

# A little bit more about ergodicity

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ICTP

2018





# mark and recapture method

mark and recapture



● ecology



# mark and recapture method

mark and recapture



- ecology
- epidemiology



mark and recapture method

mark and recapture



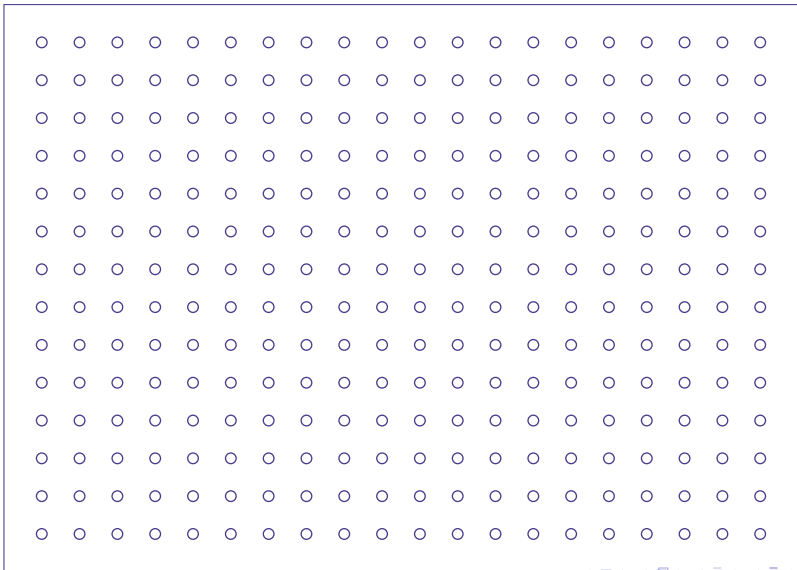
- ecology
- epidemiology



estimate populations



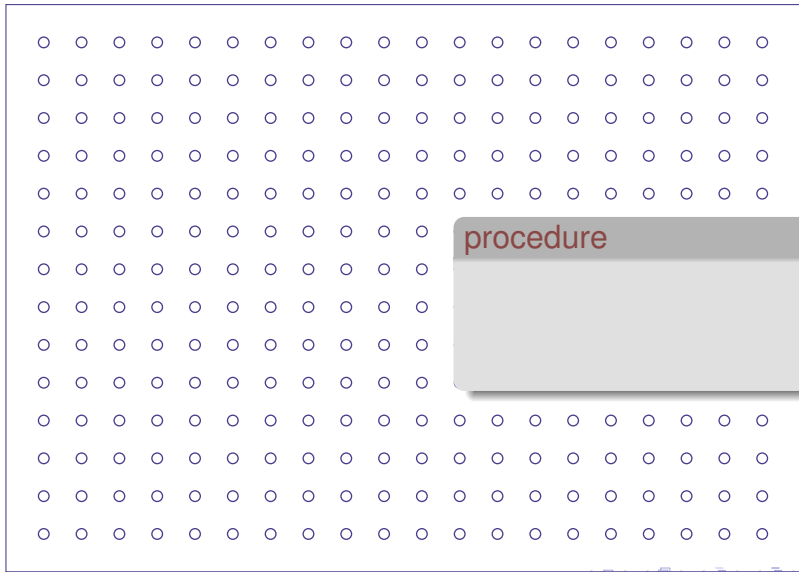
# counting fish





mark and recapture method

# counting fish

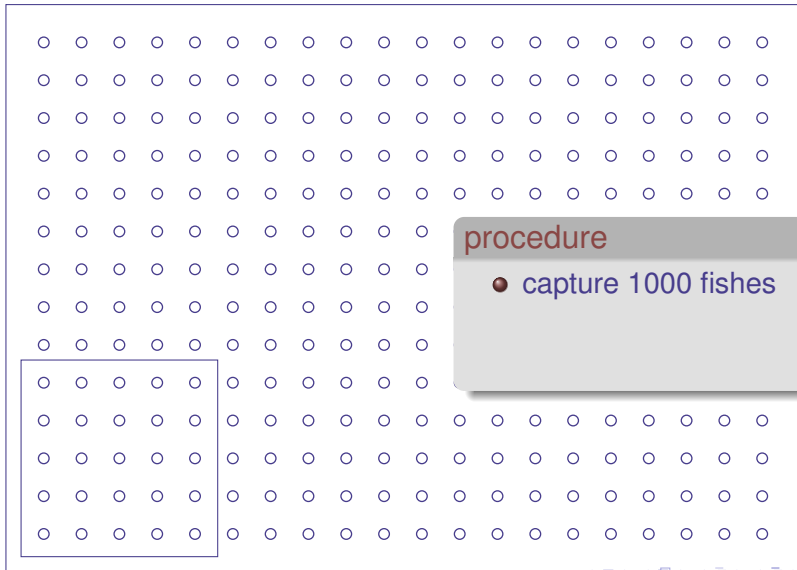


procedure

[A large grey rectangular area with a drop shadow, containing the word 'procedure' at the top. The rest of the area is blank, suggesting a space for a procedure description.]



