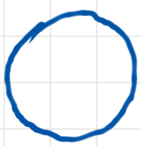
Annie lecture 4

Investigate properties of nonlinear systems.

Examples (well-understood)

1. Diffeomorphisms of $\mathbb{R}/\mathbb{Z}$.
 - rich theory, Poincaré, Denjoy...
2. Expanding maps of $\mathbb{R}/\mathbb{Z}$ e.g.



$$g_a(z) = \frac{z(z-a)}{(1-\bar{a}z)} \qquad a \in \mathbb{C}, \ |a|<1$$

$$\approx f(z) = z^2 \text{ if } |a| \approx 0.$$

g_a is a perturbation of f.

3. Anosov diffeomorphisms, e.g.

$$f(x,y) = (2x+y+\varepsilon \sin(2\pi(x+y)), x+y)$$

Expanding maps of circle

Theorem Suppose that $g: \mathbb{R}/\mathbb{Z} \circlearrowleft$ has the following properties:

- g is C^2
- $\inf_{x \in \mathbb{R}/\mathbb{Z}} g'(x) > 1$

Then for any $X \subseteq \mathbb{R}/\mathbb{Z}$, $g^{-1}(X) = X \Rightarrow \mu(X) = 0$ or 1.

In particular, if $g_*\mu = \mu$, then f is ergodic (μ = Lebesgue)

Pf Do for perturbations of $x \mapsto 2x$ (see exercises)

e.g. $g = g_a$ $|a| \approx 0$. Then $\exists \lambda > 1$ s.t

$\forall x$: • $g'(x) \geq \lambda$

• $\# g^{-1}(x) = 2$.

• $\exists x_0$ $g'(x_0) = x_0$

Taking preimages of x_0, we get a nested sequence

$$\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \ldots$$

of partitions s.t.

1) $\# \mathcal{P}_m = 2^m$

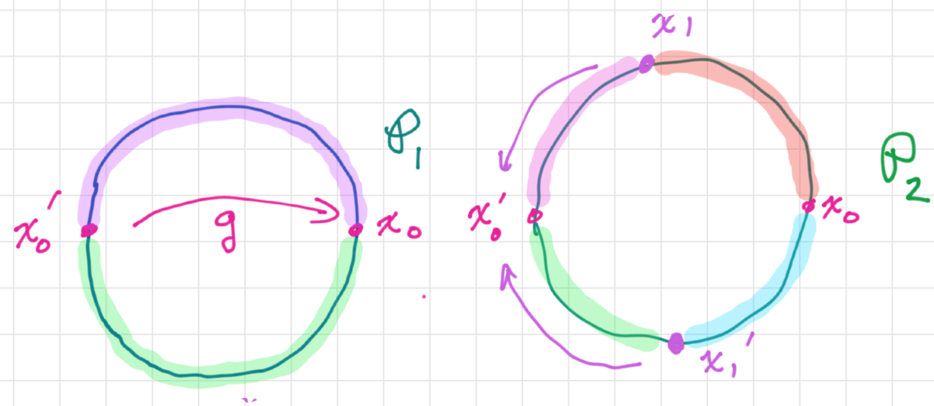
• $f^m(I) = \mathbb{R}/\mathbb{Z}$ (injective on interior)

• $I \in \mathcal{P}_m \Rightarrow |I| \leq \frac{1}{\lambda^m}$

$(I \in \mathcal{P}_m \Rightarrow f(I) \in \mathcal{P}_{m-1}$

$$|I| = \int_{f(I)} (f^{-1})'(x)\, dx$$

$$\leq \frac{1}{\lambda} |f(I)| \;)$$



Suppose $g^{-1}(X) = X$

$\mu(X) > 0$

Given $\varepsilon > 0$

Exercise 2 from Lecture 2 Shows $\exists m, I \in \mathcal{P}_m$ s.t.

$$\mu(X \mid I) > 1 - \varepsilon$$

$\Leftrightarrow$ $\mu(X' \mid I) < \varepsilon$, where $X' = \mathbb{R}/\mathbb{Z} \setminus X$

applying g^m as last time, and using $g^{-1}(X') = X'$, we get

$$\mu(X' \mid \mathbb{R}/\mathbb{Z}) = \mu(g^m(X') \mid g^m(I))$$

$$\leq \frac{\sup_{x \in X' \cap I} (g^m)'(x)}{\inf_{x \in I} (g^m)'(x)} \cdot \mu(X' \mid I)$$

$$\leq \frac{\sup_{x \in I} (g^m)'(x)}{\inf_{x \in I} (g^m)'(x)} \cdot \varepsilon$$

Distortion thm: $\exists C \leq 1$ s.t. $\forall m \geq 1$ $\forall I \in \mathcal{P}_m$:

$$\frac{\sup_{x \in I} (g^m)'(x)}{\inf_{x \in I} (g^m)'(x)} \leq C$$

Applying this, we have

$$\mu(X') = \mu(X' | \mathbb{R}/\mathbb{Z}) \leq C \cdot \varepsilon.$$

as $\varepsilon > 0$ was arbitrary

$$\mu(X') = 0 \Rightarrow \mu(X) = 1$$

$\boxtimes$

Proof of Distortion theorem:

Define $\psi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}_+$ by

$$\psi(x) = \log(g'(x)).$$

Properties of ψ:

1. $\psi \geq \log \lambda > 0$

2. ψ is C^1. In particular it is Lipschitz. $\exists M > 0$ s.t.

$$|\psi(x) - \psi(y)| \leq M \cdot |x - y|$$

3. $\forall m \geq 0$

$$\log (g^m)'(x) = \log g'(g^{m-1}x) \cdots g'(x)$$

$$= \sum_{j=0}^{m-1} \psi(g^j x).$$

Let $x, y \in I \in \mathcal{P}_m$ be given.

want to show

$$\frac{g'(x)}{g'(y)} \leq C$$

$$\Leftrightarrow \left|\log {g^m}'(x) - \log {g^m}'(y)\right| \leq \log(C)$$

$$\text{LHS} \leq \left|\sum_{j=0}^{m-1} \psi(g^j(x)) - \psi(g^j(y))\right|$$

$$\leq \sum \left|\psi(g^j(x)) - \psi(g^j(y))\right|$$

$$\leq M \sum \left|g^j(x) - g^j(y)\right|$$

$$\leq M \sum \left|g^j(I)\right|$$

$$\leq M \sum \lambda^{m-j} \leq \frac{M}{1-\lambda} \quad \boxtimes$$