Inducing and renormalization for rotations

We defined several tools:

- An renormalization procedure;

(induce R_α on $I^{(n)} = \Delta^{(n)} \cup \Delta^{(n+1)}$)

- An algorithm;

(cut $\Delta^{(n)}$from right, cut from left...)

- Partitions of the circle;

- Towers and cutting and stacking;

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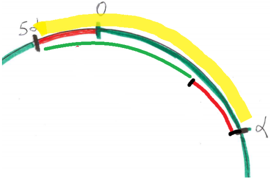

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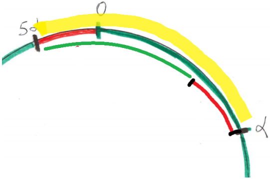

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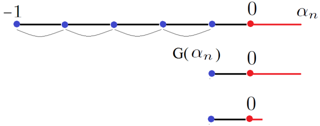

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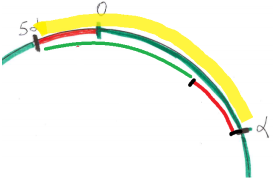

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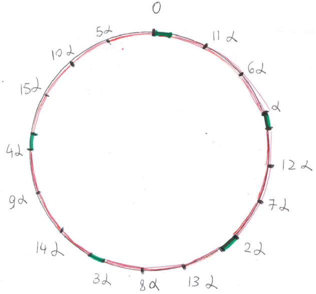

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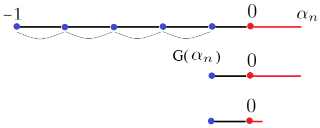

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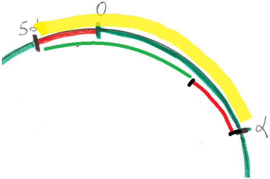

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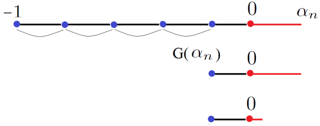

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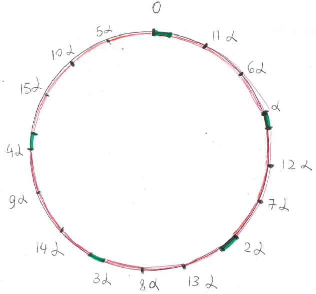

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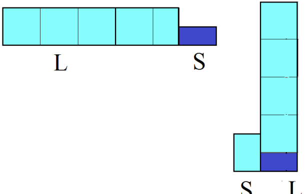

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- Three gaps theorem (Steinhaus theorem) for rotations;
  (using towers)
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  (using the renormalization procedure) [*Ref: van Strien-de Melo book*]
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Homeomorphisms and diffeomorphisms of the circle

Let $f : S^1 \to S^1$ be a homeomorphism of the circle.

- Rotation number $\rho(f) := \lim_{n\to\infty} \frac{F^n(x)-x}{n}$,

  (where F is lift of f and x any point in S^1)

[*Rk*: $\rho(f) \in \mathbb{Q}$ if and only if f has periodic points.]

Theorem (Poincaré)

If f has no periodic points, there exists an (irrational) rotation R_α (where $\alpha = \rho(f)$) and a semi-conjugacy h between f and R_α (h monotone and surjective).

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \downarrow h & & \downarrow h \\ S^1 & \xrightarrow{R_\alpha} & S^1 \end{array}$$

Theorem (Denjoy)

If in addition $f \in \mathscr{C}^2$ (or $\mathscr{C}^1$ and f' has bounded variation), then h is a conjugacy.

Theorem (Herman, see also Sinai-Khanin)

If in addition $f \in \mathbb{C}^{2+\nu}$ for some $\nu > 0$ and $\alpha = [a_0, a_1, \dots]$ satisfies $a_n \le Cn^\gamma$ for some $C, \gamma > 0$ (true for a.e. α) then the conjugacy h is $\mathscr{C}^1$.

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Corinna Ulcigrai

A Central Limit Theorem for cocycles over rotations

(based on joint work with Michael Bromberg)

University of BRISTOL

Deterministic walk driven by a rotation

CLT (Central Limit Theorem): central feature in probability theory / hyperbolic dynamical systems; what about entropy zero dynamics?

