Amie Wilkinson Exercises

Lecture 5

(1) Let $f_\epsilon\colon \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$f_\epsilon(x, y) := (2x + y - \epsilon \sin(2\pi(x + y)), x + y) = h_\epsilon \circ f_A(x, y),$$

where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and $h_\epsilon(x, y) = (x - \epsilon \sin(2\pi y), y)$.

(**Note the "minus sign" in this definition!**)

(a) Prove that for all $\epsilon \in \mathbb{R}$, f_ϵ preserves Lebesgue measure.
Hint: Show that for all x, y and all ϵ:

$$\det\left(D_{(x,y)} f_\epsilon\right) = 1.$$

Use the chain rule.

(b) Consider the cone fields on $\mathbb{T}^2$ defined by

$$\mathcal{C}^u(x, y) = \{(u, v) \in \mathbb{R}^2 : uv > 0\}, \quad \text{and } \mathcal{C}^s(x, y) = \{(u, v) \in \mathbb{R}^2 : uv < 0\}.$$

Show that $\mathcal{C}^u$ is invariant under $Df_A = A$ and that vectors in $\mathcal{C}^u$ are expanded in length under Df_A by a definite factor $\sqrt{2}$. Similarly show that $\mathcal{C}^s$ is invariant under $Df_A^{-1} = A^{-1}$ and that vectors in $\mathcal{C}^s$ are expanded in length under Df_A^{-1} by the definite factor $\sqrt{2}$.

(c) Find a value of ϵ_0 so that f_ϵ is an Anosov diffeomorphism for all $\epsilon < \epsilon_0$.
Hint: Cone fields. Use the exercise from Khadim War's lecture yesterday.

(d) Show that if $g\colon \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism, and if $(x_0, y_0) \in \mathbb{T}^2$ satisfies

$$g^k(x_0, y_0) = (x_0, y_0),$$

for some $k \geq 1$, then the eigenvalues of the matrix $D_{(x_0,y_0)} g^k$ are λ_1, λ_2, where $|\lambda_1| < 1 < |\lambda_2|$.

(e) Find a value of $\epsilon_1 > 0$ such that f_ϵ is not Anosov for $\epsilon \geq \epsilon_1$ but close to ϵ_1.
Hint: Consider the derivative at $(0, 0)$. Recall the relationship between trace and eigenvalues for 2×2 matrices of determinant 1.

(f) Prove that there exists $\bar{\epsilon} > 0$ such that f_ϵ is Anosov for $\epsilon < \bar{\epsilon}$ and $f_{\bar{\epsilon}}$ is not Anosov.

(g) *** Find $\bar{\epsilon}$ (use a computer?) Is $f_{\bar{\epsilon}}$ ergodic? If f_ϵ Anosov for $\epsilon > \bar{\epsilon}$?