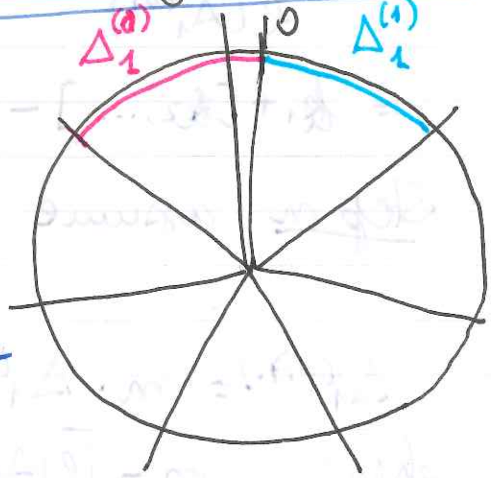
Partition of S^1 for continued fraction

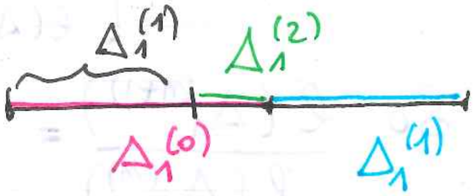
PCF1

- Ref: Sinai – Topics in Ergodic Theory –

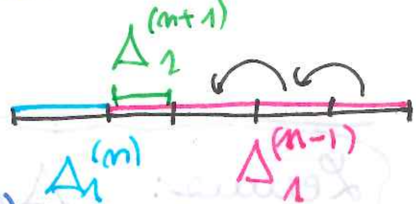


- Construction:

Step ⓪ Consider the rotation by α. Call $\Delta_1^{(0)} = [0, \alpha]$. Iterate till the last step before overpassing zero. Call the reminder $\Delta_1^{(1)}$.



Step ① Cut $\Delta_1^{(1)}$ from the opposite vertex of $\Delta_1^{(0)}$ (not 0) as much as possible and call $\Delta_1^{(2)}$ the reminder



Step n Suppose you have $\Delta_1^{(n)}$ $\Delta_1^{(n-1)}$ lying on opposite sides of 0, $\ell(\Delta_1^{(n-1)}) > \ell(\Delta_1^{(n)})$. Cut $\Delta_1^{(n)}$ from the opposite side of $\Delta_1^{(n-1)}$ as much as possible, call $\Delta_1^{(n+1)}$ the reminder.

- Denote by $[a_1, \dots, a_n \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots \cfrac{1}{a_n + \dots}}}$ the continued fraction expansion.

Lemma: $\dfrac{\ell(\Delta_1^{(n)})}{\ell(\Delta_1^{(n-1)})} = T^n \alpha = [a_n, a_{n+1}, \dots]$

Gauss

Proof: $1 = a_1 \cdot \ell(\Delta_1^{(0)}) + \ell(\Delta_1^{(1)})$

So $$\frac{\ell(\Delta_1^{(1)})}{\ell(\Delta_1^{(0)})} = \frac{1 - a_1 \alpha}{\alpha} = \frac{1 - a_1 \cdot \frac{1}{a_1 + [a_2, \dots]}}{\frac{1}{a_1 + [a_2, \dots]}}$$

$$= a_1 + [a_2, \dots] - a_1 = [a_2, \dots]$$ ok

Step n assume $\dfrac{\ell(\Delta_1^{(n)})}{\ell(\Delta_1^{(n-1)})} = [k_n, \dots]$

$\Delta_1^{(n-1)} = m \cdot \Delta_1^{(n)} + \Delta_1^{(n+1)}$ by construction

where $m = \left[\dfrac{\ell(\Delta_1^{n-1})}{\ell(\Delta_1^{(n)})}\right] = \left[\dfrac{1}{T^n \alpha}\right] \overset{\text{def}}{=} k_n$

So $$\frac{\ell(\Delta_1^{(n+1)})}{\ell(\Delta_1^{(n)})} = \frac{\ell(\Delta_1^{(n-1)}) - k_n \ell(\Delta_1^{(n)})}{\ell(\Delta_1^{(n)})} =$$

$$= \frac{\ell(\Delta_1^{(n-1)})}{\ell(\Delta_1^{(n)})} - k_n = \cancel{k_n} + [k_{n+1}, \dots] - \cancel{k_n}$$ qed