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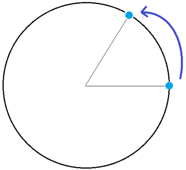

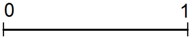

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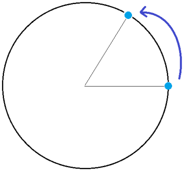

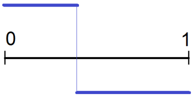

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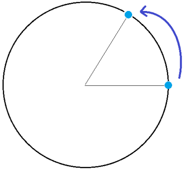

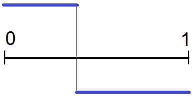

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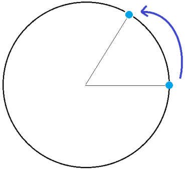

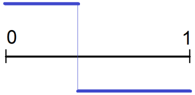

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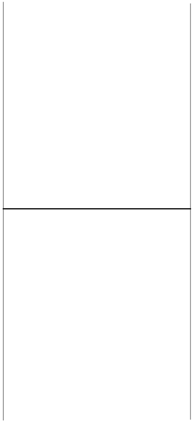

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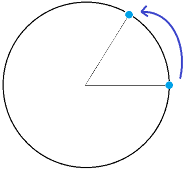

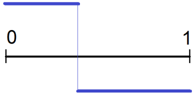

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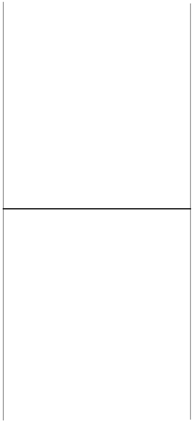

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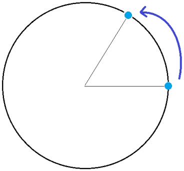

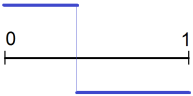

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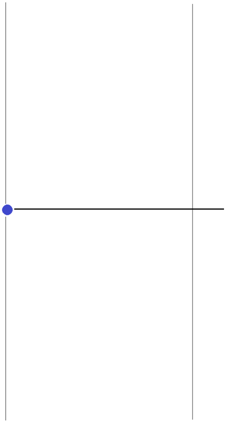

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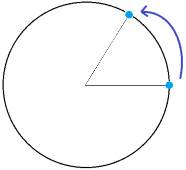

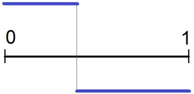

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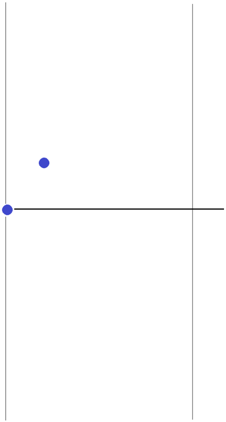

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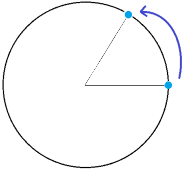

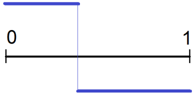

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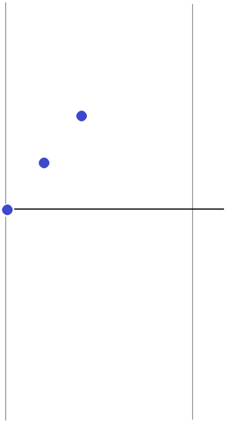

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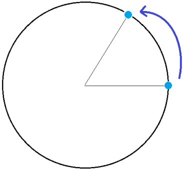

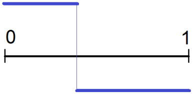

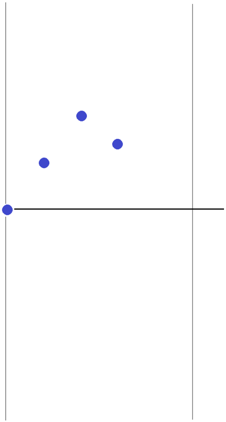

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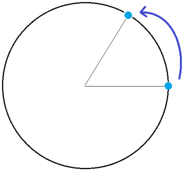

