Numerical computations of Maass cusp forms

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Maass cusp forms

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The modular surface $X = \Gamma_0(N) \backslash \mathbb{H}$ is a finite volume non-compact surface with Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

We also have the measure

$$\frac{dx\, dy}{y^2}.$$

Level 1 - SL$(2, \mathbb{Z})$

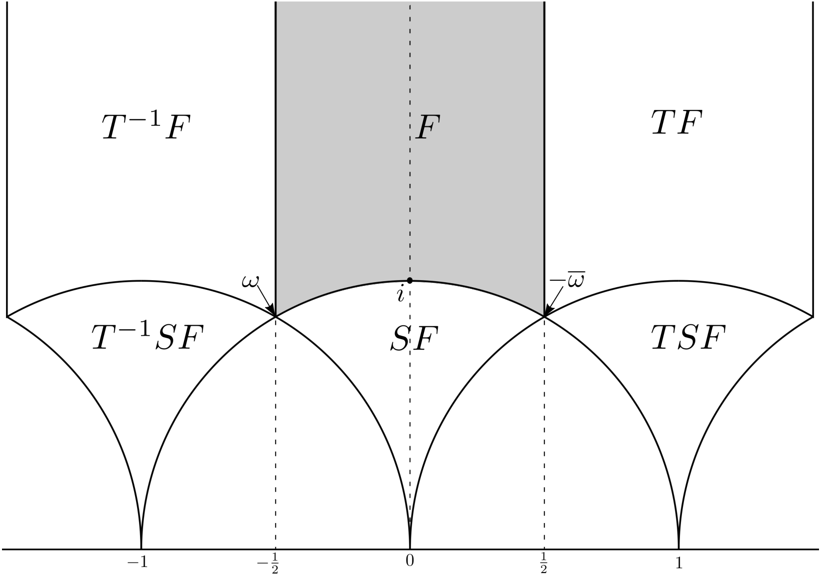
$\Gamma_0(1) = \mathrm{SL}(2, \mathbb{Z})$ is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This gives the following fundamental domain for this action

$$F = \left\{ z \in \mathbb{H} : |z| \geq 1 \text{ and } |\mathrm{Re}(z)| \leq \frac{1}{2} \right\}.$$

Fundamental domain



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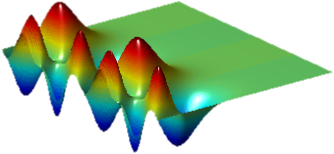
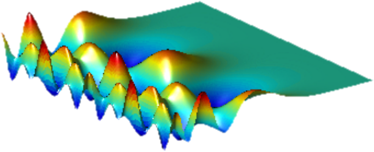
We will denote the space of Maass cusp forms of level N and Laplace eigenvalue λ by $\mathcal{S}_\lambda(N)$.
The set of functions that just satisfy points (2), (3) and (4) we shall denote as $L^2_{\text{cusp}}(X)$.

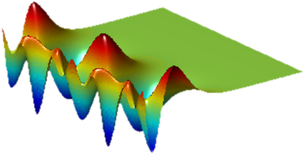
Pictures of Maass forms



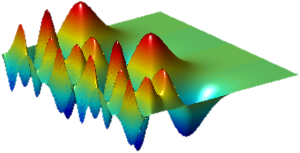
(a) Level 1, $\lambda = 91.141345\ldots$



(b) Level 1, $\lambda = 190.131547\ldots$



(c) Level 2, $\lambda = 79.867724\ldots$



(d) Level 3, $\lambda = 182.713668\ldots$

Figure: Images of Maass forms from the LMFDB.

Hecke operators

For any $f \in \mathcal{S}_\lambda(N)$ and any non-zero integer n coprime to N, we define the **Hecke operator** T_n by

$$T_n f(z) = \frac{1}{\sqrt{|n|}} \sum_{\substack{ad=n \\ (a,N)=1 \\ d>0}} \sum_{j=0}^{d-1} \begin{cases} f\left(\dfrac{az+j}{d}\right) & \text{if } n > 0, \\ f\left(\dfrac{a\bar{z}+j}{d}\right) & \text{if } n < 0. \end{cases}$$

This will map $\mathcal{S}_\lambda(N) \to \mathcal{S}_\lambda(N)$.

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This will map $\mathcal{S}_\lambda(N) \to \mathcal{S}_\lambda(N)$.
Now a famous result then tells us that there exists an orthogonal basis $\{f_j\}$ in $L^2_{\text{cusp}}(X)$ consisting of eigenfunctions to all Hecke operators T_n with $(n, N) = 1$.

