Amie Wilkinson Exercises

Lecture 1

1. Consider the tripling map $g\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by
$$g(x) = 3x \pmod 1.$$
   (a) Prove that $g_*\mu = \mu$, where μ is the Lebesgue probability measure on $\mathbb{R}/\mathbb{Z}$
   (b) Let $\mathcal{C} \subset [0,1]$ be the middle-thirds Cantor set. Show that $g(\mathcal{C}) = \mathcal{C}$.

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and let $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the automorphism:
$$f_A(x) = Ax \pmod{\mathbb{Z}^2}.$$
   (a) Show that for all $x \in \mathbb{T}^2$:
$$f_A^k(x) = x, \quad \text{for some } k \geq 2 \iff x = \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \in \mathbb{Q}^2/\mathbb{Z}^2.$$
   (b) Show that $k \leq (q_1 q_2)^2$ in part (a)

   Hint: Part (b) is a hint for part (a). Use the fact that f_A is invertible as well as the pigeonhole principle.

3. Find two Borel probability measures μ_1 and μ_2 on $\mathbb{R}/\mathbb{Z}$ and an integer $m \geq 2$ such that the following properties all hold:
   - $f_*\mu_i = \mu_i$, for $i = 1, 2$, where
$$f(x) = mx \pmod 1.$$
   - For every nonempty open interval I and $i = 1, 2$:
$$\mu_i(I) > 0$$
   - $\mu_1 \perp \mu_2$, meaning that there exists a Borel set $X \in \mathcal{B}$ such that $\mu_1(X) = 0$ and $\mu_2(X) = 1$.

   Hint: There are several ways to do this. One way (for $m = 2$) is to code the doubling map by a shift (off of a countable set of points). Consider two different Bernoulli measures on the shift... The Birkhoff ergodic theorem (or the law of large numbers) is useful here.