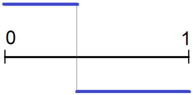

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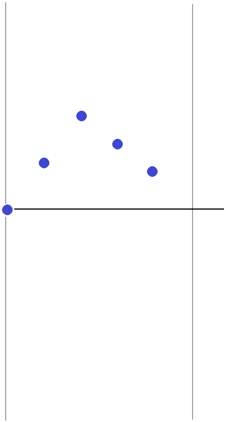

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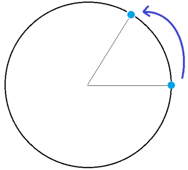

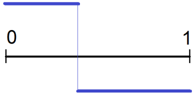

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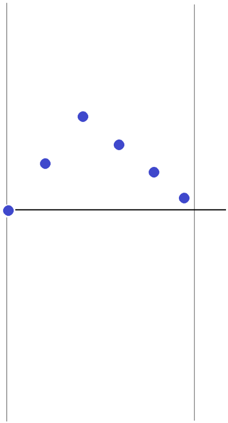

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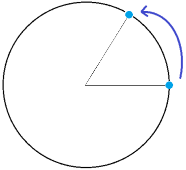

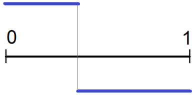

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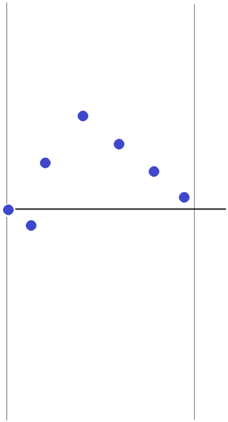

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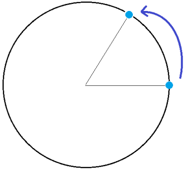

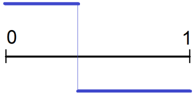

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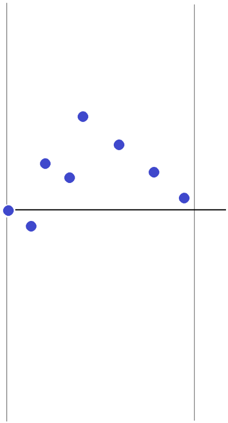

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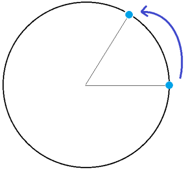

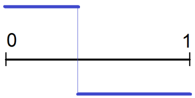

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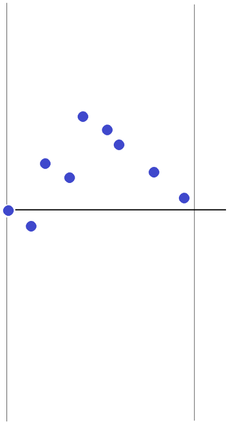

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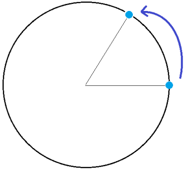

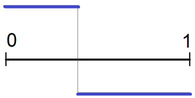

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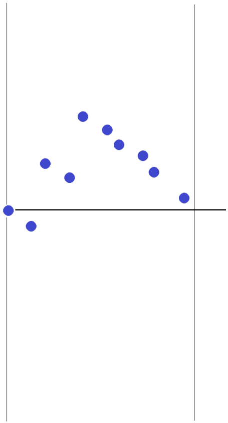

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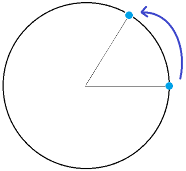

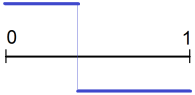

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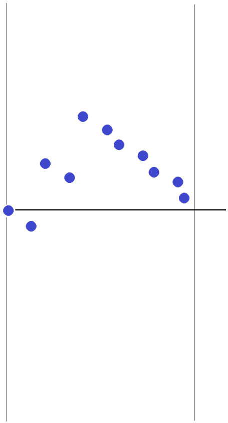

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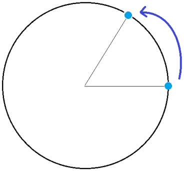

