing's Method

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Theorem 1 - Von Mangoldt

Let $N(T)$ be the number of zeros of $\zeta(s)$ in the critical strip up to some height $T > 0$. Then

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + error$$

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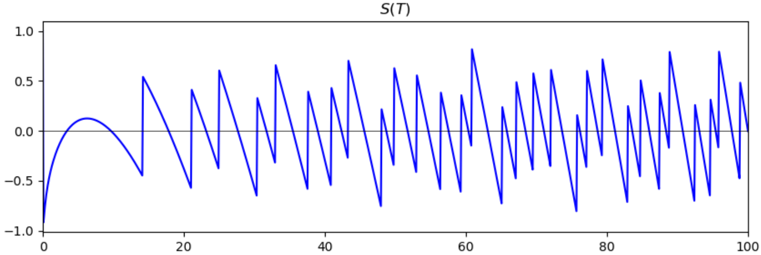

Turing's Method

Theorem 2 - Littlewood/Turing

$S(T)$ is 0 on average as $T \to \infty$ and we have the bound

$$\left| \int_T^{T+h} S(t) dt \right| \leq 2.3 + 0.128 \log \left(\frac{T+h}{2\pi} \right)$$

for $h > 0$ and $T > 168\pi$.

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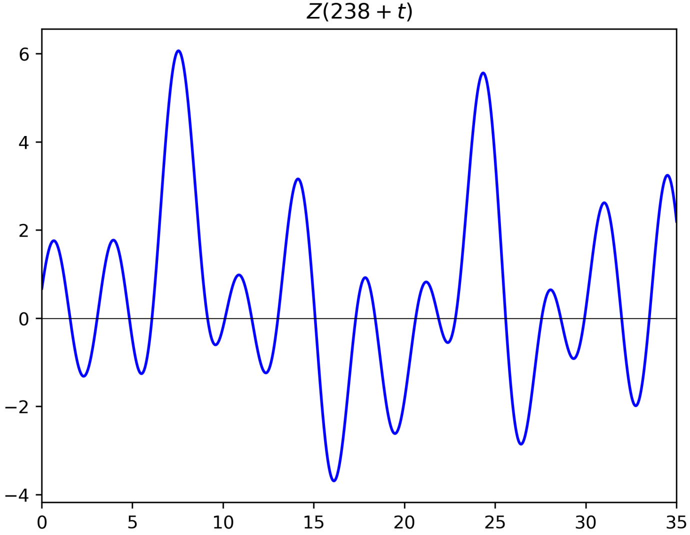
The idea is to compute a bunch of zeros in some interval $(T, T+h)$, then assume we missed a zero. Then $N(T+h) - N(T)$ is just the number of zeros that we computed $+1$. Then using this we compute $S(T)$ in this region via

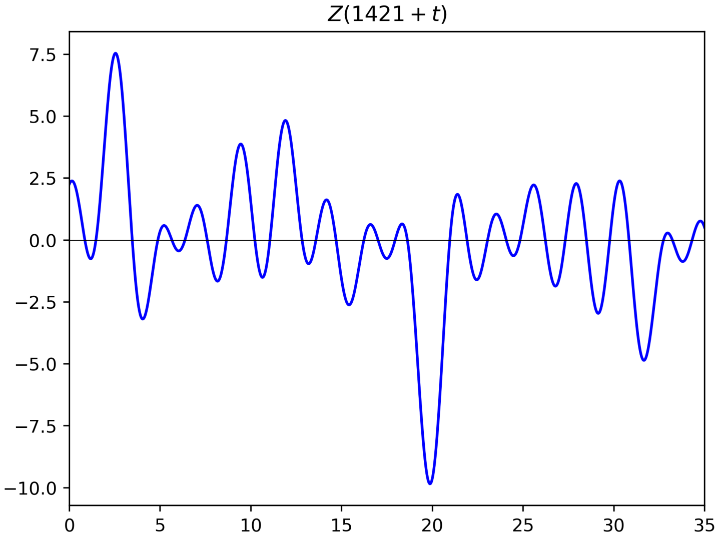
$$S(T) = N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) + \frac{T}{2\pi} - \frac{7}{8}.$$

If we didn't miss a zero then $S(T)$ will be on average 1 since we over-counted by 1, which will eventually contradict the above bound.

