Amie lecture 5

Last time: Khadim showed that if $g: \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeomorphism with

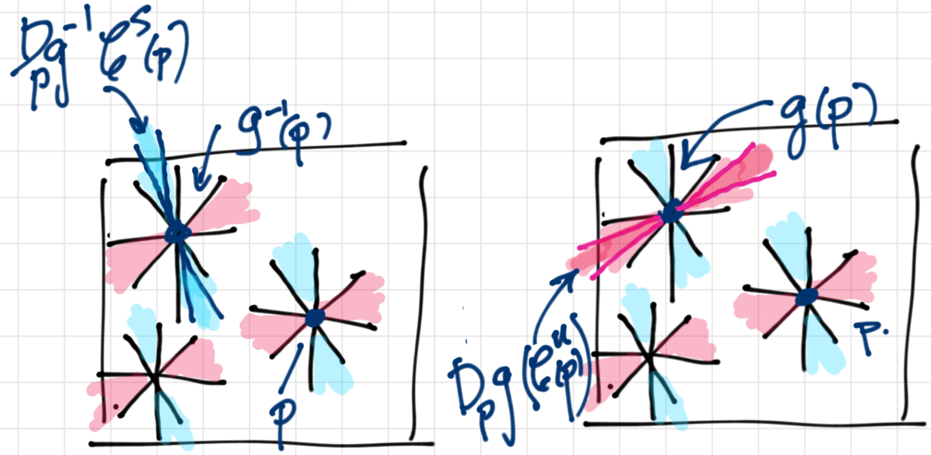
$$d_{C^1}(g, f_A) = \sup_{p \in \mathbb{T}^2} |g(p) - f_A(p)| + \sup_{p \in \mathbb{T}^2} \|D_p g - D_p f_A\|$$

sufficiently small, then g is Anosov:

$\exists$ cone fields $\mathcal{C}^u$, $\mathcal{C}^s$:

$$D_p g\, \mathcal{C}^u \subseteq \mathcal{C}^u(g(p))$$

$$D_p g^{-1} \mathcal{C}^s(p) \subseteq \mathcal{C}^s(g^{-1}(p))$$



$\exists \lambda > 1$ s.t.

$$v \in \mathcal{C}^u(p) \Rightarrow \|D_p g\, v\| \geq \lambda \|v\|$$

$$v \in \mathcal{C}^s(p) \Rightarrow \|D_p g^{-1} v\| \geq \lambda \|v\|$$

He also explained:

If $\gamma_u : [0,1] \to \mathbb{T}^2$ is "tangent to $\mathcal{C}^u$":

i.e. $\gamma_u'(t) \in \mathcal{C}^u(\gamma_u(t)) \quad \forall t$

then $g \circ \gamma_u : [0,1] \to \mathbb{T}^2$ is:

1) "even more tangent to $\mathcal{C}^u$"

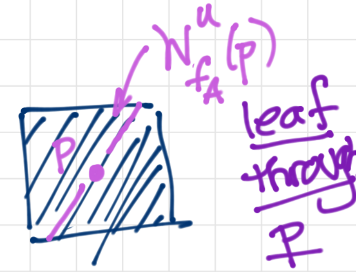
$$(g\circ\gamma_u)' = Dg(\gamma_u') \in \frac{1}{\lambda^2}\mathcal{C}^u(g\circ\gamma_u(t))$$

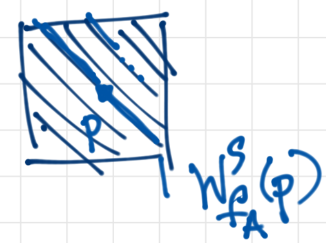
2) longer:

$$\ell(g\circ\gamma_u) \geq \lambda\, \ell(\gamma_u).$$

Use these facts to construct <u>foliations</u>.

$\mathcal{W}^u_g$, $\mathcal{W}^s_g$

Recall $\mathcal{W}^u_{f_A} =$ 

$\mathcal{W}^s_{f_A} =$ 

<u>Theorem</u> If $g : \mathbb{T}^2 \circlearrowleft$ is Anosov, then $\exists$ foliations $\mathcal{W}^u, \mathcal{W}^s$, with smooth

leaves, such that

$q, q' \in W^s(p) \Rightarrow$

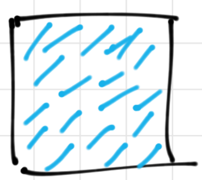
$d(g^n(q), g^n(q')) \xrightarrow{n \to \infty} 0$