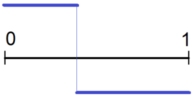

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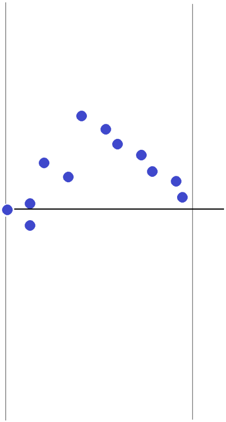

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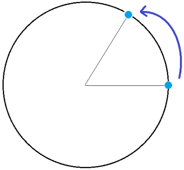

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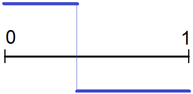

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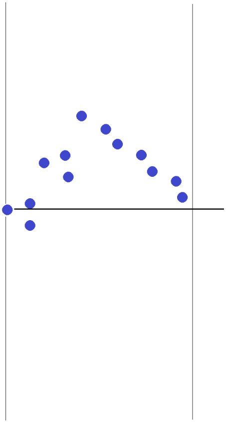

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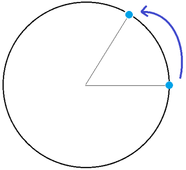

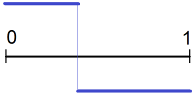

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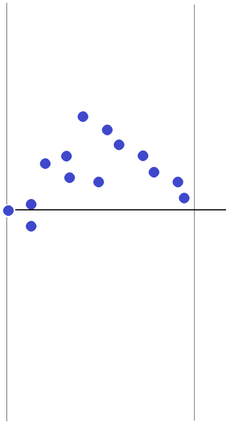

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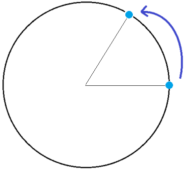

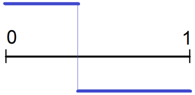

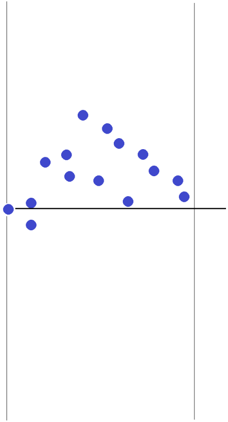

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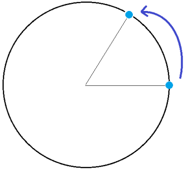

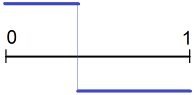

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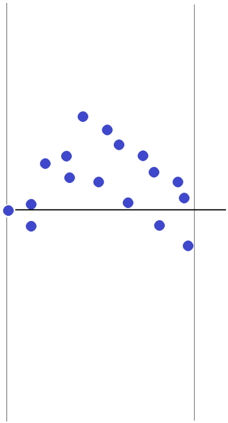

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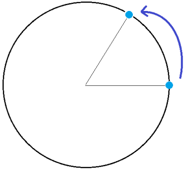

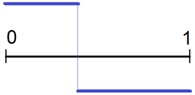

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Walk on $\mathbb{R}$ driven by R_α and f:

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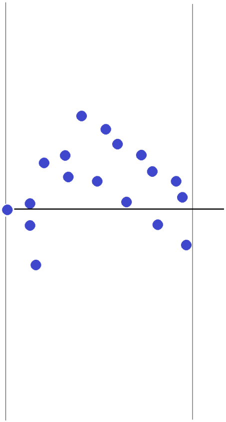

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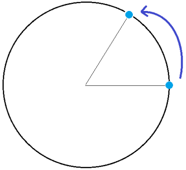

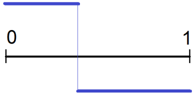

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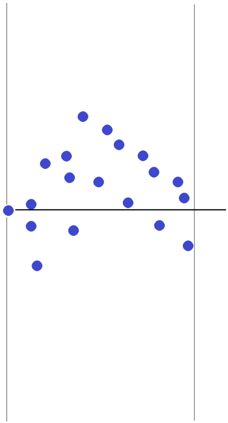