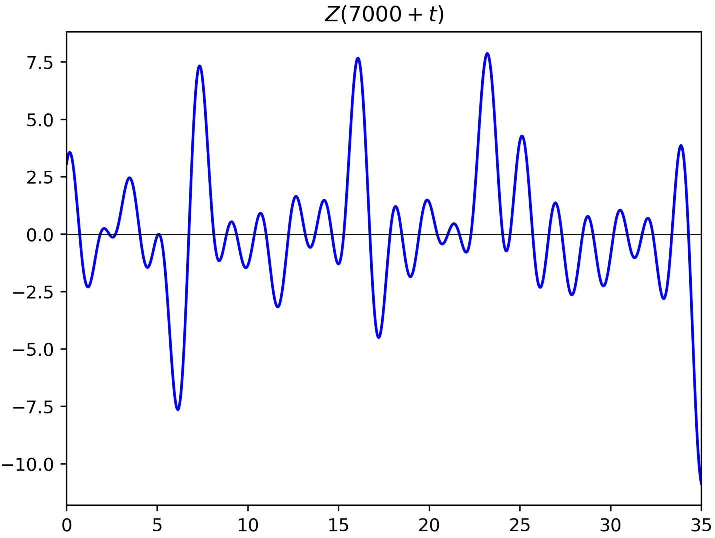
Some pictures of $Z(t)$, $t \approx 238$



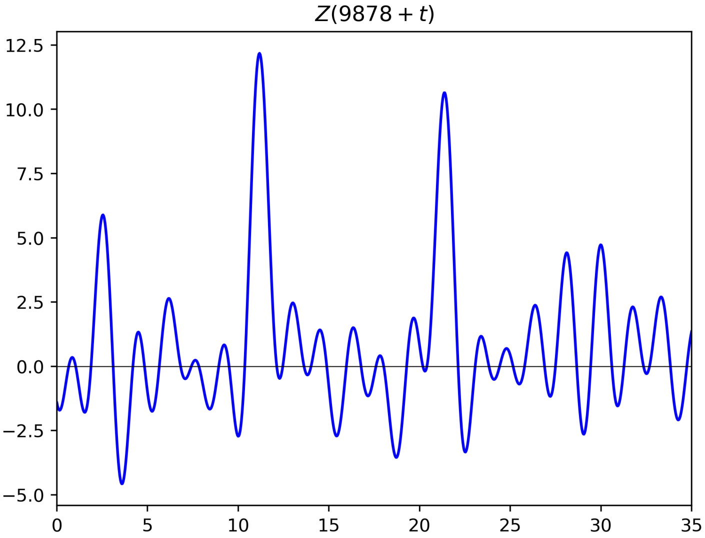
Some pictures of $Z(t)$, $t \approx 1421$



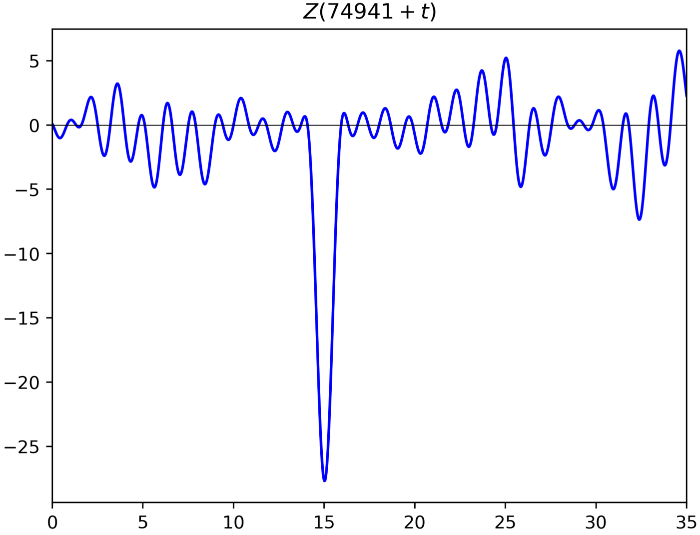
Some pictures of $Z(t)$, $t \approx 7000$



