1 Introduction to Dynamical Systems

Dynamical systems is an exciting and very active field in pure and applied mathematics, that involves tools and techniques from many areas such as analyses, geometry and number theory and has applications in many fields as physics, astronomy, biology, meterology, economics.

The adjective *dynamical* refers to the fact that the systems we are interested in is *evolving in time*. In applied dynamics the *systems* studied could be for example a box containing molecules of gas in physics, a species population in biology, the financial market in economics, the wind currents in metereology. In pure mathematics, a dynamical system can be obtained by iterating a function or letting evolve in time the solution of equation.

Discrete dynamical systems are systems for which the time evolves in discrete units. For example, we could record the number of individuals of a population every year and analyze the growth year by year. The time is parametrized by a discrete variable n which assumes integer values: we will denote natural numbers by $\mathbb{N}$ and integer numbers by $\mathbb{Z}$. In a *continuous* dynamical system the time variable changes continuously and it is given a real number t. We will denote real numbers by $\mathbb{R}$.

Our main examples of *discrete dynamical systems* are obtained by iterating a map. Let X be a space. For example, X could be the unit interval $[0,1]$, the unit square $[0,1]\times[0,1]$, a circle (but also the surface of a doughnut or a Cantor set). Let $f : X \to X$ be a map. We can think as f as the map which gives the time evolution of the points of X. If $x \in X$, consider the iterates $x, f(x), f(f(x)), \ldots$.

Notation 1.1. *For $n > 0$ we denote by $f^n(x)$ the n^{th} iterate of f at x, i.e. $f \circ f \circ \cdots \circ f$, n times.*[1] *In particular, $f^1 = f$ and by convention f^0 : is the identity map, which will be denoted by Id ($Id(x) = x$ for all $x \in X$).*

We can think of $f^n(x)$ as the status of the point x at time n. We call forward *orbit*[2] the evolution of a point x.

Definition 1.1. *We denote by $\mathcal{O}_f^+(x)$ the forward orbit of a point $x \in X$ under iterates of the map f, i.e.*

$$\begin{aligned}\mathcal{O}_f^+(x) &:= \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}\\ &= \{f^n(x), \quad n \in \mathbb{N}\}.\end{aligned}$$

This gives an example of a discrete dynamical system parametrized by $n \in \mathbb{N}$.

Example 1.1. *Let $X = [0,1]$ be the unit interval. Let $f : X \to X$ be the map $f(x) = 4x(1-x)$. For example*

$$\mathcal{O}_f\left(\frac{1}{3}\right) = \left\{\frac{1}{3}, \frac{4}{3}\left(1-\frac{1}{3}\right) = \frac{8}{9}, 4\cdot\frac{8}{9}\left(1-\frac{8}{9}\right) = \frac{32}{81}, \ldots\right\}.$$

Example 1.2. *Let X be a circle of radius 1. An example of map $f : X \to X$ is the (clockwise) rotation by an angle $2\pi\alpha$, which maps each point on the circle to the point obtained by rotating clockwise by an angle $2\pi\alpha$.*

If f is invertible, we have a well defined inverse $f^{-1} : X \to X$ and we can also consider backwards iterates $f^{-1}(x), f^{-2}(x), \ldots$.

[1] Do not confuse this notation with the n^{th} derivative, which will be denoted by $f^{(n)}$, or by the n^{th} power, which will not be used!

[2] The name orbit cames from astronomy. The first dynamical system studied were indeed the solar system, where trajectory of a point (in this case a planet or a star) is an orbit.

Notation 1.2. *If f is invertible and $n < 0$, we denote by $f^n(x)$ the n^{th} iterate of f^{-1} at x, i.e. $f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}$, n times. Remark that even if f is not invertible, we will often write $f^{-1}(A)$ where $A \subset X$ to denote the set of premimges of A, i.e. the set of $x \in X$ such that $f(x) \in A$.*

Definition 1.2. *If f is invertible, we denote by $\mathcal{O}_f(x)$ the (full) orbit of a point $x \in X$ under forward and backward iterates of f, i.e.*

$$\begin{aligned}\mathcal{O}_f(x) &:= \{\dots, f^{-k}(x), \dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots, f^k(x), \dots\} \\ &= \{f^k(x), \quad k \in \mathbb{Z}\}.\end{aligned}$$

In this case, we have an example of discrete dynamical system in which we are interested in both past and future and the time is indexed by $\mathbb{Z}$.