# counting fish



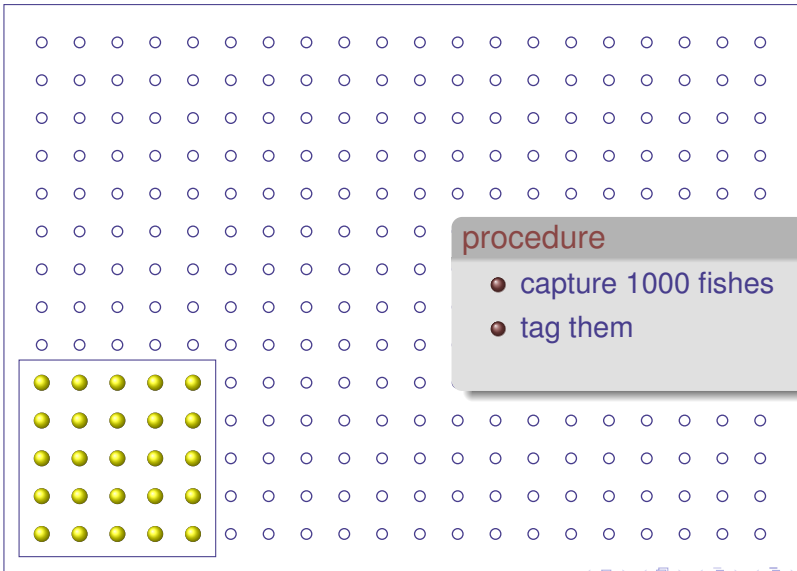
**procedure**

- capture 1000 fishes



mark and recapture method

# counting fish



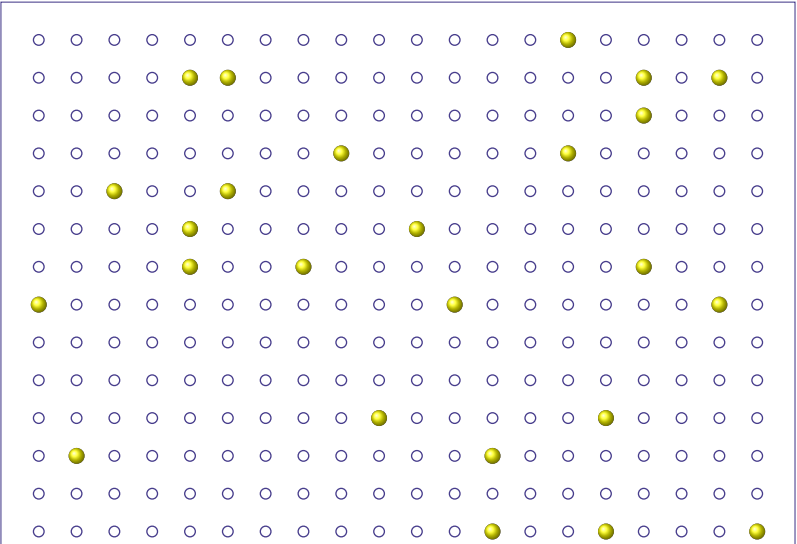
**procedure**

- capture 1000 fishes
- tag them

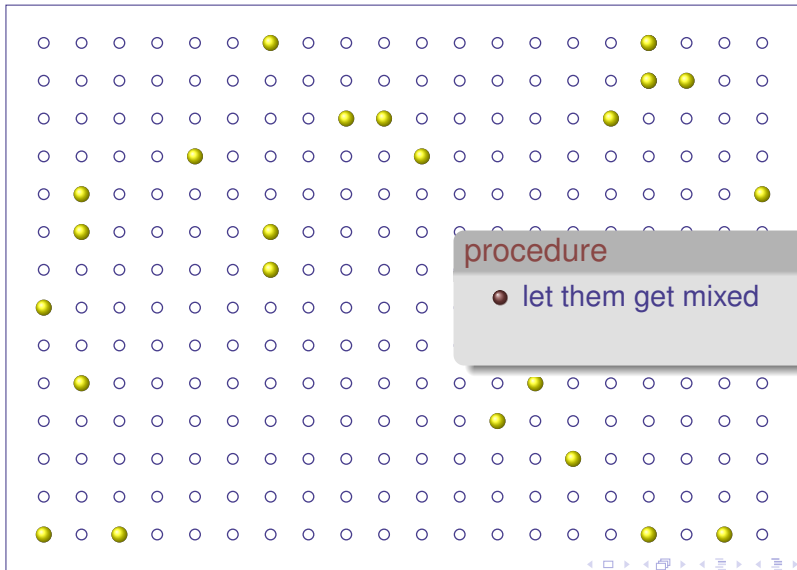


mark and recapture method

# counting fish



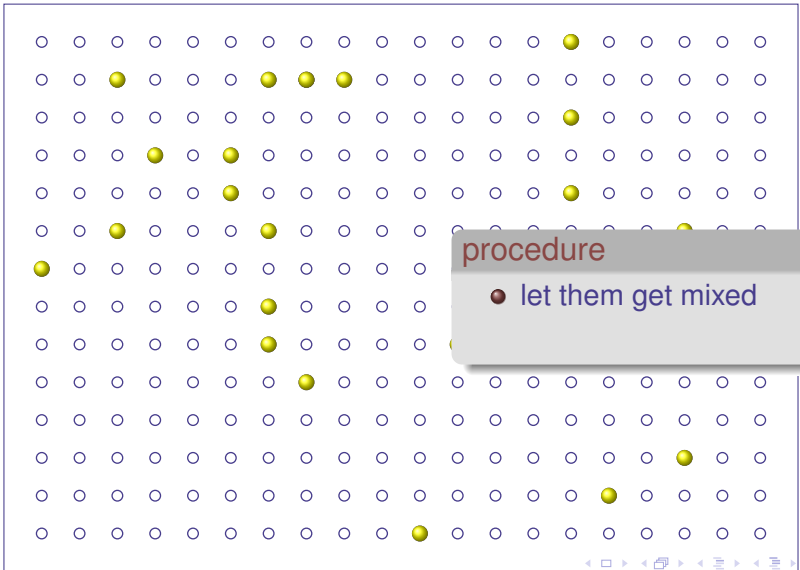
# counting fish



procedure

- let them get mixed

# counting fish



procedure  
● let them get mixed





# counting fish



procedure

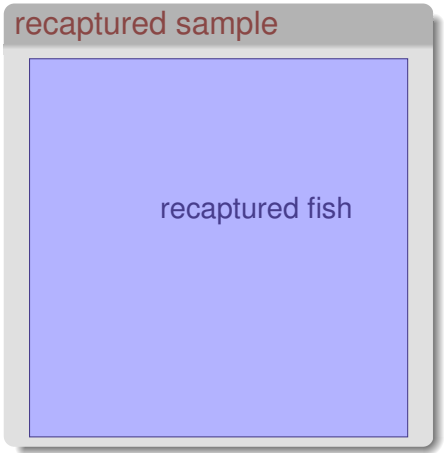