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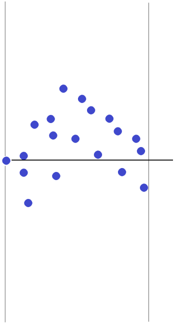

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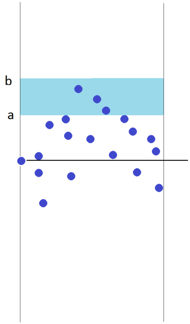

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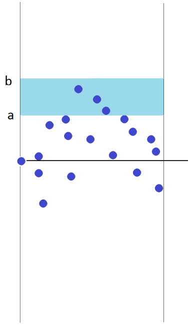

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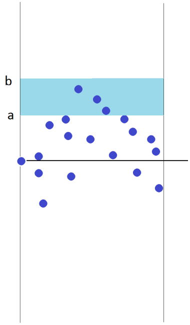

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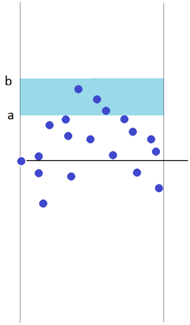

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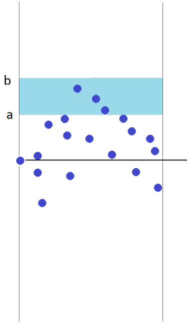

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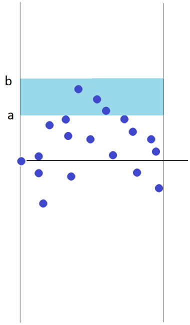

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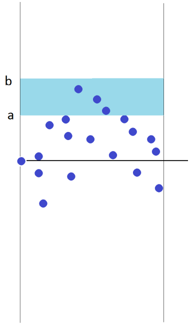

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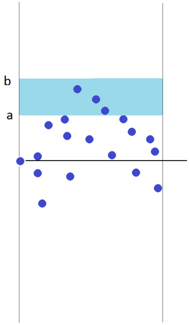

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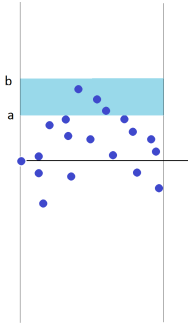

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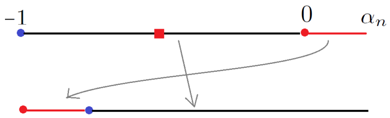

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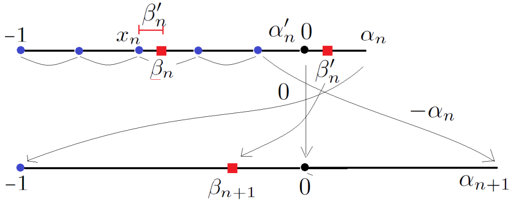

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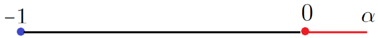

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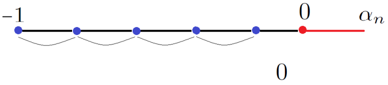

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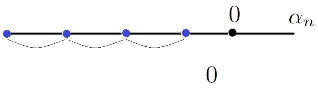

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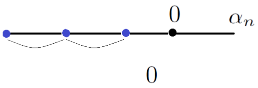

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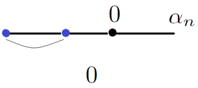

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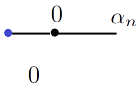

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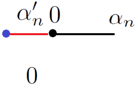

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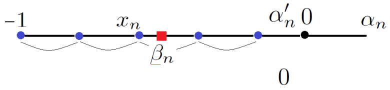

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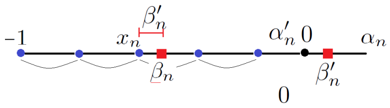

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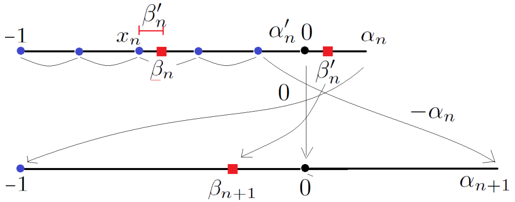