Even if the rule of evolution is deterministic, the long term behavior of the system is often "'*chaotic*"'. For example, even if two points x, y are very close, there exists a large n such that $f^n(x)$ and $f^n(y)$ are far apart. This property (which we will express formally later) is known as *sensistive dependence of initial conditions.* There are various mathematical definitions of *chaos*, but they all include sensitive dependence of initial conditions. Different branches of dynamical systems, in particular topological dynamics and ergodic theory, provide tools to quantify *how chaotic* is a systems and to predict the asymptotic behaviour. We will see that often even if one cannot predict the behaviour of each single orbit (since even if deterministic it is too complicated), one can predict the *average* behaviour.

The main objective in dynamical systems is to understand the behaviour of all (or almost all) the orbits. Orbits can be fairly complicated even if the map is quite simple. A first basic question is whether orbits are finite or infinite. Even if the index run through a infinite set (as $\mathbb{N}$ or $\mathbb{Z}$) it could happen that $\mathcal{O}_f(x)$ is finite, for example if the points in the orbit repeat each other. This is the simplest type of orbit.

Definition 1.3. *A point $x \in X$ is* periodic *if there exists $n \in \mathbb{N}\backslash\{0\}$, such that $f^n(x) = x$. If $n = 1$, so that we have $f(x) = x$, we say that x is a* fixed point*. More in general, if $f^n(x) = x$ we say that x is periodic of periodic n or that n is a period for x. In particular, $f^{n+j}(x) = f^j(x)$ for all $j \geq 0$.*

Example 1.3. *In example 1.1, the point $x = 3/4$ is a fixed point, since $f(3/4) = 4 \cdot 3/4(1 - 3/4) = 3/4$.*

Example 1.4. *In example 1, $\alpha = 1/4$, i.e. we consider the rotation by $\pi/2$, all points are periodic with period 4 and all orbits consist of four points: the initial points are the points obtained rotating it by $\pi/2$, π and $3\pi/2$.*

Definition 1.4. *If x is a periodic point, the* minimal period *of x is the minimum integer $n \geq 1$ such that $f^n(x) = x$.*

In particular, if n is the minimal period of x, the points $f(x), \dots, f^{n-1}(x)$ are all different than x. Be aware that in some textbook the *period* of a periodic point x means the *minimal period.*

Definition 1.5. *A point $x \in X$ is* preperiodic *if there exists $k, n \in N$ such that $f^{n+k}(x) = f^k(x)$. In this case $f^{n+j}(f^k(x)) = f^j(f^k(x))$ for all $j \in \mathbb{N}$.*

Exercise 1.1. *Show that if f is invertible every preperiodic point is periodic.*

Examples of questions that are investigated in dynamical systems are:

Q1 Are there fixed points? Are there periodic points?

Q2 Are periodic points dense?

Q3 Is there an orbit which is dense, i.e. an orbit which gets arbitrarily close to any other point in X?

Q4 Are all orbits dense?

We will answer these questions for the first examples in the next lectures. More in general, these properties are studied in *topological dynamics*[3], see Chapter 2.

If an orbit is dense, it visits every part of the space. A further natural question is how much time it spends in each part of the space. For example, let $A \subset X$ be a subset of the space. We can count the number of visits of a segment $\{x, f(x), \dots, f^n(x)\}$ of the orbit $\mathcal{O}^+_f(x)$ to the set A. If χ_A denotes the characteristic function of A, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if} x \ \in A \\ 0 & \text{if } x \notin A \end{cases},$$

let us count the number of visits

$$Card\{0 \leq k < n, \quad \text{such that } f^k(x) \in X\} = \sum_{k=0}^{n-1} \chi_A(f^k(x))$$

and divide by n to get the *frequency* of visits in time n:

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_A(f^k(x)). \tag{1}$$

Intuitively, orbit $\mathcal{O}^+_f(x)$ is *equidistributed* if the frequency in (1) is getting closer and closer, as n increases, to the *volume* of A (or the lenght, or the area, ...)[4]. This means that the orbit asymptotically spends in each part of the space a time proportional to the volume.

Q1 Are orbits equidistributed?

This last question is a main quesition in ergodic theory[5] A priory, not even the existence of a limit of the frequency (1) is guaranteed. One of the main theorems that we will see in Chapter 4, the Birkhoff ergodic theorem, will show that for *almost all* points the limit exists and guarantee that if the system is enough chaotic (more precisely, *ergodic*), than the frequency converge to the expected limit. As we will see, questions related to equidistributions have many connections and applications in number theory.