- let them get mixed
- recapture

counting fish  
○○●○

mixing properties  
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

mark and recapture method

# estimate





# estimate

recaptured sample

1000

tagged

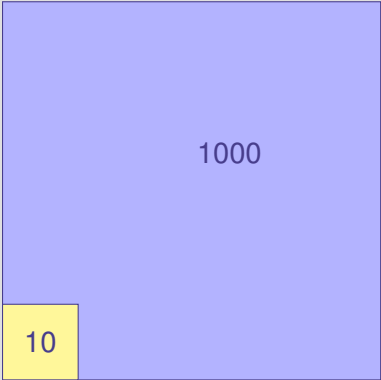




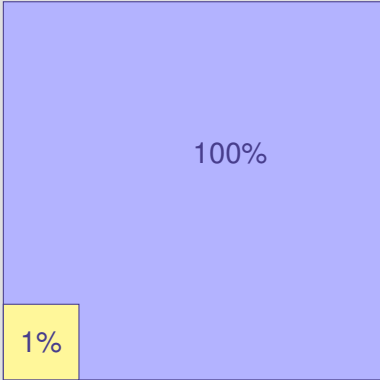


# estimate

recaptured sample



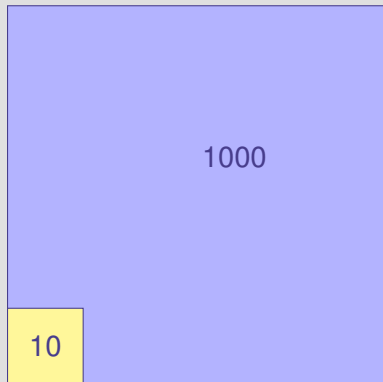
estimated population



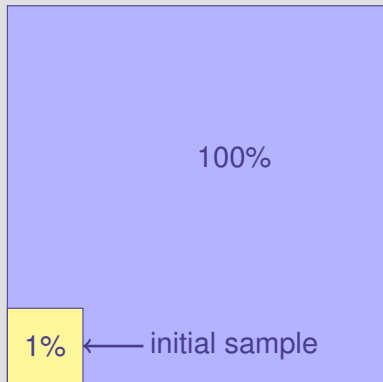


## estimate

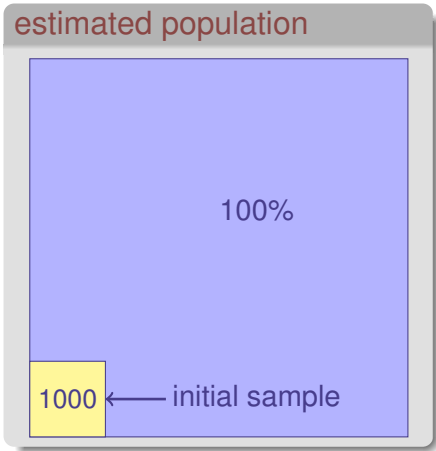
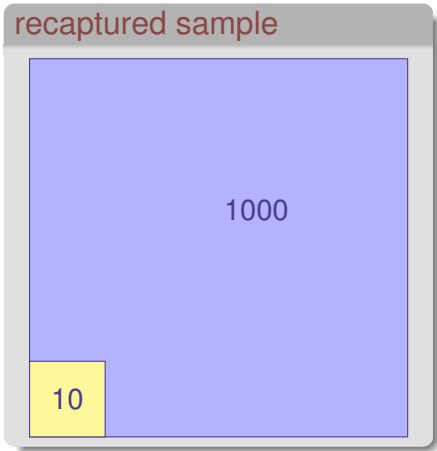
recaptured sample