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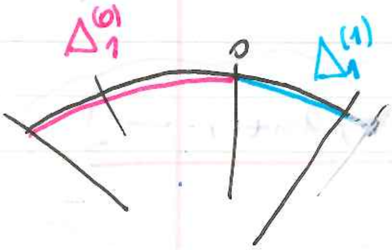
Lemma: $$\Delta_1^{(n)} = \begin{cases} [0, R_\alpha^{q_n} 0] & n \text{ odd} \\ [R_\alpha^{q_n} 0, 0] & n \text{ even} \end{cases}$$

Proof: $\boxed{n=0}$ $\Delta_1^{(0)} = [0, \alpha] = [0, R_\alpha \cdot 0]$ $\quad q_0 = 1$

$\boxed{n=1}$ $\Delta_1^{(1)} = [R_\alpha^{q_1} \cdot 0, 0]$ $\quad q_1 = \left[\frac{1}{\alpha}\right]$ maximum number of iteration before going over 0

$\boxed{n=2}$ $R_\alpha \cdot \Delta_1^{(1)}$ is the first interval starting from the left endpoint of $\Delta_1^{(0)}$



Than we have

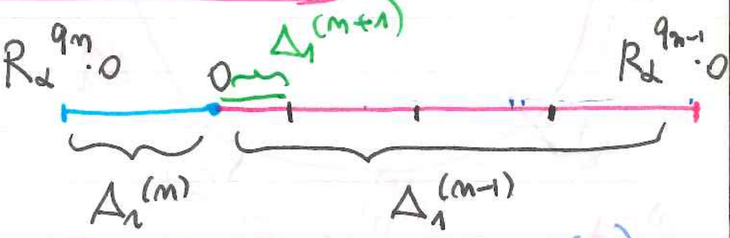
$R_\alpha \cdot R_\alpha^{q_1} \Delta_1^{(1)}, \dots, R_\alpha \cdot R_\alpha^{q_1 (k_2 - 1)}$

The left endpoint of the last is:

$$R_\alpha \cdot R_\alpha^{q_1(k_2-1)} \cdot \underbrace{R_\alpha^{q_1}\cdot 0}_{\text{left endpoint of } \Delta_1^{(1)}} = R_\alpha^{q_1 \cdot k_2 + \underset{1}{\overset{\shortparallel}{q_0}}}\, 0 = R_\alpha^{q_2}\cdot 0$$

By using $\boxed{q_{n+1} = a_{n+1}\, q_n + q_{n-1}}$ ok

$\boxed{\text{Inductive step:}}$



$R_\alpha^{q_{n-1}} \cdot \Delta_1^{(n)} =$

$= R_\alpha^{q_{n-1}} [R_\alpha^{q_n}\cdot 0, 0]$ is the first copy of $\Delta_1^{(n)}$ starting from the opposite endpoint of $\Delta_1^{(n-1)}$

The following are:

$$R_\alpha^{q_{n-1}+q_n}\Delta_1^{(n)},\ R_\alpha^{q_{n-1}+2q_n}\cdot\Delta_1^{(n)},\ \dots,\ R_\alpha^{q_{n-1}+(a_{n+1}-1)q_n}\Delta_1^{(n)}$$

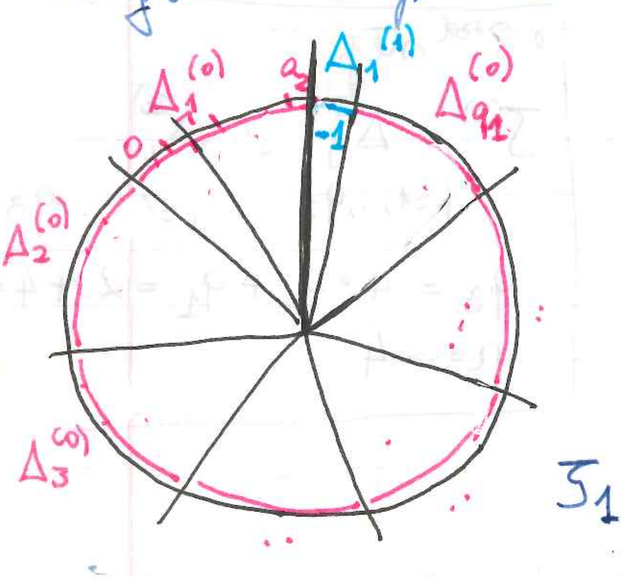
So the endpoint of the remainder $\Delta_1^{(n+1)}$ is

