classification

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# Dynamical systems Expanding maps on the circle. Classification

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ICTP

2018

coding ●○○○○○○

semiconjugacy

classification

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# expanding maps are factors of $\sigma$

classification

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semiconjugacy

# expanding maps are factors of $\sigma$

#### theorem

•  $f : \mathbb{S}^1 \to \mathbb{S}^1$  expanding map

classification

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semiconjugacy

# expanding maps are factors of $\sigma$

- $f : \mathbb{S}^1 \to \mathbb{S}^1$  expanding map
- deg(*f*) = 2

classification

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semiconjugacy

# expanding maps are factors of $\sigma$

- $f : \mathbb{S}^1 \to \mathbb{S}^1$  expanding map
- deg(*f*) = 2
- $\Rightarrow$  *f* is a factor of  $\sigma$  on  $\Sigma_2^+$

classification

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semiconjugacy

# expanding maps are factors of $\sigma$

- $f : \mathbb{S}^1 \to \mathbb{S}^1$  expanding map
- deg(*f*) = 2
- $\Rightarrow$  *f* is a factor of  $\sigma$  on  $\Sigma_2^+$
- $\exists h : \Sigma_2^+ \to \mathbb{S}^1$  such that  $f^n(h(\underline{x})) \in \Delta_{x_n}$  for all  $n \ge 0$

points of non-injectivity

classification

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# points of non-injectivity

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points of non-injectivity

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# points of non-injectivity

#### points of non-injectivity

• if 
$$h(\underline{x}) = h(\underline{y}) = x$$

classification

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points of non-injectivity

# points of non-injectivity

#### points of non-injectivity

- if  $h(\underline{x}) = h(y) = x$
- then there exists  $n \ge 0$

classification

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points of non-injectivity

# points of non-injectivity

#### points of non-injectivity

- if  $h(\underline{x}) = h(\underline{y}) = x$
- then there exists  $n \ge 0$
- such that

 $f^n(x) = p$ 

classification

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points of non-injectivity

# points of non-injectivity

#### points of non-injectivity

- if  $h(\underline{x}) = h(\underline{y}) = x$
- then there exists  $n \ge 0$
- such that

 $f^n(x) = p$ 

• where f(p) = p

points of non-injectivity

classification

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proof

points of non-injectivity

proof

#### comments on the proof - points of non-injectivity

• f is injective on  $\Delta_i^o$ 



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proof

# comments on the proof - points of non-injectivity

• f is injective on  $\Delta_i^o$ 

• 
$$f(\partial \Delta_i) = p$$

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proof

- *f* is injective on  $\Delta_i^o$
- $f(\partial \Delta_i) = p$
- if  $x \in \Delta_0^o \cup \Delta_1^o$

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proof

- f is injective on  $\Delta_i^o$
- $f(\partial \Delta_i) = p$
- if  $x \in \Delta_0^o \cup \Delta_1^o$
- then first symbol of  $\underline{x}$  such that  $h(\underline{x}) = x$

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proof

- f is injective on  $\Delta_i^o$
- $f(\partial \Delta_i) = p$
- if  $x \in \Delta_0^o \cup \Delta_1^o$
- then first symbol of  $\underline{x}$  such that  $h(\underline{x}) = x$
- is 0 or 1 (no ambiguity)

proof



classification

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proof



points of non-injectivity

proof

proof - points of injectivity

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classification

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proof

# **proof** - points of injectivity • let $h(\underline{x}) = h(y) = x$ with $\underline{x} \neq y$

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proof

- let  $h(\underline{x}) = h(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- let *N* be the first integer such that  $x_N \neq y_N$

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proof

# proof - points of injectivity

- let  $h(\underline{x}) = h(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- let *N* be the first integer such that  $x_N \neq y_N$

• then

$$h(\underline{x}) = h(\underline{y}) = x \in \bigcap_{n=0}^{N-1} f^{-n}(\Delta_{x_n})$$