estimated population



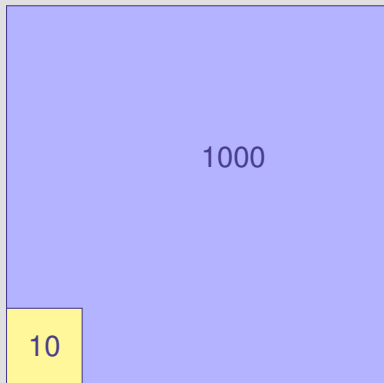
# estimate



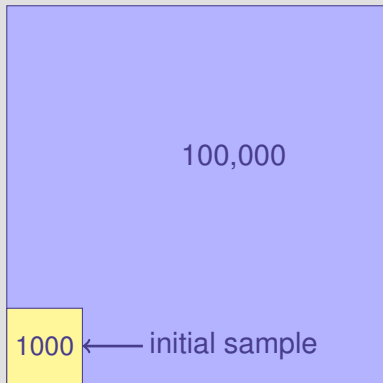


# estimate

## recaptured sample



## estimated population



# assumptions

## implicit hypotheses

- fish amount or measure is not altered



# assumptions

## implicit hypotheses

- fish amount or measure is not altered
- tagged fish move around the lake and come back
- tagged and non-tagged fish get well mixed



counting fish

ooooo

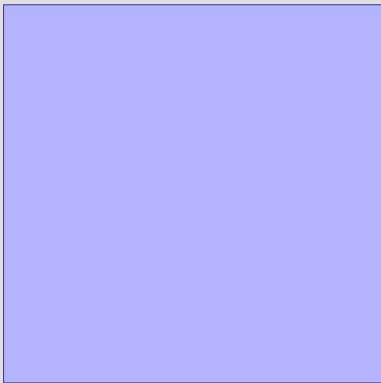
mixing properties

●○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

possible failure 1

# estimate

recaptured sample





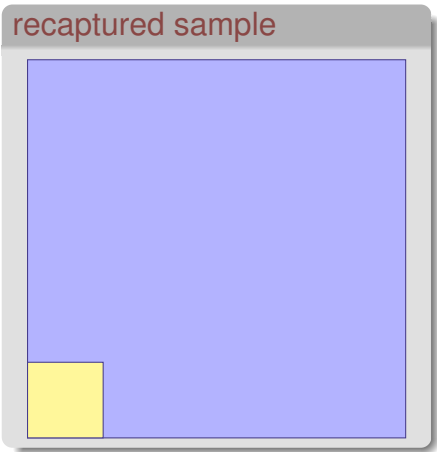
counting fish  
○○○○○

mixing properties  
●○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

possible failure 1

# estimate

recaptured sample

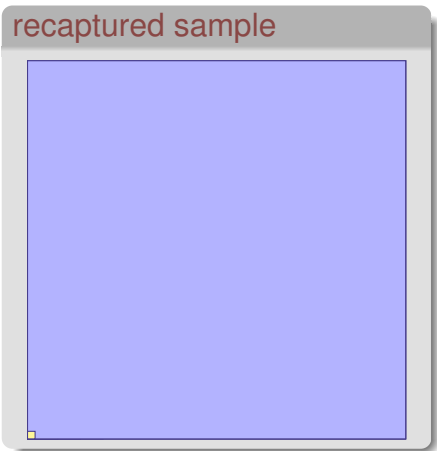






possible failure 1

# estimate





counting fish

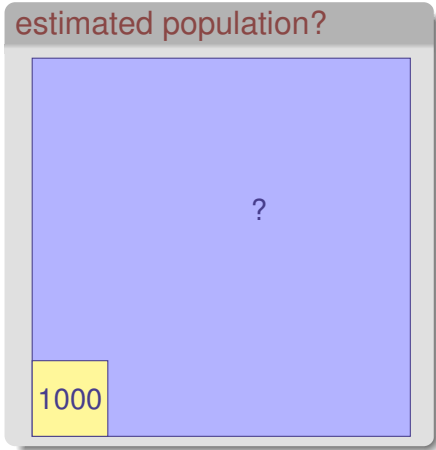
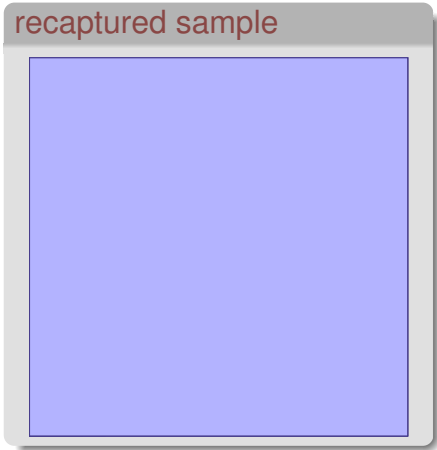
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mixing properties