$$R_\alpha^{q_{n-1}+(a_{n+1}-1)q_n}\cdot R_\alpha^{q_n}\cdot 0 = R_\alpha^{\underbrace{q_{n-1}+a_{n+1}\cdot q_n}}\cdot 0$$
$$= R_\alpha^{q_{n+1}}\cdot 0 \qquad \text{qed}$$

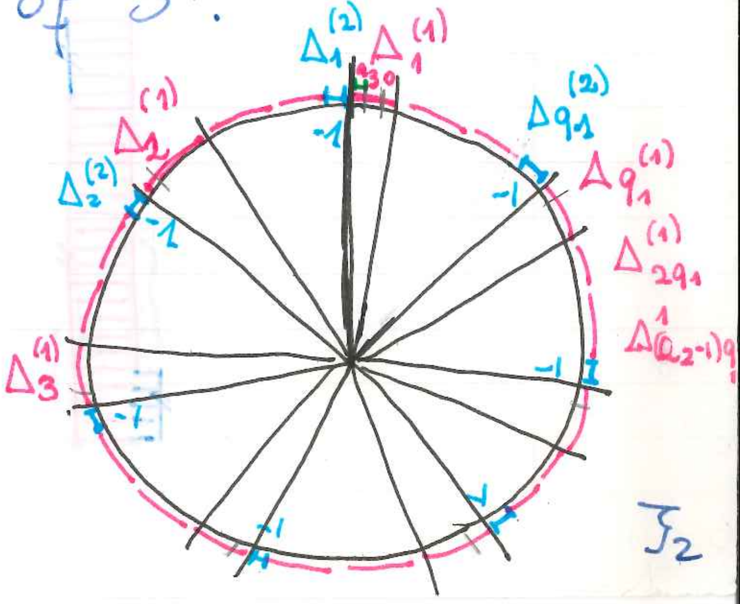
- Partitions: Put $\Delta_i^{(n)} \doteqdot R_\alpha^{i-1}\cdot\Delta_1^{(n)}$

$$\boxed{\mathcal{J}_n = \{\Delta_i^{(n-1)},\ 1\le i\le q_n\}\cup\{\Delta_i^{(n)},\ 1\le i\le q_{n-1}\}}$$

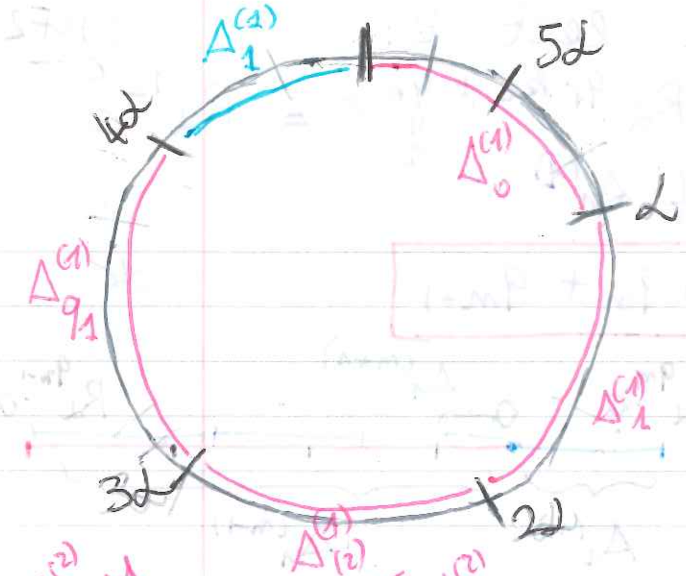
It gives a partition of S^1.