1.1 Extra: Continuous Dynamical Systems

A continous dynamical system can be given by a 1−parameter family of maps $f_t : X \to X$ where $t \in \mathbb{R}$. The main example is given by solutions of a differential equation. Let $X \subset \mathbb{R}^n$ be a space, $g : X \to \mathbb{R}^n$ a function, $x_0 \in X$ an initial condition and

$$\begin{cases} \dot{x}(t) = g(x) \\ x(0) = x_0 \end{cases} \tag{2}$$

be a differential equation. If the solution $x(x_0, t)$ is well defined, unique and exists for all t and all initial conditions $x_0 \in X$, if we set $f_t(x_0) := x(x_0, t)$ we have an example of a continous dynamical system. In this case, an orbit is given by the trajectory described by the solution:

[3]Topological Dynamics is a branch of dynamics that investigate the properties of *continous* maps.

[4]More in general, we will have a measure on X (lenght, area and volume are all examples of measures) which is preserved by the map f and we will ask if the frequency tends to the measure of A. See Chapter 4.

[5]Ergodic Theory is a branch of dynamics which investigate the chaotic properties of maps which preserves a *measure*.

Definition 1.6. *If* $\{f_t\}_{t\in\mathbb{R}}$ *is a continuous dynamical system, we denote*

$$\mathcal{O}_{f_t}(x) := \{f_t(x), \quad t \in \mathbb{R}\}.$$

More in general, a 1-parameter family $\{f_t\}_{t\in\mathbb{R}}$ is called a *flow* if f_0 is the identity map and for all $t, s \in \mathbb{R}$ we have $f_{t+s} = f_t \circ f_s$, i.e.

$$f_{t+s}(x) = f_t(f_s(x)) = f_s(f_t(x)), \quad \text{for all } x \in X.$$

1.2 Extra: Dynamical systems as actions

A more formal way to define a dynamical system is the following, using the notion of *action.*

Let X be a space and G group (as $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{R}^d$) or a semigroup (as $\mathbb{N}$).

Definition 1.7. *An* action *of* G *on* X *is a map* $\psi : G \times X \to X$ *such that, if we write* $\psi(g, x) = \psi_g(x)$ *we have*

(1) If e *id the identity element of* G, $\psi_e : X \to X$ *is the identity map;*

(2) For all $g_1, g_2 \in G$ *we have* $\psi_{g_1} \circ \psi_{g_2} = \psi_{g_1 g_2}$.[6]

A discrete dynamical systems is then defined as an action of the group $\mathbb{Z}$ or of the semigroup $\mathbb{N}$. A continous dynamical system is an action of $\mathbb{R}$. There are more complicated dynamical systems defined for example by actions of other groups (for example $\mathbb{R}^d$).

Exercise 1.2. *Prove that the iterates of a map* $f : X \to X$ *give an action of* $\mathbb{N}$ *on* X*. The action* $\mathbb{N} \times X \to X$ *is given by*

$$(n, x) \to f^n(x).$$

Prove that if f *is invertible, one has an action of* $\mathbb{Z}$*.*

Exercise 1.3. *Prove that the solutions of a differential equation as (2) (assuming that for all points* $x_0 \in X$ *the solutions are unique and well defined for all times) give an action of* $\mathbb{R}$ *on* X*.*

2 Rotations of the circle: periodic points and dense orbits

Consider a circle of unit radius. More precisely, we will denote by S^1 the set

$$S^1 = \{(x, y)| \quad \sqrt{x^2 + y^2} = 1\} \subset \mathbb{R}^2.$$

Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we can also write

$$S^1 = \{e^{2\pi i\theta}, \quad 0 \le \theta < 1\} \subset \mathbb{C}.$$

Consider a rotation R_α of angle $2\pi\alpha$ on the circle (see Figure 1). It is given by

$$R_\alpha(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)} = e^{2\pi i\alpha} e^{2\pi i\theta}.$$

Since complex numbers in S^1 are multiplied by $e^{2\pi i\alpha}$, this is known as *multiplicative notation* for the rotation R_α.

[6] If X has an additional structure (for example X is a topological space or X is a measured space), we can ask the additional requirement that for each $g \in G$, $\psi_g : X \to X$ preserves the structure of X (for example ψ_g is a continous map if X is a topological space or ψ_g preserves the measure. We will see more precisely these definitions in Chapters 2 and 4.