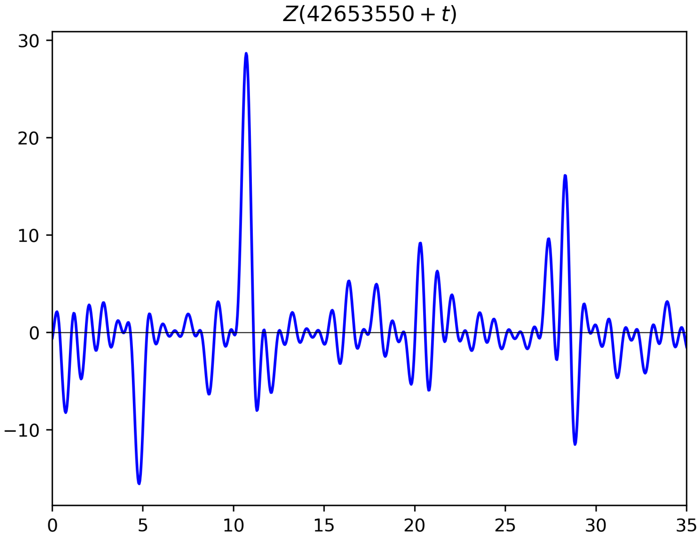
Some pictures of $Z(t)$, $t \approx 9878$



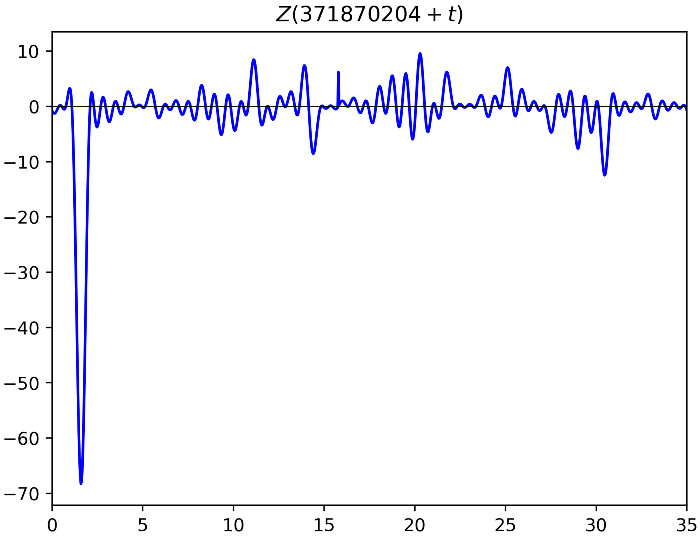
Some pictures of $Z(t)$, $t \approx 74941$



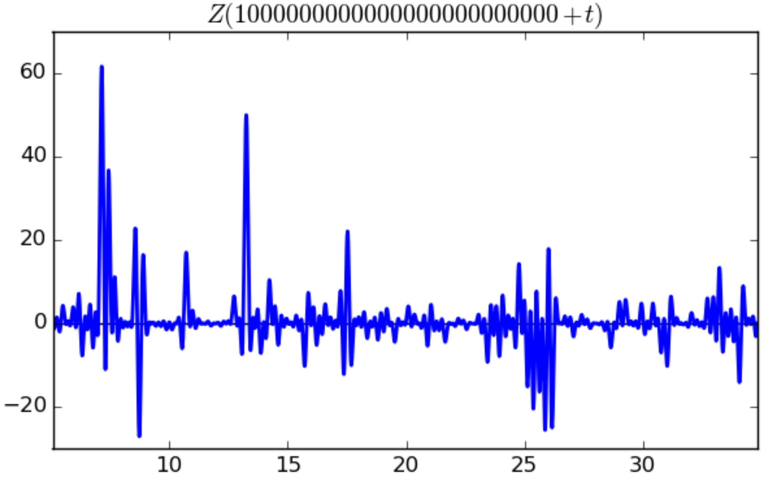
Some pictures of $Z(t)$, $t \approx 42653550$