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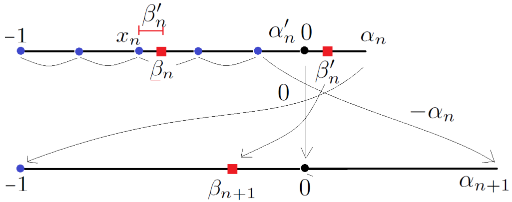

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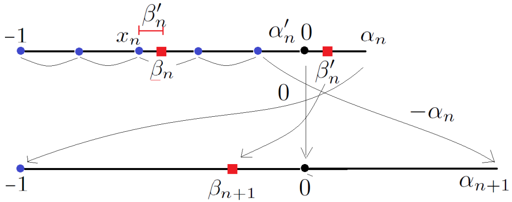

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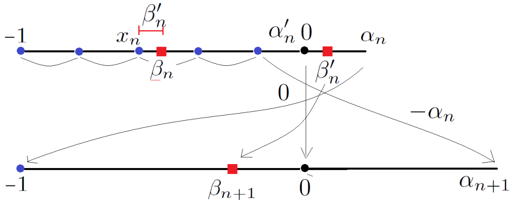

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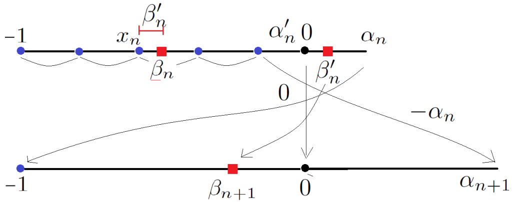

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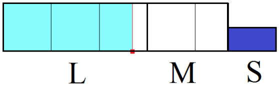
Towers and cutting and stacking for Ostrowski towers

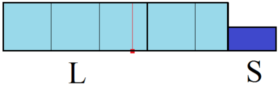
The Ostrowsky renormalization algorithm gives a presentation of R_α as a sequence of 3 Rohlin towers over the induced maps.



- For the *rotation*, there are only two towers (of heigts q_n and q_{n+1}): a *large* one (L) and a *small* (S) one.
- Cut them into 3 towers by the position of β_n.
- Call them $\{L, M, S\}$ for *large*, *middle*, *small*.
- From stage n to $n+1$, do *cutting and stacking*.

Towers and cutting and stacking for Ostrowski towers

The Ostrowsky renormalization algorithm gives a presentation of R_α as a sequence of 3 Rohlin towers over the induced maps.



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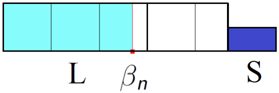

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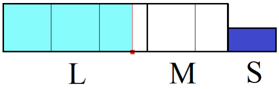

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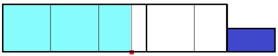

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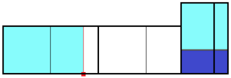

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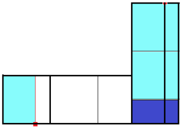

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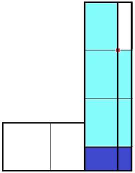

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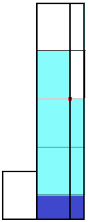

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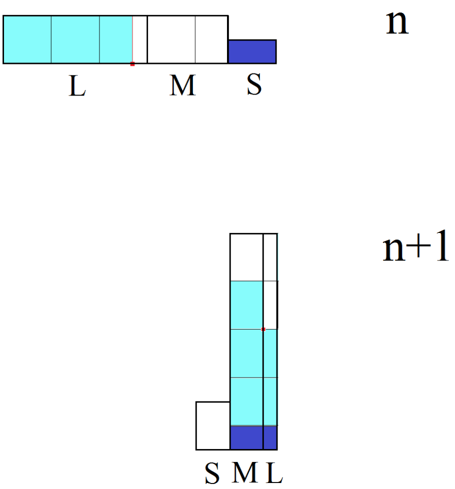

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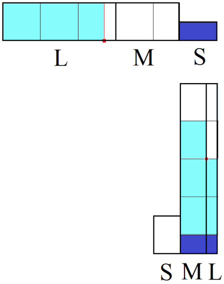