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proof

# proof - points of injectivity

- let  $h(\underline{x}) = h(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- let *N* be the first integer such that  $x_N \neq y_N$

• then

$$h(\underline{x}) = h(\underline{y}) = x \in \bigcap_{n=0}^{N-1} f^{-n}(\Delta_{x_n})$$

• which is an interval  $\Delta_{x_0...x_{N-1}}$ 

points of non-injectivity

proof

proof - points of injectivity

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proof

## proof - points of injectivity

• now *N* is the first integer such that  $x_N \neq y_N$ 

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proof

- now *N* is the first integer such that  $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$

proof

# proof - points of injectivity

- now *N* is the first integer such that  $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$

•  $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$ 

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proof

- now *N* is the first integer such that  $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$

- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$
- but  $\Delta_0 \cap \Delta_1 = \{p, q\}$

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proof

- now *N* is the first integer such that  $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$

- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$
- but  $\Delta_0 \cap \Delta_1 = \{p, q\}$
- $\bullet \Rightarrow f^{N+1}(x) = \rho$

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proof

- now *N* is the first integer such that  $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$

- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$
- but  $\Delta_0 \cap \Delta_1 = \{p, q\}$
- $\Rightarrow f^{N+1}(x) = p_{\square}$

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theorem

classification

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# classification

theorem

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# classification

# • let $f, g: \mathbb{S}^1 \to \mathbb{S}^1$ be expanding maps
theorem

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## classification

#### theorem

- let  $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  be expanding maps
- such that deg(f) = deg(g) = 2

theorem

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# classification

#### theorem

- let  $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  be expanding maps
- such that deg(f) = deg(g) = 2
- $\Rightarrow$  f and g are topologically conjugate

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#### classification





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corollary

#### classification

classification

classification

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corollary

# **classification** • $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map

classification

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corollary

# classification • $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map

• deg(*f*) = 2

classification

classification

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# corollary

#### classification

- $f: \mathbb{S}^1 \to \mathbb{S}^1$  expanding map
- deg(*f*) = 2
- $\Rightarrow$  *f* topologically conjugate to *E*<sub>2</sub>

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#### proof

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proof

proof - theorem

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proof

#### proof

#### proof - theorem

•  $f, g: \mathbb{S}^1 \to \mathbb{S}^1$  expanding maps

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proof

# proof

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  expanding maps
- with  $\deg(f) = \deg(g) = 2$

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proof

## proof

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  expanding maps
- with  $\deg(f) = \deg(g) = 2$
- $\bullet \ \Rightarrow \exists \ h_f, h_g: \Sigma_2^+ \to \mathbb{S}^1 \text{ semiconjugacies}$

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proof

# proof

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  expanding maps
- with  $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$  semiconjugacies
- such that

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proof

# proof

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  expanding maps
- with  $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$  semiconjugacies
- such that

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proof

# proof

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$  expanding maps
- with  $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$  semiconjugacies
- such that

$$\begin{array}{l}
\bullet & f \circ h_f = h_f \circ \sigma \\
\bullet & g \circ h_g = h_g \circ \sigma
\end{array}$$

proof

proof

### definition of h

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proof

#### proof

# definition of *h*

# • let us define $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that

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proof

#### proof

# definition of h• let us define $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that • $h(x) = h_g(h_f^{-1}(x))$

proof

proof

classification

h is well-defined, case 1

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proof

#### proof

### h is well-defined, case 1

• Case 1:  $h_f^{-1}(x)$  consists of a single point

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proof

#### proof

- Case 1:  $h_f^{-1}(x)$  consists of a single point
- $\Rightarrow$  *h* is well-defined