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possible failure 1

# estimate



# preservation of the measure

## measure preserving process

- $f : L \rightarrow L$  process where

# preservation of the measure

## measure preserving process

- $f : L \rightarrow L$  process where
- for any sample  $A \subset L$









counting fish

ooooo

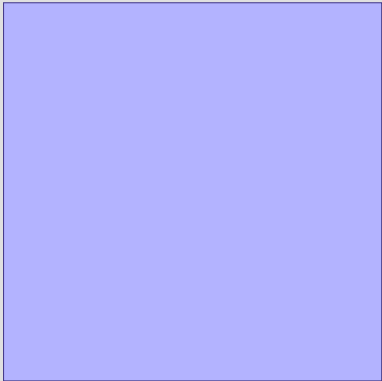
mixing properties

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possible failure 2

# possibility 2

recaptured sample



counting fish

ooooo

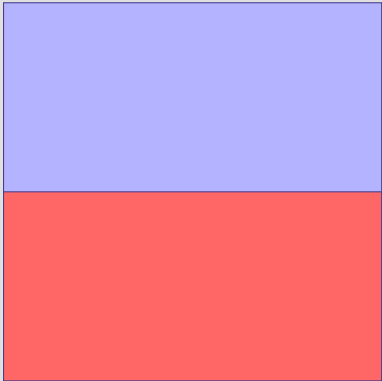
mixing properties

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possible failure 2

# possibility 2

recaptured sample



counting fish

ooooo

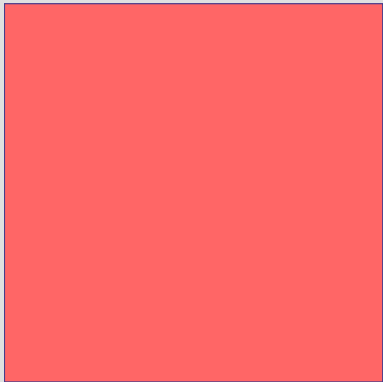
mixing properties

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possible failure 2

# possibility 2

recaptured sample





counting fish

ooooo

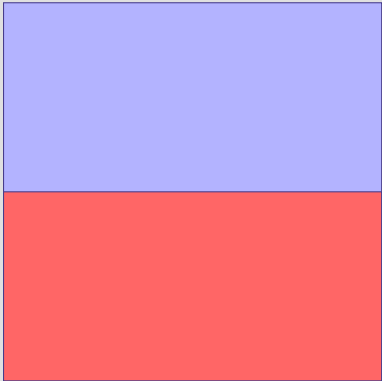
mixing properties

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possible failure 2

# possibility 2

recaptured sample





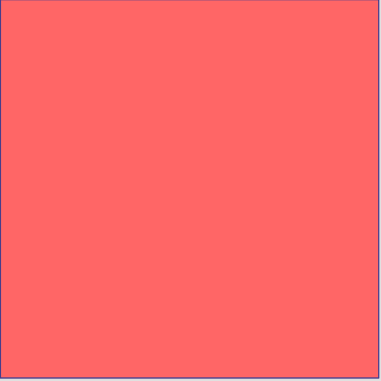
counting fish  
○○○○○

possible failure 2

mixing properties  
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# possibility 2

recaptured sample



A rectangular box with a grey border and a light grey header. The header contains the text "recaptured sample" in a dark brown font. The main body of the box is a solid, bright red color, representing a sample that is entirely composed of one type of fish.

