# ICTP Summer School on Dynamical Systems
# Rotations of the circle and renormalization
# Homework 5

**Exercise A1** Let $R_\alpha$ be an irrational rotation, where $\alpha = [a_1, \dots, ]$ and consider the inducing intervals $I^{(n)}$ defined in class. Consider the towers representation of $R_\alpha$ as towers over $\Delta^{(n-1)}$ and $\Delta^{(n)}$. Let $\underline{h}^{(n)}$ and $\underline{\lambda}^{(n)}$ be vectors which give the heights and widths of the towers (from left to right) at step $n$, namely

$$\text{if } n \text{ is odd:} \qquad \underline{h}^{(n)} := \begin{pmatrix} q_{n-1} \\ q_n \end{pmatrix}, \qquad \underline{\lambda}^{(n)} := \begin{pmatrix} \lambda_n \\ \lambda_{n-1} \end{pmatrix},$$
$$\text{if } n \text{ is even:} \qquad \underline{h}^{(n)} := \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}, \qquad \underline{\lambda}^{(n)} := \begin{pmatrix} \lambda_{n-1} \\ \lambda_n \end{pmatrix},$$

and let $A_n$ be the $2 \times 2$ matrix given by

$$A_n := \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \quad if\ n \text{ is even, or} \qquad A_n := \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix} \quad if\ n \text{ is odd.}$$

(a) Verify that $\underline{h}^{(n)}$ satisfy the following matrix relation:

$$\underline{h}^{(n)} = A_n A_{n-1} \dots A_1 \underline{h}^{(0)}, \qquad \underline{h}^{(0)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Show that the widths $\underline{\lambda}^{(n)}$ satisfy

$$\underline{\lambda}^{(n)} = (A_n^t)^{-1} (A_{n-1}^t)^{-1} \cdots (A_1^t)^{-1} \underline{\lambda}^{(0)}, \qquad \underline{\lambda}^{(0)} := \begin{pmatrix} \{1/\alpha\} \\ \alpha \end{pmatrix},$$

where $A^t$ denotes the transpose of the matrix $A$.

**Exercise A2** Let $I^{(n)} = \Delta^{(n)} \cup \Delta^{(n-1)}$ be the inducing intervals for the rotation $R_\alpha : S^1 \to S^1$. Let $f : I^{(0)} \to \mathbb{R}$ be a bounded function. Define the sequence of induced functions by

$$f^{(n)}(x) := \begin{cases} \sum_{i=0}^{q_n - 1} f(R_\alpha^i(x)) & if\ x \in \Delta^{(n-1)}; \\ \sum_{i=0}^{q_{n-1}-1} f(R_\alpha^i(x)) & if\ x \in \Delta^{(n)}. \end{cases}$$

[This function associate to a point in the base the Birkhoff sum of the function $f$ *along the tower*, i.e. along the piece of the orbit that goes from the base point up to the top of the tower to which it belongs. They are also called *special Birkhoff sums.*]

(a) Show that if $f$ is constant on $\Delta^{(0)}$ and constant on $\Delta^{(-1)} := S^1 \backslash \Delta^{(0)}$, then $f^{(n)}$ is constant on $\Delta^{(n)}$ and constant on $\Delta^{(n-1)}$,

(b) Assume that $f$ is integrable. Prove that $\int_{I^{(n)}} f^{(n)}(x) dx = \int_{I^{(0)}} f(x) dx$.

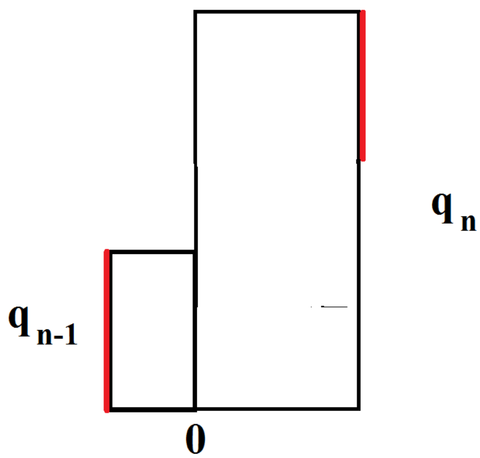
(c) Assume that $f$ has bounded variation. Let $Var_I^{(n)}(f)$ be defined by $Var_\Delta^{(n)}(f) + Var_\Delta^{(n-1)}(f)$ where $Var_\Delta^{(n)}(f)$ denotes the variaton of $f$ on $\Delta^{(n)}$. Prove that

$$Var_{I^{(n)}}(f^{(n)}) \leq Var_{I^{(0)}}(f).$$

**Exercise A3** Consider the tower representation of an irrational rotation.

(a) Show that the points in the red segments on the boundary of the towers shown in the figure are identified.

[*Hint*: You can try to think about cutting and stacking and how the towers are obtained from the previous level towers.]




(b) Finish the proof of the Denjoy-Koksma inequality using towers started in class (i.e. discuss the remaining cases according to the position of $x$ in the towers).