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proof

#### proof

- Case 1:  $h_f^{-1}(x)$  consists of a single point
- $\Rightarrow$  *h* is well-defined  $\sqrt{}$

proof

proof

classification

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proof

#### proof

# *h* is well-defined, case 2 • $\exists \underline{x} \neq y$ such that $h_f(\underline{x}) = h_f(y) = x$

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proof

#### proof

- $\exists \underline{x} \neq \underline{y}$  such that  $h_f(\underline{x}) = h_f(\underline{y}) = x$
- $\Rightarrow \exists N \text{ such that } f^N(x) = p_f, \text{ with } f(p_f) = p_f$

proof

classification

h is well-defined, case 2

proof

• 
$$f^N(x) = p_f$$

proof

- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$

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- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$
- $\iff \sigma^N(\underline{x}) = 0000000...$  or  $\sigma^N(\underline{x}) = 1111111...$

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#### h is well-defined, case 2

- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$
- $\iff \sigma^N(\underline{x}) = 0000000...$  or  $\sigma^N(\underline{x}) = 1111111...$
- otherwise

 $f^n(x) = p \in \Delta_{01} \cup \Delta_{10}$ 

for some  $n \ge N$ 

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#### h is well-defined, case 2

- $f^{N}(x) = p_{f}$
- $h_f(\underline{x}) = x$
- $\iff \sigma^N(\underline{x}) = 0000000...$  or  $\sigma^N(\underline{x}) = 1111111...$
- otherwise

 $f^n(x) = p \in \Delta_{01} \cup \Delta_{10}$ 

for some  $n \ge N \rightarrow$  contradiction

proof

proof

classification

proof

# proof

• 
$$h_f(\underline{x}) = h_f(\underline{y}) = x$$
 with  $\underline{x} \neq \underline{y}$ 

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proof

# proof

- $h_f(\underline{x}) = h_f(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- $N \ge 0$  the first such that  $f^N(x) = p$

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proof

## proof

- $h_f(\underline{x}) = h_f(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- $N \ge 0$  the first such that  $f^N(x) = p$
- $\Rightarrow$   $x_n = 0$  and  $y_n = 1$  for all  $n \ge N$

proof

# proof

#### h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- $N \ge 0$  the first such that  $f^N(x) = p$

• 
$$\Rightarrow$$
  $x_n = 0$  and  $y_n = 1$  for all  $n \ge N$ 

• 
$$x_{N-1} = 1$$
 and  $y_{N-1} = 0$ 

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proof

## proof

#### h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$  with  $\underline{x} \neq \underline{y}$
- $N \ge 0$  the first such that  $f^N(x) = p$
- $\Rightarrow$   $x_n = 0$  and  $y_n = 1$  for all  $n \ge N$

• 
$$x_{N-1} = 1$$
 and  $y_{N-1} = 0$ 

•  $x_n = y_n$  for all  $n \le N - 2$ 

coding

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h is well defined, case 2

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coding

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proof

### proof

## h is well defined, case 2

• let us show  $h_g(\underline{x}) = h_g(\underline{y})$ 

## proof

#### h is well defined, case 2

• let us show  $h_g(\underline{x}) = h_g(\underline{y})$ 

• 
$$h_g(\underline{x}) \in \Delta^g_{x_0...x_{N-2}10000..}$$

## proof

- let us show  $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0...x_{N-2}10000...}$
- $h_g(\underline{y}) \in \Delta^g_{x_0...x_{N-2}011111...}$

### proof

- let us show  $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0...x_{N-2}10000...}$
- $h_g(\underline{y}) \in \Delta^g_{x_0...x_{N-2}011111...}$
- $\Rightarrow$   $h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1...x_{N-2}} = [a_{N-2}, b_{N-2}]$

### proof

#### h is well defined, case 2

- let us show  $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0...x_{N-2}10000...}$
- $h_g(\underline{y}) \in \Delta^g_{x_0...x_{N-2}011111...}$
- $\Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1...x_{N-2}} = [a_{N-2}, b_{N-2}]$