$q, q' \in W^u(p) \Rightarrow$

$d(g^{-n}(q), g^{-n}(q')) \xrightarrow{n \to \infty} 0$

Idea of construction

start with some family of curves inside of $\mathcal{C}_e^{u}$

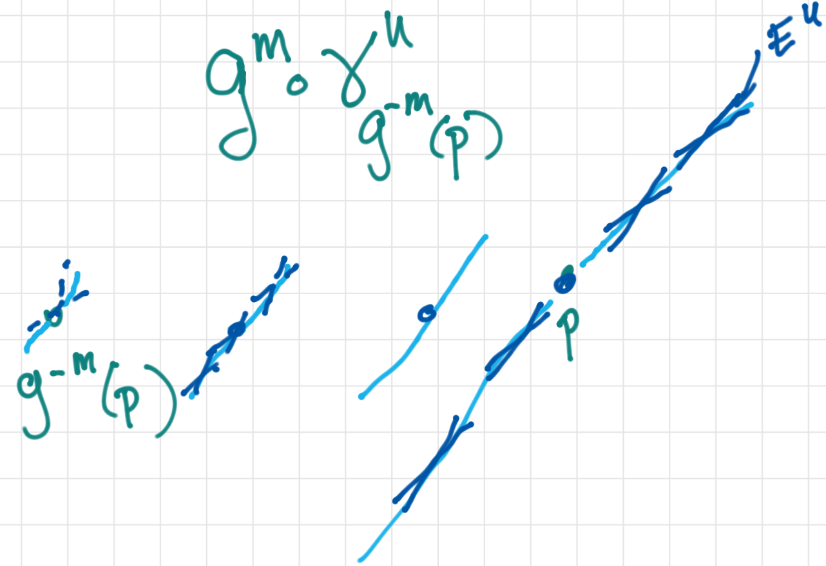


$\gamma^u_p(0) = p$

$\dot{\gamma}^u_p \in \mathcal{C}^u(\gamma^u_p(t))$

For $p \in \mathbb{T}^2$ consider

$g^m \circ \gamma^u_{g^{-m}(p)}$

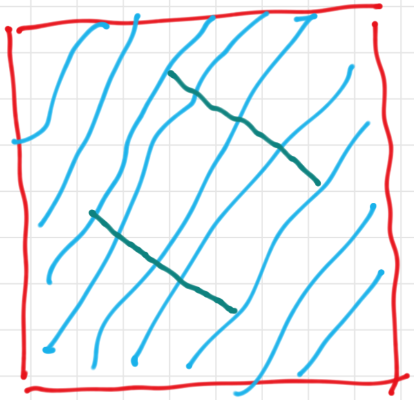


as $m \to \infty$ this will converge to $W^u_g(p)$.

Similarly construct

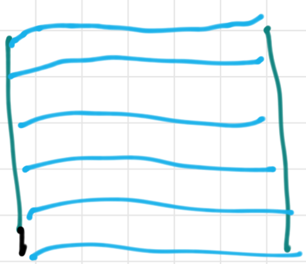
$W^s_g(p) = \text{"}\lim_{n \to \infty}\text{"}\, g^{-m} \circ \gamma^s_{g^m(p)}$

Properties of $\mathcal{W}^u$:



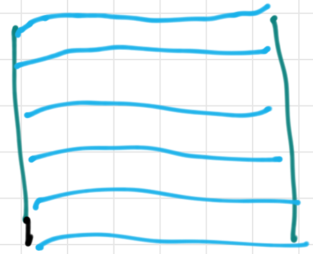
leaves of $\mathcal{W}^u$ are smooth, tangent to E^u

Let's look close up:



(plaques)

$\mathcal{W}^u_{loc}$



(plaques)

S = square

$\mathcal{W}^u_{loc}$

theorem

If $g \in C^2$, the plaques of $\mathcal{W}^{u/s}_{loc}$ are "Good enough for fubini".

Denote by μ^u_p the 1-dimensional lebesgue measure on $\mathcal{W}^u_{loc}(p)$.