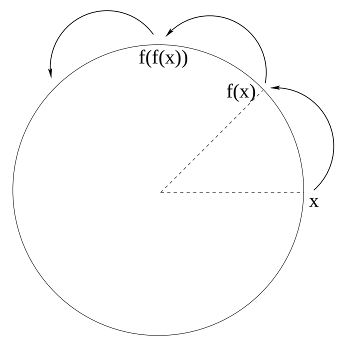

Figure 1: A rotation of S^1.

There is a natural distance $d(z_1, z_2)$ between points on S^1, which is given by the *arc lenght* distance. We will renormalized it by dividing by 2π. For example, if $0 \le \theta_1 < \theta_2$ and $2\pi(\theta_2 - \theta_1) < \pi$ we have

$$d(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) = \frac{\text{arc lenght distance} = \ 2\pi(\theta_2 - \theta_1)}{2\pi} = \theta_2 - \theta_1.$$

This is clear by the geometric meaning that, since both points are rotated by the same angle $2\pi\alpha$, this distance is preserved i.e.

$$d(R_\alpha(z_1), R_\alpha(z_2)) = d(z_1, z_2), \qquad \text{for all } z_1, z_2 \in S^1.$$

Thus, the rotation of the circle is an example of an *isometry*, i.e. a map which preserves a distance.

There is another alternative way to describe a circle, that will be often more convenient. Imagine to *cut open* the circle to obtain an interval. Let $I/\sim$ denote the unit interval with the endpoints identified: the symbol $\sim$ recalls us that $0 \sim 1$ are glued together. Then $I/\sim$ is equivalent to a circle. More formally, consider $\mathbb{R}/\mathbb{Z}$, i.e. the space whose points are equivalence classes $x + \mathbb{Z}$ of real numbers x up to integers: two reals $x_1, x_2 \in \mathbb{R}$ are in the same equivalence class iff there exists $k \in \mathbb{Z}$ such that $x_1 = x_2 + k$. Then $\mathbb{R}/\mathbb{Z} = I/\sim$ since the unit interval $I = [0, 1]$ contains exactly one representative for each equivalence class with the only exception of 0 and 1, which belong to the same equivalence class, but are identifyed.

The map $\Psi : \mathbb{R}/\mathbb{Z} \to S^1$ given by

$$x \xrightarrow{\Psi} \Psi(x) = e^{2\pi i x} \tag{3}$$

establishes a one-to-one correspondence between $\mathbb{R}/\mathbb{Z}$ and S^1. The distance given by arc lenght divided by 2π, gives the following distance on $\mathbb{R}/\mathbb{Z}$:

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}. \tag{4}$$

Thus, we will use the same symbol d for both distances.

Exercise 2.1. *Check that the arc lenght distance divided by 2π becomes the distance in (4) on $\mathbb{R}/\mathbb{Z}$ under the identification given by Ψ, i.e.*

$$d(x, y) = \frac{\textit{arc lenght \ between } \Psi(x) \textit{ and } \Psi(y)}{2\pi}.$$

The rotation R_α, under this identification between S^1 and $\mathbb{R}/\mathbb{Z}$ becomes the map R_α : $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by

$$R_\alpha = x + \alpha \mod 1,$$

where $\mod 1$ means that we subtract the integer part (for example $3.14 \mod 1 = 0.14$), hence taking the representative of the equivalence class $x + \alpha + \mathbb{Z}$ which lies in $[0, 1)$. We call α the rotation number of R_α (remark that the rotation angle is $2\pi\alpha$). More explicitely, if $\alpha \in [0, 1]$ we have

$$R_\alpha = \begin{cases} x + \alpha & \text{if } x + \alpha < 1 \\ x + \alpha - 1 & \text{if } x + \alpha \geq 1 \end{cases}$$

We call this *additive notation* (since here the rotation becomes addition $\mod 1$).

Rotations of the circle display a very different behaviour according if the rotation number α is rational ($\alpha \in \mathbb{Q}$) or irrational ($\alpha \in \mathbb{R}\backslash\mathbb{Q}$). Recall that

Definition 2.1. *The orbit $\mathcal{O}_f(z_1)$ is* dense *if for all $z_2 \in S^1$ and for all $\epsilon > 0$ there exists $n > 0$ such that $R_\alpha^n(z_1) \in B(z_2, \epsilon)$ where $B(z_2, \epsilon)$ is the ball of radius ϵ and center z_2, i.e. $B(z_2, \epsilon) = \{z \in S^1 | \quad d(z, z_2) < \epsilon\}$.*

Theorem 2.1 (Dichotomy for Rotations)**.** *Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rotation of the circle.*

(1) If $\alpha = p/q$ is rational, with $p, q \in \mathbb{Z}$, all orbits are periodic of period q;

(2) If α is irrational, for every point $z \in S^1$ the orbit $\mathcal{O}_{R_\alpha}(z)$ is dense.