•  $g^{N-1}$  is injective in  $(a_{N-2}, b_{N-2})$ 

### proof

- let us show  $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0...x_{N-2}10000...}$
- $h_g(\underline{y}) \in \Delta^g_{x_0 \dots x_{N-2} 011111 \dots}$
- $\Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1...x_{N-2}} = [a_{N-2}, b_{N-2}]$
- $g^{N-1}$  is injective in  $(a_{N-2}, b_{N-2})$
- $\exists ! r \in (a_{N-2}, b_{N-2})$  such that  $g^{N-1}(r) = q_g$

#### proof

### proof

#### h is well defined, case 2

• let us show  $h_g(\underline{x}) = h_g(\underline{y})$ 

• 
$$h_g(\underline{x}) \in \Delta^g_{x_0...x_{N-2}10000...}$$

•  $h_g(\underline{y}) \in \Delta^g_{x_0...x_{N-2}011111...}$ 

$$\bullet \Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1...x_{N-2}} = [a_{N-2}, b_{N-2}]$$

- $g^{N-1}$  is injective in  $(a_{N-2}, b_{N-2})$
- $\exists ! r \in (a_{N-2}, b_{N-2})$  such that  $g^{N-1}(r) = q_g$
- $h_g(\underline{x}) \in [r, b_{N-2}]$  and  $h_g(\underline{y}) \in [a_{N-2}, r]$

coding

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proof

proof

h is well defined, case 2

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proof

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# *h* is well defined, case 2 • $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \ge N} g^{-N}(\Delta_0)$

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# *h* is well defined, case 2 • $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \ge N} g^{-N}(\Delta_0) = r$

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proof

## proof

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \ge N} g^{-N}(\Delta_0) = r$
- $h_g(\underline{y}) \in [a_{N/2}, r] \cap \bigcap_{n \ge N} g^{-N}(\Delta_1)$

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proof

### proof

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \ge N} g^{-N}(\Delta_0) = r$
- $h_g(\underline{y}) \in [a_{N/2}, r] \cap \bigcap_{n \ge N} g^{-N}(\Delta_1) = r$

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proof

### proof

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \ge N} g^{-N}(\Delta_0) = r$
- $h_g(\underline{y}) \in [a_{N/2}, r] \cap \bigcap_{n \ge N} g^{-N}(\Delta_1) = r$
- $\Rightarrow$  *h* is well-defined.

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proof

## proof

## *h* is continuous

• let x be such that  $f^n(x) \neq p_f$  for all  $n \ge 0$ 

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proof

## proof

- let x be such that  $f^n(x) \neq p_f$  for all  $n \ge 0$
- take N > 0 such that  $d(\underline{x}, \underline{y}) < \frac{1}{3^N} \Rightarrow d(h_g(\underline{x}), h_g(\underline{y})) < \varepsilon$

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proof

## proof

- let x be such that  $f^n(x) \neq p_f$  for all  $n \ge 0$
- take N > 0 such that  $d(\underline{x}, y) < \frac{1}{3^N} \Rightarrow d(h_g(\underline{x}), h_g(y)) < \varepsilon$
- $x = \bigcap_{n=0}^{\infty} f^{-n}(\Delta_{x_n})$  is in the interior of  $\bigcap_{n=0}^{N} f^{-n}(\Delta_{x_n})$

#### proof

### proof

- let x be such that  $f^n(x) \neq p_f$  for all  $n \ge 0$
- take N > 0 such that  $d(\underline{x}, y) < \frac{1}{3^N} \Rightarrow d(h_g(\underline{x}), h_g(y)) < \varepsilon$
- $x = \bigcap_{n=0}^{\infty} f^{-n}(\Delta_{x_n})$  is in the interior of  $\bigcap_{n=0}^{N} f^{-n}(\Delta_{x_n})$
- $\Rightarrow$  there is  $\delta > 0$  such that  $d(x, y) < \delta \Rightarrow d(h(x), h(y)) < \varepsilon$