& μ^s_p 1-dim leb. on $\mathcal{W}^s_{loc}(p)$

Then

1) If $Z \subseteq S$ is any set with $\mu(Z) = 0$, then for μ-a.e. $p \in Z$

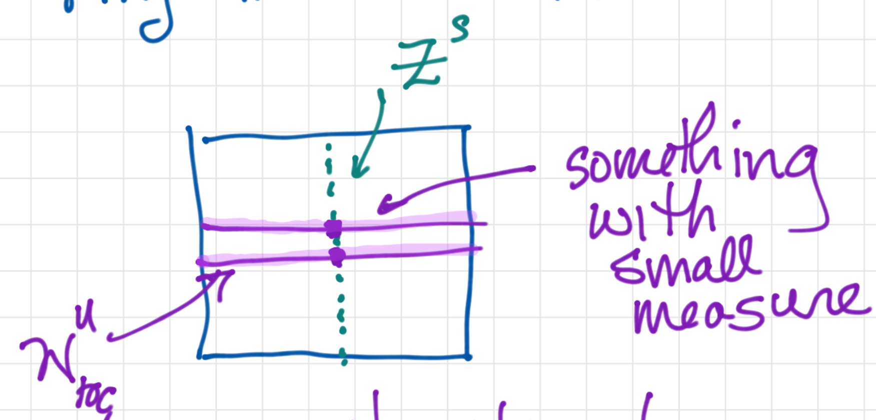
$$\mu^c_p(Z) = 0$$

2) For any $q \in \mathbb{T}^2$, if $Z^s \subseteq W^s_{loc}$ is any set with $\mu^s_q(Z^s) = 0$ then

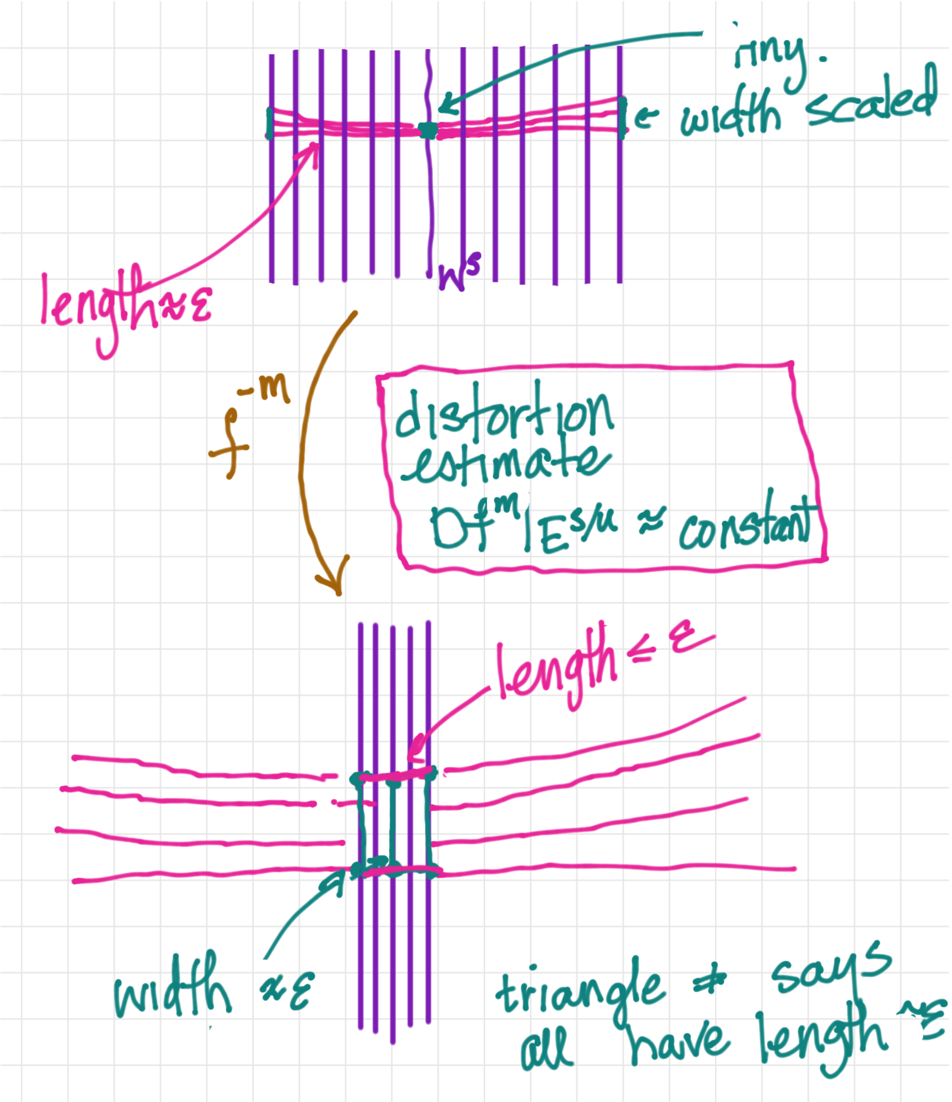
$$\mu\Big(\bigcup_{p \in Z^s} W^u_{loc}(p)\Big) = 0$$

The same results hold for W^s_{loc}, with the roles of s & u switched.

Why this is true.



Iterate, etc.


tiny. width scaled
length ≈ ε
W^s
f^{-m}
distortion estimate $Df^m|_{E^{s/u}} \approx$ constant
length ≤ ε
width ≈ ε
triangle ≠ says all have length ≈ ε