Hecke eigenvalues

A Maass cusp form f of level N and with Laplace eigenvalue $\lambda = \frac{1}{4} + R^2$ has a Fourier expansion (at ∞) of the form

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{iR}(2\pi |n| y) e(nx)$$

where $e(nx) = \exp(2\pi inx)$ and $K_\nu(u)$ is the K-Bessel function.

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We call a Maass form f **even** if $a(-n) = a(n)$ or **odd** if $a(-n) = -a(n)$.

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We call a Maass form f **even** if $a(-n) = a(n)$ or **odd** if $a(-n) = -a(n)$.
If f is also a Hecke eigenfunction for all Hecke operators T_n with $(n, N) = 1$, i.e $T_n f = \lambda(n) f$, then we can normalise such that $a(1) = 1$ and we have

$$a(n) = \lambda(n)$$
$$a(-n) = \varepsilon \lambda(n)$$

where ε is 1 is f is even and -1 if f is odd.

Hejhal's Algorithm

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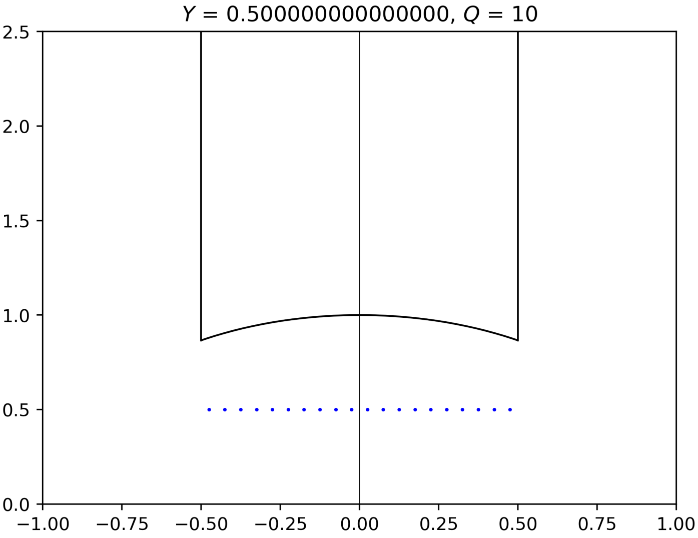
4. We then use a non-linear search strategy to zoom in on an Laplace eigenvalue.

Picture of z_m



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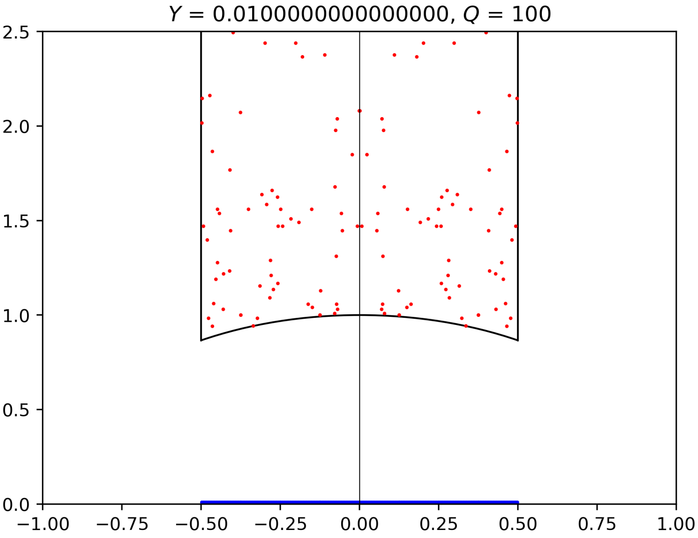

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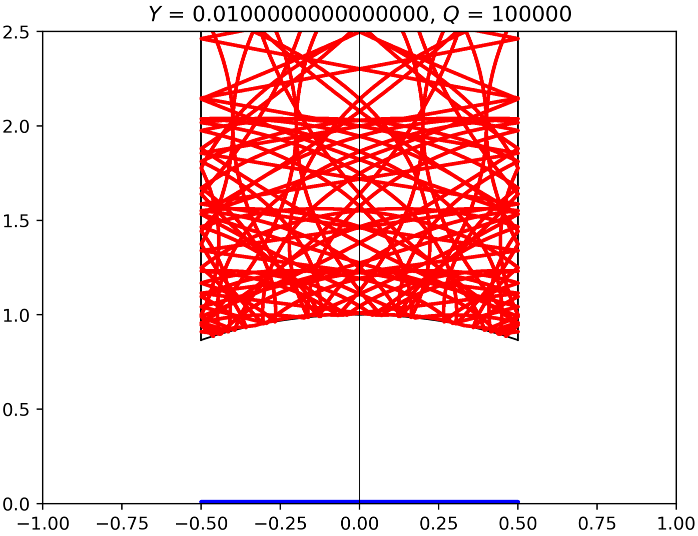