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# proof

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## proof

#### *h* is continuous

• let x be such that  $f^{K}(x) = p_{f}$  for some K > 0

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#### proof

## proof

#### *h* is continuous

• let x be such that  $f^{K}(x) = p_{f}$  for some K > 0

• 
$$\Rightarrow$$
  $h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$  such that

#### proof

#### h is continuous

• let x be such that  $f^{K}(x) = p_{f}$  for some K > 0

• 
$$\Rightarrow$$
  $h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$  such that

•  $\underline{x} = x_0 \dots x_{K-2} 011111 \dots$  and  $\underline{y} = x_0 \dots x_{K-2} 100000 \dots$ 

#### proof

### proof

- let *x* be such that  $f^{K}(x) = p_{f}$  for some K > 0
- $\Rightarrow$   $h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$  such that
- $\underline{x} = x_0 \dots x_{K-2} 011111 \dots$  and  $y = x_0 \dots x_{K-2} 100000 \dots$
- take  $\varepsilon > 0$  and N > 0 and take y > x

#### proof

### proof

- let x be such that  $f^{K}(x) = p_{f}$  for some K > 0
- $\Rightarrow$   $h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$  such that
- $\underline{x} = x_0 \dots x_{K-2} 011111 \dots$  and  $y = x_0 \dots x_{K-2} 100000 \dots$
- take  $\varepsilon > 0$  and N > 0 and take y > x
- if  $y \in \Delta_{x_0...x_{K-2}1000}$  (with *N* subsymbols)

#### proof

### proof

- let *x* be such that  $f^{K}(x) = p_{f}$  for some K > 0
- $\Rightarrow$   $h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$  such that
- $\underline{x} = x_0 \dots x_{K-2} 011111 \dots$  and  $y = x_0 \dots x_{K-2} 100000 \dots$
- take  $\varepsilon > 0$  and N > 0 and take y > x
- if  $y \in \Delta_{x_0...x_{K-2}1000}$  (with *N* subsymbols)
- then  $d(h(x), h(y)) < \varepsilon$

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## proof

### *h* is continuous

• analogously, we take y < x

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proof

## proof

- analogously, we take y < x
- if  $y \in \Delta_{x_0...x_{K-2}011111}$  (with *N* subsymbols)

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proof

## proof

- analogously, we take y < x
- if  $y \in \Delta_{x_0...x_{K-2}011111}$  (with *N* subsymbols)
- then  $d(h_f^{-1}(y), \underline{y}) < \frac{1}{3^N}$  and then

proof

## proof

- analogously, we take y < x
- if  $y \in \Delta_{x_0...x_{K-2}011111}$  (with *N* subsymbols)
- then  $d(h_f^{-1}(y), \underline{y}) < \frac{1}{3^N}$  and then
- $d(h(y), h(x)) = d(h_g(h_f^{-1}(y)), h_g(\underline{y})) < \varepsilon$

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## conclusion

### proof

#### conclusion

• it is easy to see that  $h^{-1}$  is well defined and continuous.

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## proof

### conclusion

- it is easy to see that  $h^{-1}$  is well defined and continuous.
- and

 $h \circ f =$ 

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proof

## proof

### conclusion

• it is easy to see that  $h^{-1}$  is well defined and continuous.

and

$$h \circ f = h_g \circ h_f^{-1} \circ f =$$
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#### proof

## proof

## conclusion

• it is easy to see that  $h^{-1}$  is well defined and continuous.

and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} =$$

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proof

# proof

## conclusion

- it is easy to see that  $h^{-1}$  is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} = g \circ h_g \circ h_f^{-1} =$$

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proof

# proof

## conclusion

- it is easy to see that  $h^{-1}$  is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} = g \circ h_g \circ h_f^{-1} = g \circ h$$

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# proof

## conclusion

• it is easy to see that  $h^{-1}$  is well defined and continuous.

and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} = g \circ h_g \circ h_f^{-1} = g \circ h$$