counting fish

ooooo

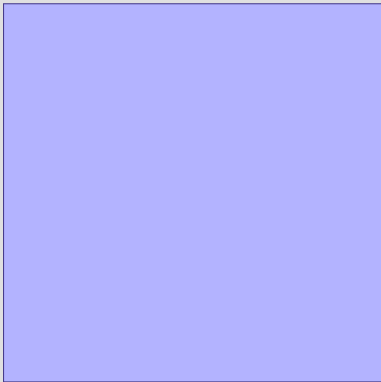
possible failure 2

mixing properties

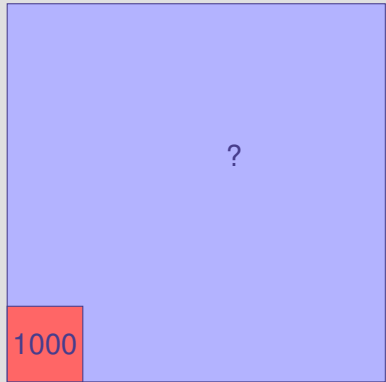
ooooo●ooooooooooooooooooooooooooooo

## possibility 2

recaptured sample



estimated population



# possibility 3

possibility 3

possible failure 3

# possibility 3

possibility 3 - trace

counting fish

ooooo

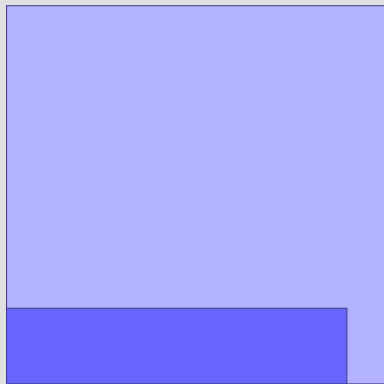
possible failure 3

mixing properties

oooooooo●oooooooooooooooooooo

# estimate

recaptured sample



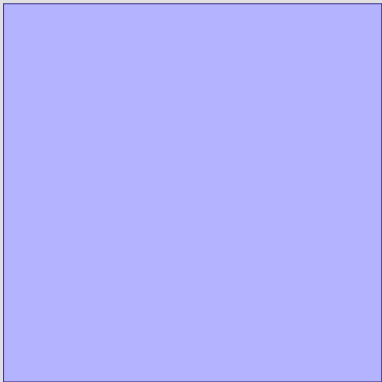
counting fish  
○○○○○

mixing properties  
○○○○○○○○●○○○○○○○○○○○○○○○○○○○○

possible failure 3

# estimate

recaptured sample



counting fish

ooooo

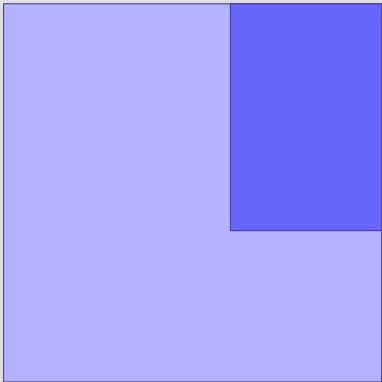
mixing properties

oooooooo●oooooooooooooooooooo

possible failure 3

# estimate

recaptured sample



counting fish

ooooo

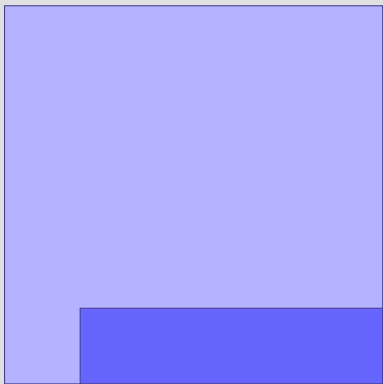
possible failure 3

mixing properties

oooooooo●oooooooooooooooooooo

# estimate

recaptured sample





counting fish

ooooo

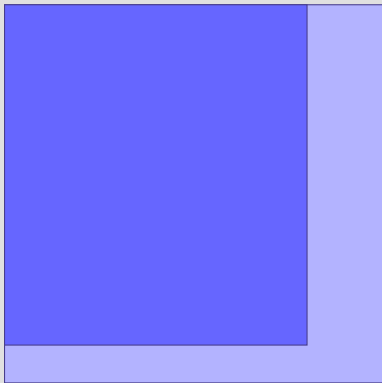
possible failure 3

mixing properties

oooooooo●oooooooooooooooooooo

# estimate

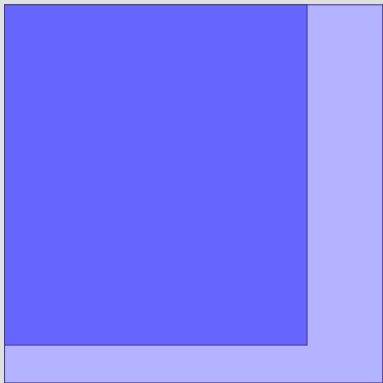
recaptured sample



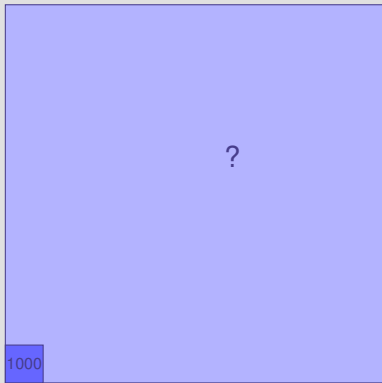
possible failure 3

# estimate

recaptured sample