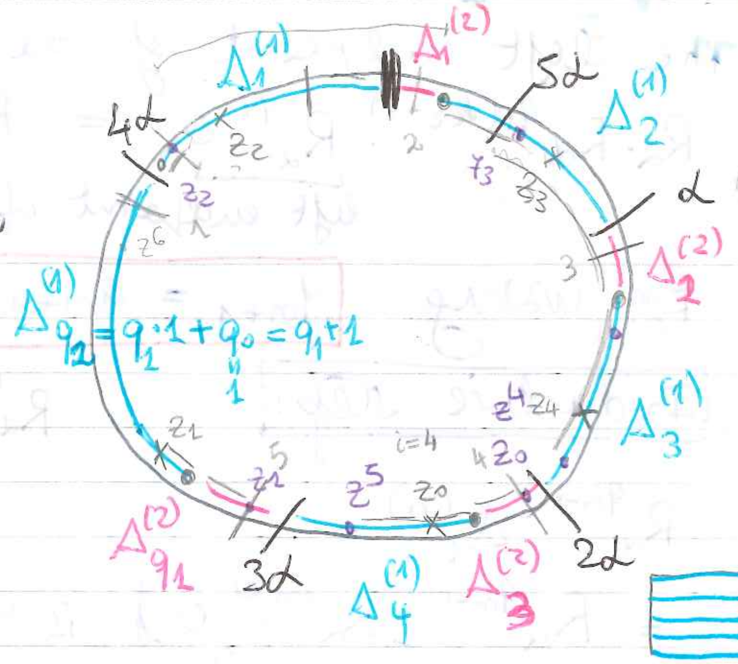


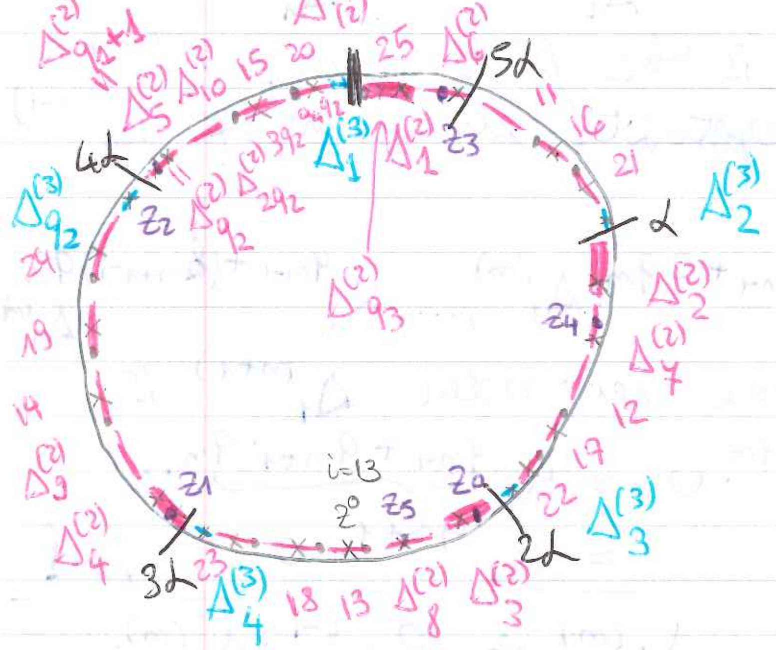
$\mathcal{J}_1$



$\mathcal{J}_2$







$\# \text{rosa} = a_2'$

- **1st step**: $a_0 =$

$q_0 = 1 \qquad q_1 = 4$