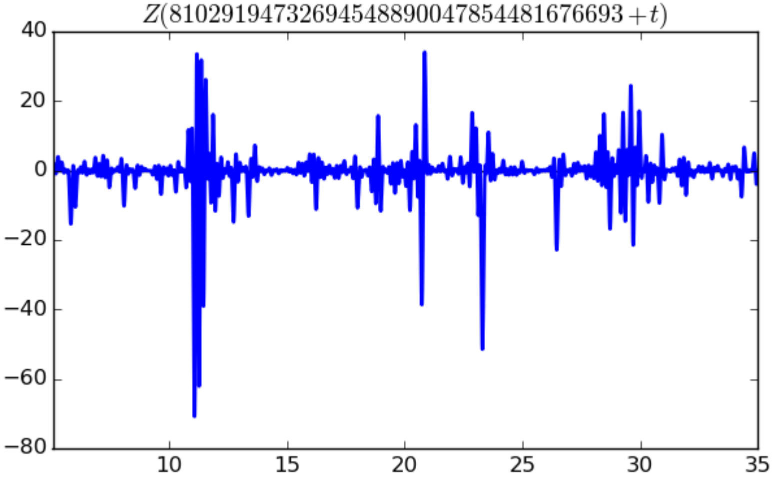
Some pictures of $Z(t)$, $t \approx 371870204$



Some pictures of $Z(t)$, $t \approx 1.0 \times 10^{24}$



Some pictures of $Z(t)$, $t \approx 8.10291947327 \times 10^{34}$



Consequences of the computations - Mertens Conjecture

Definition - Möbius function

Let $n \in \mathbb{N}$. Then the **Möbius function** $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if a square divides } n, \\ (-1)^k & \text{if } n = p_1 p_2 \dots p_k. \end{cases}$$

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Mertens Conjecture

Let

$$M(n) = \sum_{k=1}^{n} \mu(k).$$

Then for all $n > 1$ we have

$$|M(n)| < \sqrt{n}.$$

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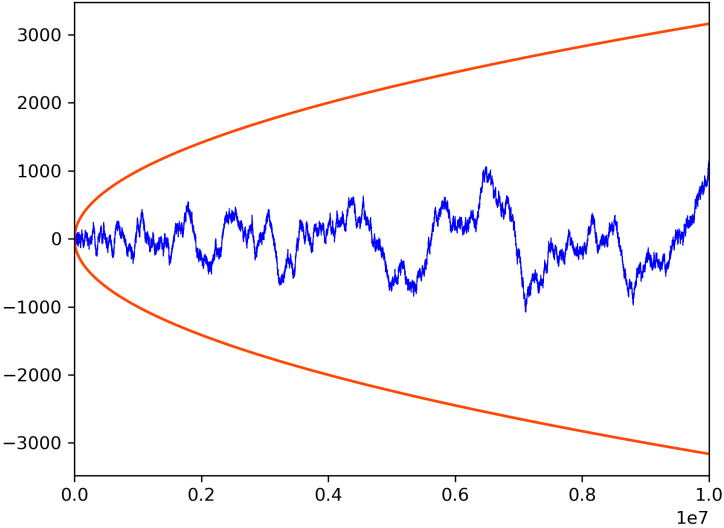

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Sadly, in 1985 Odlyzko and te Riele disproved Mertens conjecture. Their proof doesn't show an explicit counterexample and instead shows that

$$\limsup_{n\to\infty} M(n) n^{-1/2} > 1.06;$$
$$\liminf_{n\to\infty} M(n) n^{-1/2} < -1.009.$$

These bounds were attained by computing a bunch of zeros of the Riemann zeta function to high accuracy. Although no explicit counterexample has been found, we know it must be between 10^{14} and $10^{10^{40}}$.

Consequences of the computations - Computing $\pi(x)$

Theorem (Platt - 2012)

We have

$$\pi(10^{24}) = \#\{\text{primes } p \leq 10^{24}\} = 18,435,599,767,349,200,867.$$

To compute this, the first $103,800,788,359$ zeros of $\zeta(s)$ were calculated to an accuracy of roughly 25 decimal places.
This also agrees with earlier results that required the Riemann Hypothesis.

Thanks for listening!