estimated population



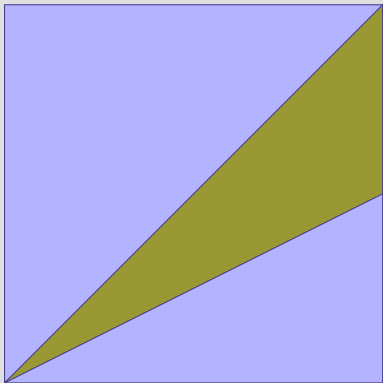
# possibility 4

## possibility 4

mixing lake population

# estimate - mixing situation

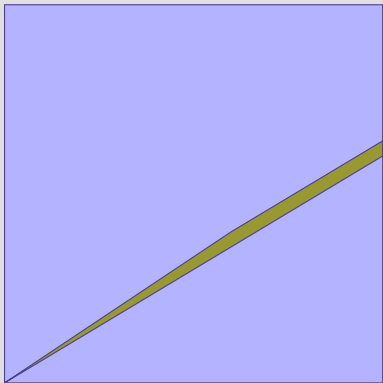
recaptured sample



mixing lake population

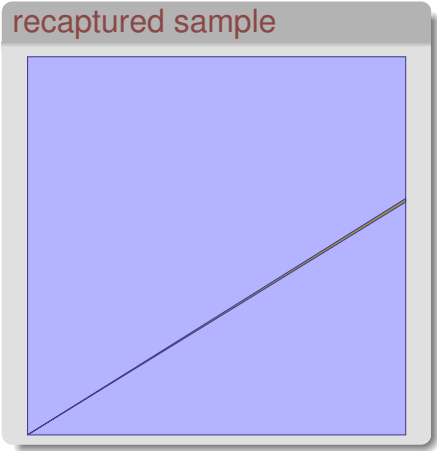
# estimate - mixing situation

recaptured sample



mixing lake population

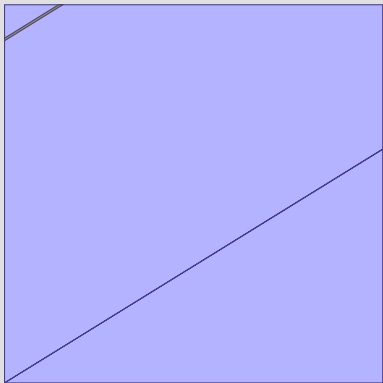
# estimate - mixing situation



mixing lake population

# estimate - mixing situation

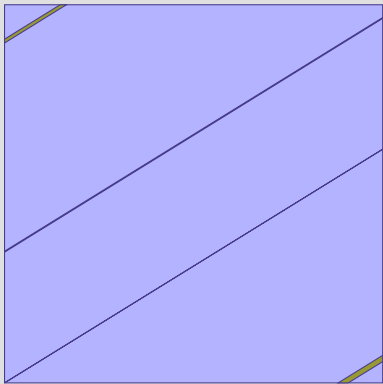
recaptured sample



mixing lake population

# estimate - mixing situation

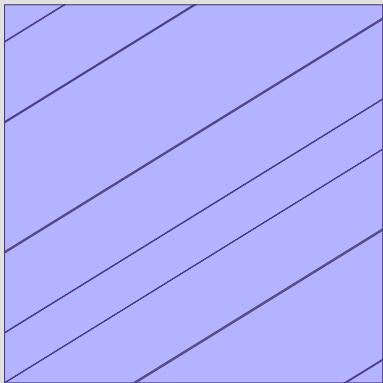
recaptured sample



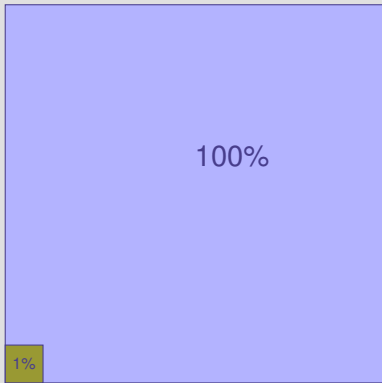


# estimate - mixing situation

recaptured sample

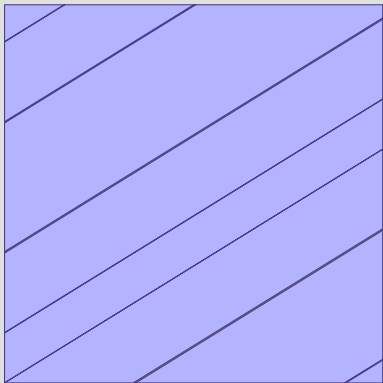


estimated population

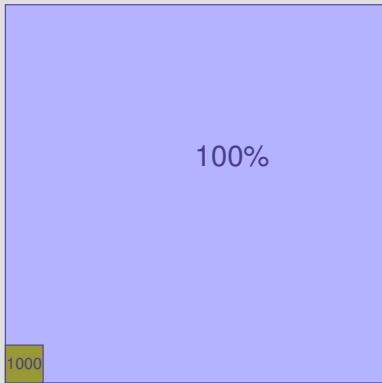


# estimate - mixing situation

recaptured sample

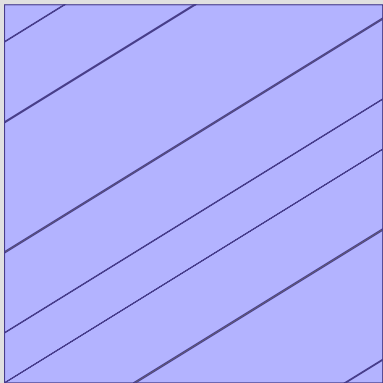


estimated population

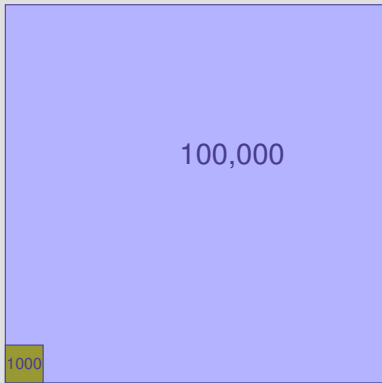


# estimate - mixing situation

recaptured sample



estimated population



# mixing property

## mixing property

- $f : L \rightarrow L$  is mixing

# mixing property

## mixing property

- $f : L \rightarrow L$  is mixing
- if every pair of samples  $A, B \subset L$  satisfy