$\mathfrak{J}^{(0)} = \Delta^{(1)}_1 \cup \Delta^{(2)}_i$

$i = 1, \dots, q_1$

- **2nd step**

$\mathfrak{J}^{(1)} = \Delta^{(2)}_i \cup \Delta^{(1)}_j$

$i = 1, \dots, q_1 \qquad j = 1, \dots, q_2$

$a_1 = 1 \qquad q_2 = 1 \cdot q_1 + \underset{1}{q_0} = 5$

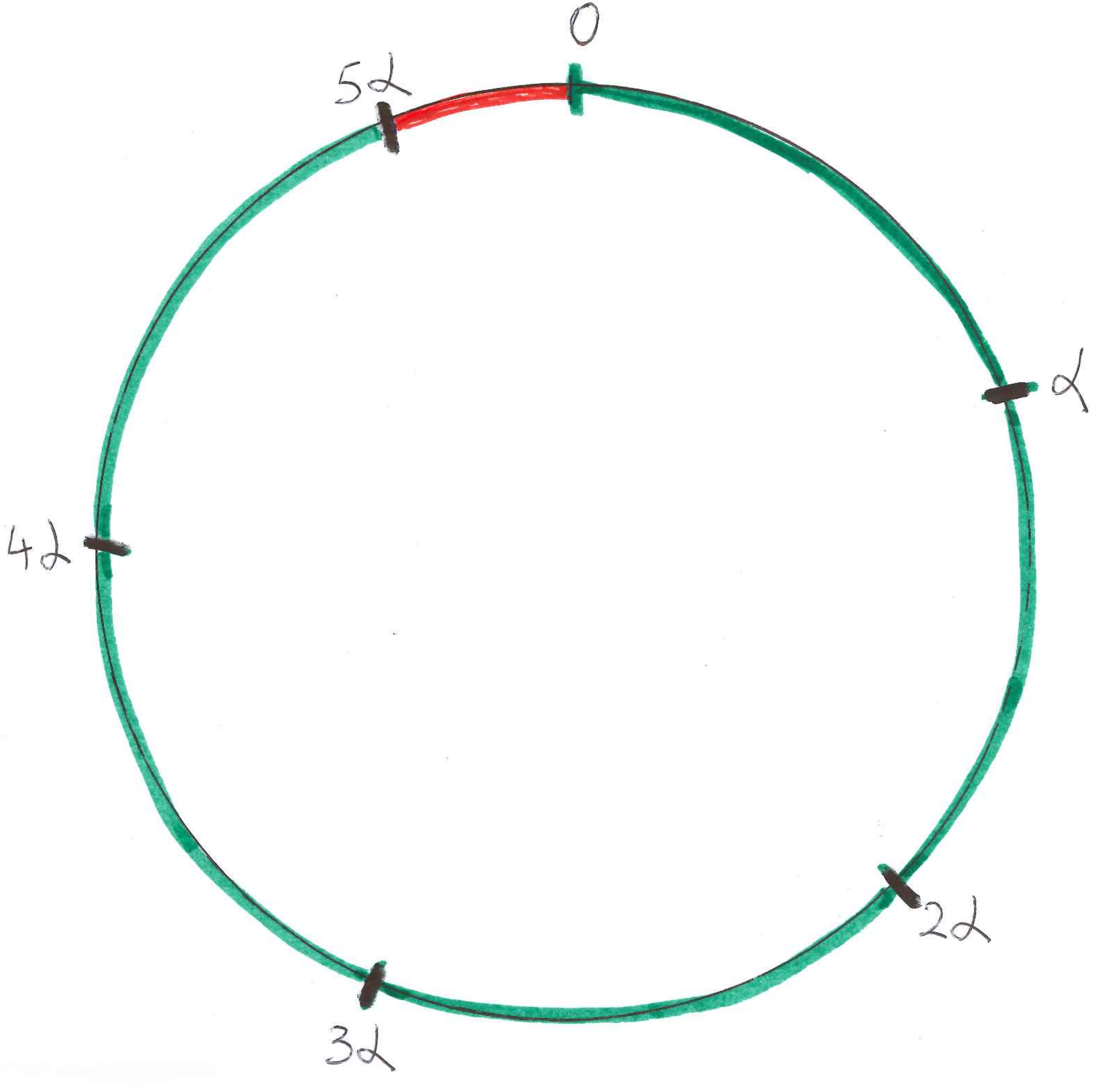
- **3rd step**

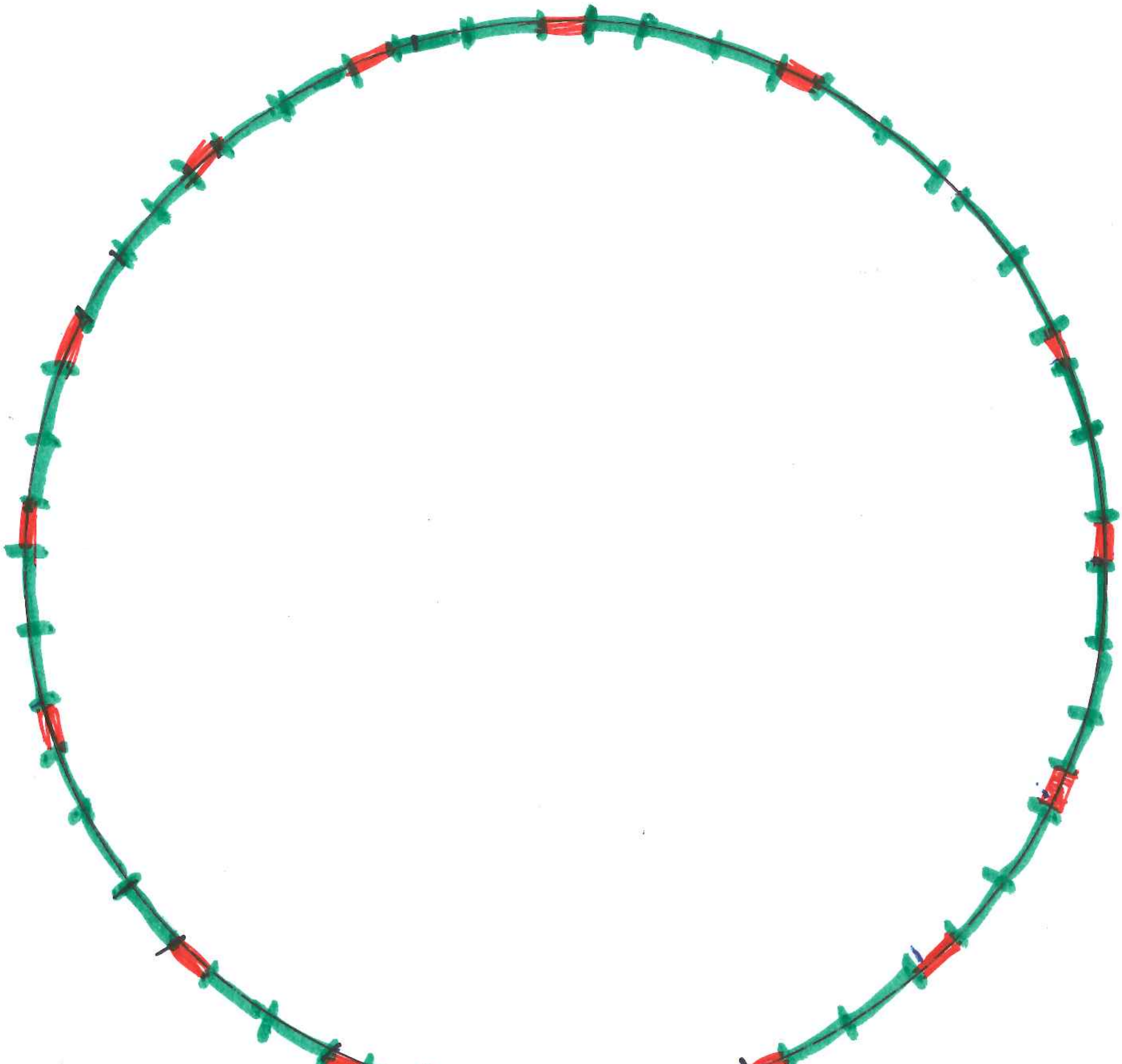
$\mathfrak{J}^{(2)} = \Delta^{(3)}_j \cup \Delta^{(2)}_i$