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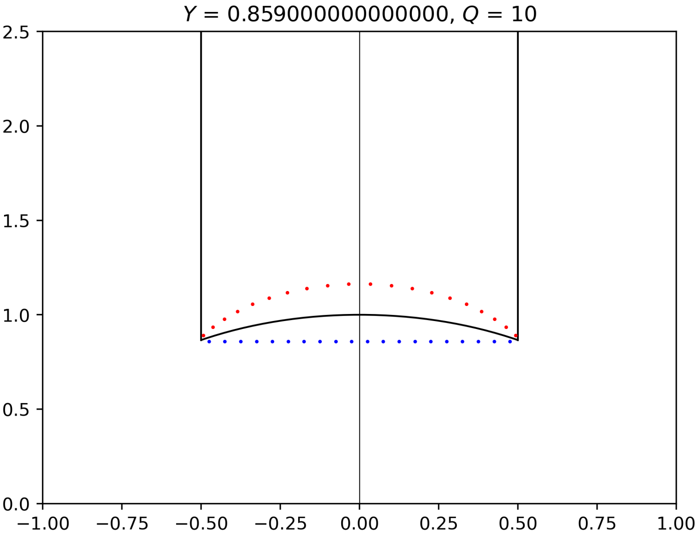

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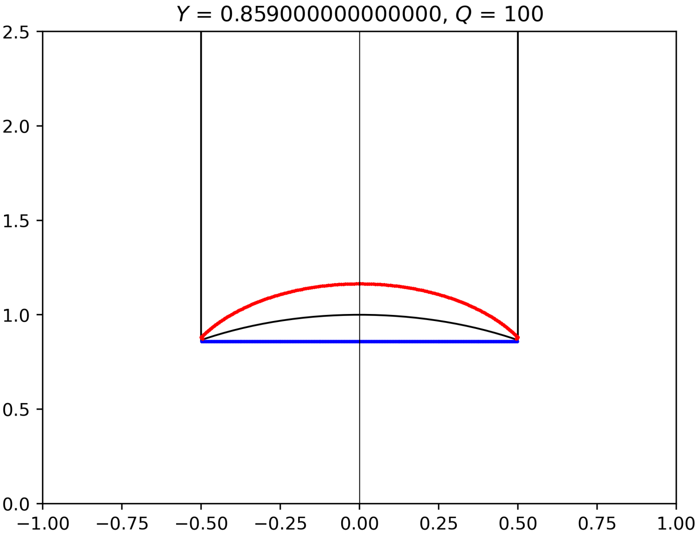

List of eigenvalues for $\mathrm{SL}(2, \mathbb{Z})$

R_1	$9.53369526135\ldots$
R_2	$12.1730083247\ldots$
R_3	$13.7797513519\ldots$
R_4	$14.3585095183\ldots$
R_5	$16.1380731715\ldots$
R_6	$16.6442592019\ldots$
R_7	$17.7385633811\ldots$
R_8	$18.1809178345\ldots$
R_9	$19.4234814708\ldots$
R_{10}	$19.4847138547\ldots$

Table: List of first 10 eigenvalues R on $\mathrm{SL}(2, \mathbb{Z})$. Data from the LMFDB.

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- There are nearly 15000 examples of these Maass cusp forms computed and stored on the LMFDB.
- There is also a phase 2 to this algorithm which allows us to compute more Fourier coefficients once we have a good approximation to R and its first few Fourier coefficients.

Verification methods

- In 2006, Booker, Strömbergsson and Venkatesh proved that it is possible to certify whether a candidate Maass form is "close" to a true Maass form. Roughly, suppose you have a computed eigenvalue $\tilde{\lambda} = \frac{1}{4} + \tilde{R}^2$ and the coefficients of a suspected Maass form $\tilde{f}$. Then they showed that if $\tilde{f}$ is "almost automorphic", then $\tilde{f}$ is "close" to a true Maass cusp form f. They only showed this for level 1, i.e $\mathrm{SL}(2, \mathbb{Z})$ and computed and verified the first few Maass cusp forms to a hundred digits.

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- Little plug: In order to do rigorous computations we use interval arithmetic. A library I use for this is ARB library for C which does arbitrary ball arithmetic.

Thanks for listening!