ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Homework 3

Exercise 3.1. Let R_α be an irrational rotation, let $\alpha = [a_0, a_1, \dots]$ be the continued fraction expansion of α and let $p_n/q_n = [a_0, \dots, a_n]$ be the convergents of α. Let $\Delta^{(n)}$ be the arc with endpoints 0 and $R_\alpha^{q_n}(0)$. Verify *yourself* that the lenghts λ_n of the intervals $\Delta^{(n)}$ for $n \geq 1$ satisfy

$$\frac{\lambda_n}{\lambda_{n-1}} = G^n(\alpha),$$

where G is the Gauss map and that the return times to $I^{(n)} = \Delta^{(n)} \cup \Delta^{(n-1)}$, $n \geq 1$ are q_n and q_{n+1}. Try:

- the case $n = 0$ and $n = 1$ first (drawing pictures);
- the general case (by induction).

Exercise 3.2. Let $R_\alpha : S^1 \to S^1$ be a rotation. Let $I \subset S^1$ be an arc (an interval in $[0,1]/\sim$). Let T be the Poincar'e first return map of R_α to I. Show that T is an exchange of three intervals, i.e. there exists three subintervals I_1, I_2, I_3 which partition I and such that $T(I_1), T(I_2), T(I_3)$ cover again I.

Exercise 3.3. Let G be the Gauss map and let F be the Farey map, given by

$$F(x) = \begin{cases} \frac{1-x}{x} & \text{if } x > \frac{1}{2} \\ \frac{x}{1-x} & \text{if } x \leq \frac{1}{2}. \end{cases}$$

Let F_0 and F_1 be the two branches of F, F_0 denoting the left branch (restriction to $[0,1/2]$), F_1 denoting the left branch (restriction to $[1/2,1]$).

Verify that G can be obtained accelerating F by putting together all consecutive branches of F_0 (plus the next F_1), namely

$$G(x) = \begin{cases} F_1(x) & \text{if } x > \frac{1}{2}, \\ F_1 \circ F_0^{k(x)+1}(x) & \text{if } x \leq \frac{1}{2}, \end{cases}$$

where $k(x)$ is the largest integer k such that $x \in [0,1/2]$, $F(x) \in [0,1/2], \dots F^k(x) \in [0,1/2]$.