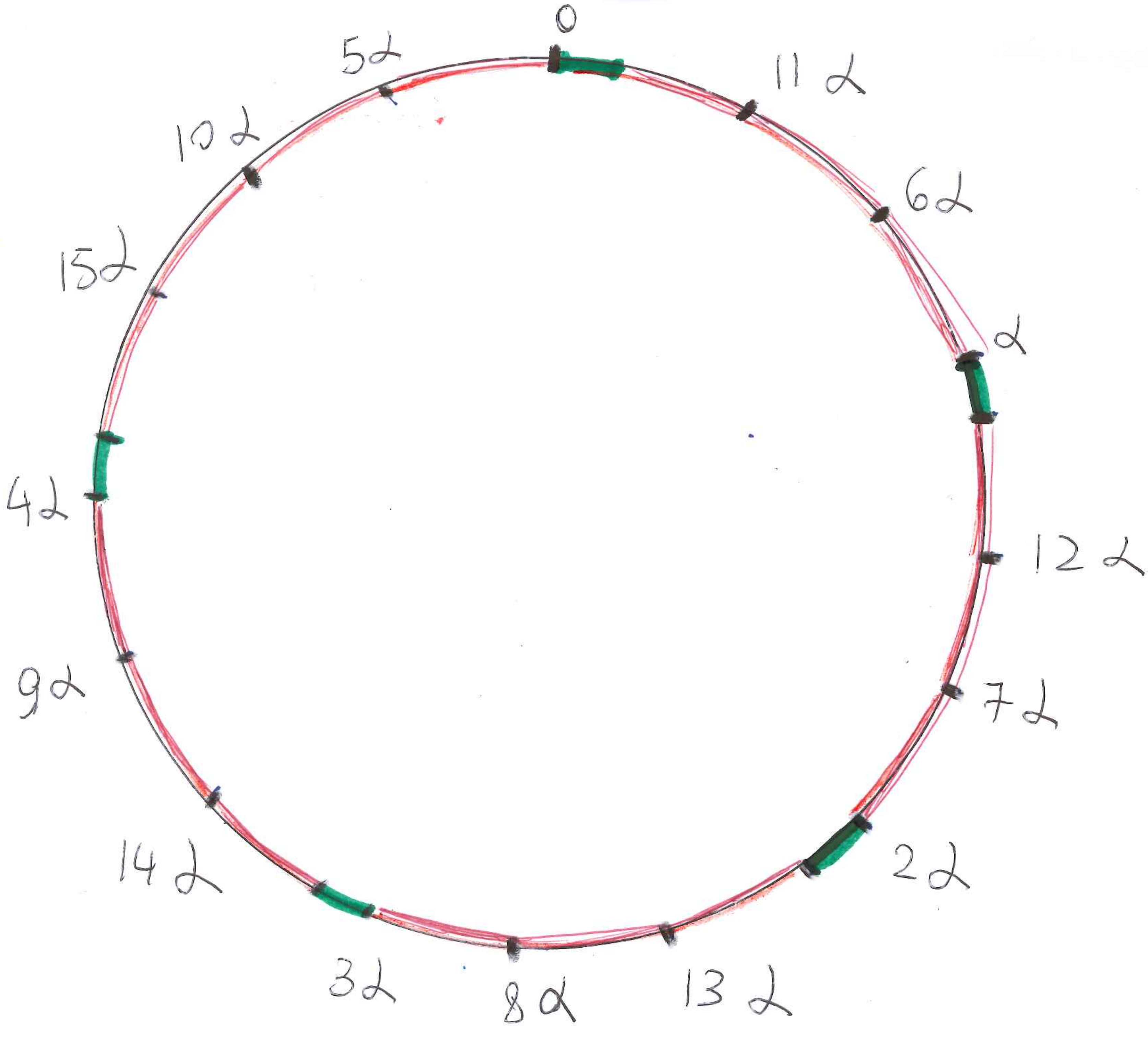
$j = 1, \dots, q_2 \qquad i = 1, \dots, q_3$

$q_3 = 4 \cdot q_2 + q_1 = 20 + 4 = 24$

$a_3 = 4$


0
α
2α
3α
4α
5α





0
11α
6α
α
12α
7α
2α
13α
8α
3α
14α
9α
4α
15α
10α
5α