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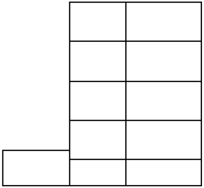

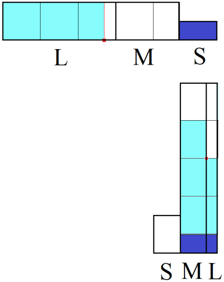

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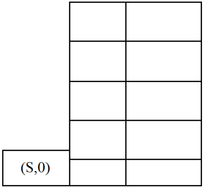

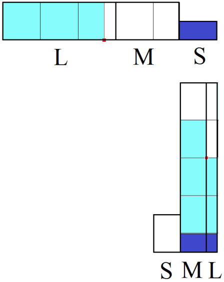

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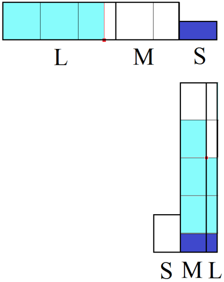

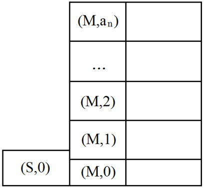

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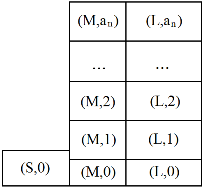

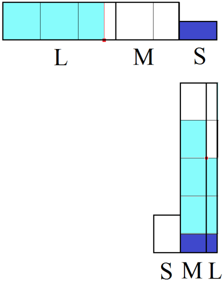

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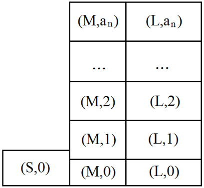

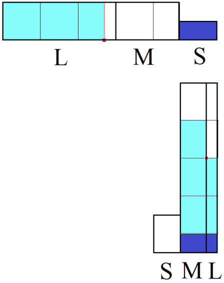

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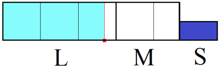

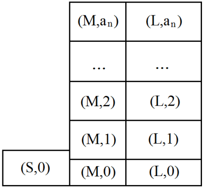

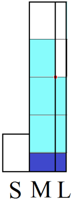

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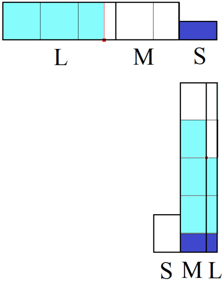

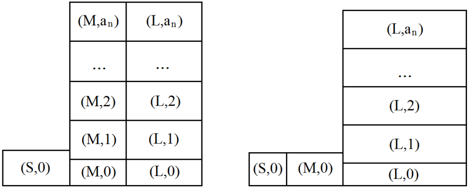

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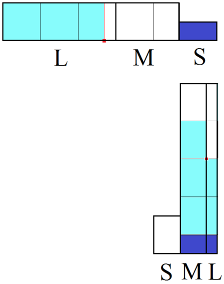

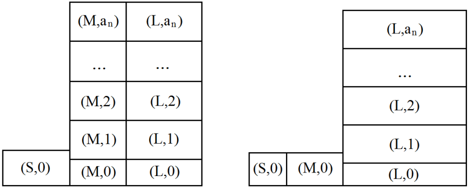

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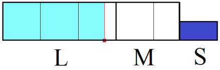

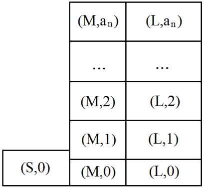

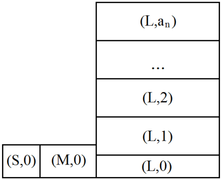

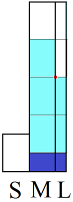

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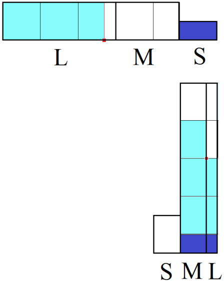

