ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Homework 1

Exercise 1.1. Let $R_\alpha : S^1 \to S^1$ where $\alpha = p/q$ and p and q are coprime, i.e., $gcd(p,q) = 1$.

(a) Draw an orbit of R_α for α for $p/q = 2/7$ and for $p/q = 5/8$. Prove that q is the minimal period, i.e. for each $x \in \mathbb{R}/\mathbb{Z}$ we have $R_\alpha^k(x) \neq x$ for each $1 \leq k < q$. What is the dynamical meaning of p?

(b) Assume now that α is irrational. Conclude the proof started in class that for every $z \in S^1$ the forward orbit $\mathscr{O}_{R_\alpha}(z)$ is *dense* in S^1.

* **Exercise 1.2.** Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be an irrational number. Prove that there are infinitely many fractions p/q where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and p, q coprime, that solve the equation

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

[*This result is known as Dirichlet theorem in Number Theory and it shows how well an irrational number can be approximated by rational numbers.*]

More Exercises

Exercise 1.3. Let $X = [0,1) \times [0,1)$ and let $f : X \to X$ be given by

$$f(x,y) = (x + y \mod 1, y).$$

(a) Show that the set $Per(f)$ of periodic points consists exactly of the points $(x,y) \in X$ such that y is rational.

(b) Are there points whose orbit is dense? Explain your answer.

[Here in the definition of density let d be the Euclidean distance on $[0,1] \times [0,1]$, that is

$$d\left((x_1,y_1),(x_2,y_2)\right) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.]$$

Exercise 1.4. Let $X = [0,1) \times [0,1)$ be the unit square and let d be the Euclidean distance. Consider the map

$$f(x_1,x_2) = (x_1 + \alpha_1 \mod 1,\ x_2 + \alpha_2 \mod 1).$$

Assume that at least one of α_1, α_2 is irrational.

(a) Write down the the n^{th}-term of the orbit $\mathcal{O}_f^+(\underline{x})$ of $\underline{x} = (x_1,x_2)$ under f and show that for any $\underline{x} = (x_1,x_2) \in [0,1)^2$ all points in the orbit $\mathcal{O}_f^+(\underline{x})$ of $\underline{x}$ under f are *distinct*.

(b) Show that for any $\underline{x} = (x_1,x_2) \in [0,1)^2$ and any N positive integer there exists $1 \leq n \leq N^2$ such that

$$d(f^n(\underline{x}), \underline{x}) \leq \frac{\sqrt{2}}{N},$$

where d is the Euclidean distance.

Exercise 1.5. Consider the map $\Psi : \mathbb{R}/\mathbb{Z} \to S^1$ given by $\Psi(x) = e^{2\pi i x}$.

(a) Check that Ψ is well-defined and establishes a one-to-one correspondence between $\mathbb{R}/\mathbb{Z}$ and S^1 (that is, show Ψ is injective and surjective).

(b) Fix $b \in \mathbb{N}$. Show that $f_b : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $f_b(x) = bx \mod 1$ is conjugate to the map $g_b : S^1 \to S^1$ given by

$$g_b(z) = z^b.$$

(c) Let d be the distance on S^1 given by the shortest arc length divided by 2π. Show that

$$d(\Psi(x), \Psi(y)) = \min_{m \in \mathbb{Z}} |x - y + m|.$$

(d) Show that g_b *expands* distances in the following sense: if $d(x, y) \leq 1/2b$, then

$$d(g_b(x), g_b(y)) = bd(x, y).$$

* **Exercise 1.6.** Let $X = [0, 1]$ and let $f : X \to X$ be differentiable and invertible. Prove that:

(a) If $f'(x) > 0$ then f has only fixed points and no periodic points;

(b) If $f'(x) < 0$ then f has a unique fixed point and all other periodic points have period two.