In both cases the dynamics of the rotation is quite simple to describe: either all orbits are periodic, or all orbits are dense.

Proof. Let $\alpha = p/q$ with $p, q \in \mathbb{Z}$. Then for each $x \in \mathbb{R}/\mathbb{Z}$

$$R_\alpha^q(x) = x + q\frac{p}{q} \mod 1 = x + p \mod 1 = x.$$

Thus every point is periodic of period q. This proves (1).

Let us now prove (2). In this case it will be convenient to work on S^1 and use multiplicative notation. Assume that α is irrational. In particular, for each $z_1 = e^{2\pi i x_1} \in S^1$, for all $m \neq n$, $R_\alpha^n(e^{2\pi i x_1}) \neq R_\alpha^m(e^{2\pi i x_1})$. Indeed, if they were equal, $e^{2\pi i(x_1+m\alpha)} = e^{2\pi i(x_1+n\alpha)}$ thus $2\pi(x_1 + m\alpha) = 2\pi(x_1 + n\alpha) + 2\pi k$ for some integer $k \in \mathbb{N}$. Thus, $m\alpha = n\alpha + k$. But this shows that $\alpha = k/(m - n)$, contradicting the assumption that α is irrational.

To show that the orbit of $z_1 \in S^1$ is dense, we have to show that for each $z_2 \in S^1$ and $\epsilon > 0$ there is a point of $\mathcal{O}_f(z_1)$ inside the ball $B(z_2, \epsilon)$. Let N be big enough so that $1/N < \epsilon$. Consider the points $z_1, R_\alpha(z_1), \ldots, R_\alpha^{N-1}(z_1)$. Since as we proved before they are all distinct, by Pigeon Hole principle, there exists n, m such that $0 \leq n < m \leq N$ and

$$d(R_\alpha^n(z_1), R_\alpha^m(z_1)) \leq \frac{1}{N} < \epsilon.$$

This means that for some θ with $|\theta| < 1/N$ we have

$$R_\alpha^m(z_1) = e^{2\pi i\theta} R_\alpha^n(z_1) \;\Leftrightarrow\; e^{2\pi i m\alpha} z_1 = e^{2\pi i\theta} e^{2\pi i n\alpha} z_1 \;\Leftrightarrow\; \frac{e^{2\pi i m\alpha}}{e^{2\pi i n\alpha}} = e^{2\pi i\theta} \tag{5}$$

Consider now R_α^{m-n}. We claim that it is again a rotation by an agle smaller than ϵ. Indeed, from (**??**) we see that

$$R_\alpha^{m-n}(z_1) = e^{2\pi i m\alpha} e^{-2\pi i n\alpha} z_1 = \frac{e^{2\pi i m\alpha}}{e^{2\pi i n\alpha}} z_1 = e^{2\pi i\theta} z_1$$

R_α^{m-n} is a rotation by θ where $|\theta| < 1/N$, so

Thus, if we consider multiples $R_\alpha^{(m-n)}(z_1)$, $R_\alpha^{2(m-n)}(z_1)$, $R_\alpha^{3(m-n)}(z_1), \ldots$ we obtain points

$$e^{2\pi i x_1}, e^{2\pi i(x_1+\theta)}, e^{2\pi i(x_1+2\theta)}, \ldots, e^{2\pi i(x_1+k\theta)}, \ldots,$$

whose spacing on S^1 is less than ϵ. Thus, there will be a $j > 0$ such that $R_\alpha^{j(m-n)}(z_1)$ enters the ball $B(z_2, \epsilon)$. □

Exercise 2.2. *Prove that if $\alpha = p/q$ and $(p, q) = 1$ i.e. p and q are coprime, than q is the minimal period, i.e. for each $x \in \mathbb{R}/\mathbb{Z}$ we have $R_\alpha^k(x) \neq x$ for each $1 \leq k < q$.*

Remark 2.1. *If $\alpha = p/q$ and $(p, q) = 1$ then $|p|$ gives the* winding number*, i.e. the number of "'turns"' that the orbit of any point does around the circle S^1 before closing up.*