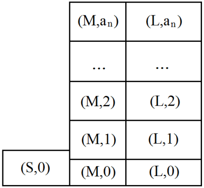

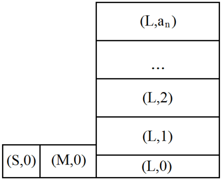

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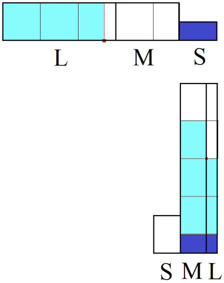

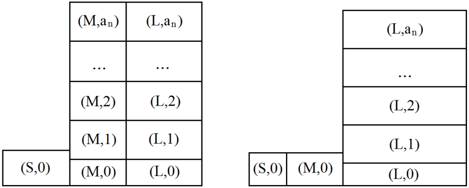

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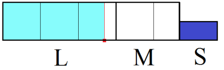

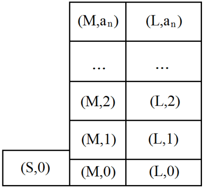

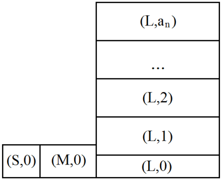

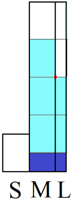

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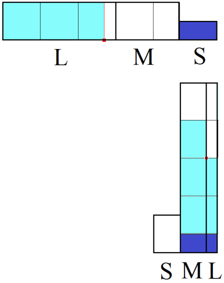

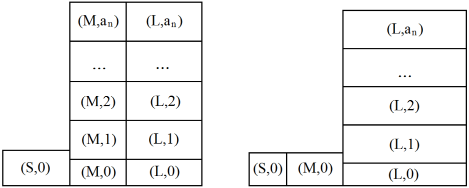

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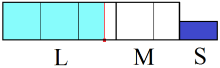

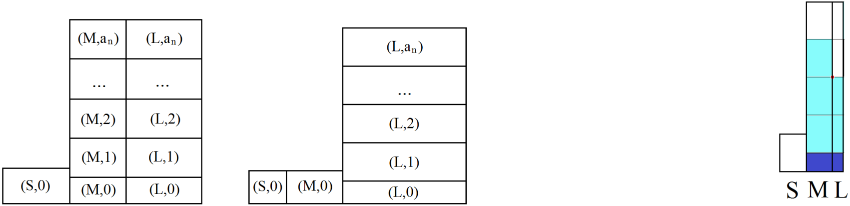

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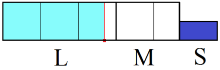

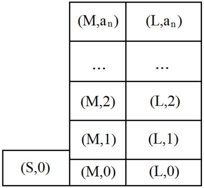

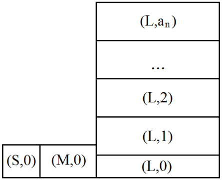

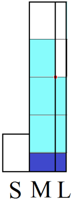

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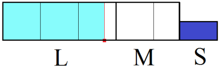

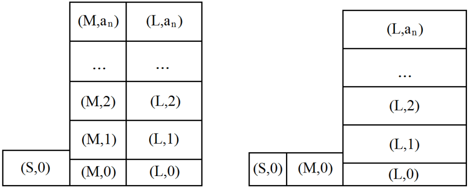

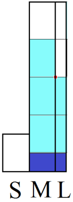

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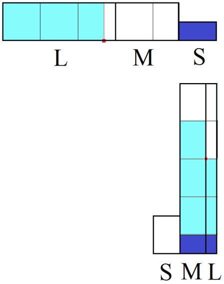

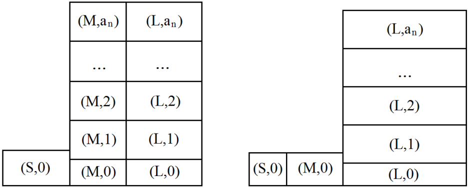

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Sample of results which can be proved using these tools:

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  (using towers)
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- Rotation numbers for homeos and diffeos of S^1
  (using the renormalization procedure) [*Ref: van Strien-de Melo book*]
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MFO Oberwolfach
Dynamische Systeme

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A Central Limit Theorem for cocycles over rotations

(based on joint work with Michael Bromberg)

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University of BRISTOL