# mixing property

## mixing property

- $f : L \rightarrow L$  is mixing
- if every pair of samples  $A, B \subset L$  satisfy
- 

$$m(A \cap f^n(B)) \rightarrow m(A)m(B)$$

# mixing property

## in our context

- if the lake population is mixing

# mixing property

## in our context

- if the lake population is mixing
- $A \subset L$  initial sample



# mixing property

## in our context

- if the lake population is mixing
- $A \subset L$  initial sample
- 

$$\frac{m(A \cap f^n(A))}{m(A)}$$

# mixing property

## in our context

- if the lake population is mixing
- $A \subset L$  initial sample
- 

$$\frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$

# mixing property

mixing property

# mixing property

mixing property



mark and recapture method works

# mixing property

NO mixing property

?

mark and recapture method works

# possibility 3

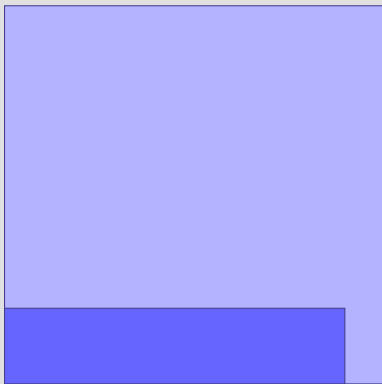
possibility 3

# possibility 3

## possibility 3 - trace

# estimate

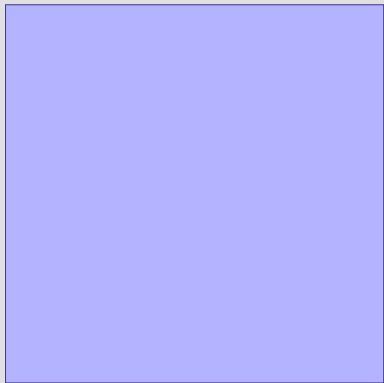
recaptured sample





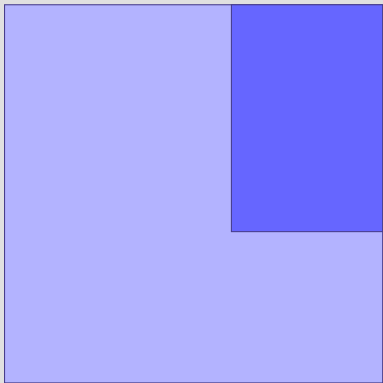
# estimate

recaptured sample



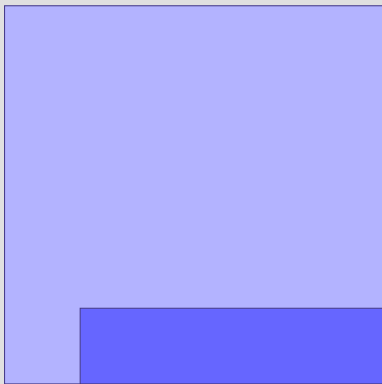
# estimate

recaptured sample



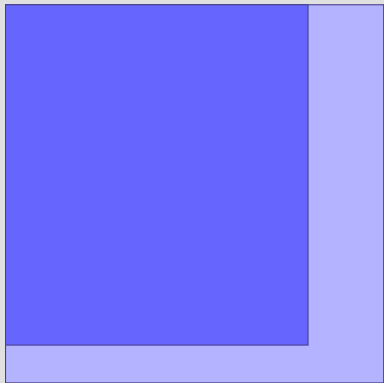
# estimate

recaptured sample



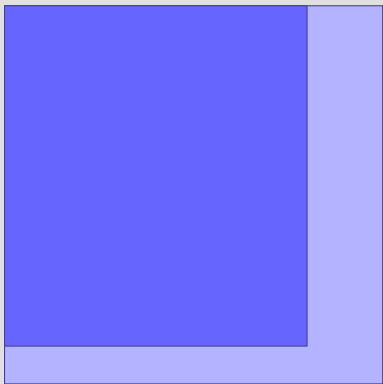
# estimate

recaptured sample

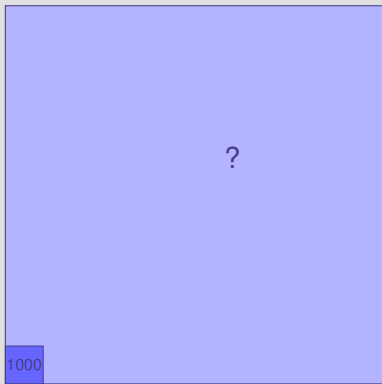


# estimate

recaptured sample



estimated population



## possibility 3

in this example

$$\exists \lim \frac{m(A \cap f^n(A))}{m(A)}$$

## possibility 3

in this example

$$\exists \lim \frac{m(A \cap f^n(A))}{m(A)}$$

what about this:

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow \textcircled{?}$$

# ergodicity

## ergodicity (exercise)

- $f : L \rightarrow L$  is ergodic



# ergodicity

## ergodicity (exercise)

- $f : L \rightarrow L$  is ergodic
- if every pair of samples  $A, B \subset L$  satisfy

# ergodicity

## ergodicity (exercise)

- $f : L \rightarrow L$  is ergodic
- if every pair of samples  $A, B \subset L$  satisfy
- 

$$\frac{1}{N} \sum_{n=0}^{N-1} m(A \cap f^n(B)) \rightarrow m(A)m(B)$$

# ergodic property

## in our context

- if the lake population is ergodic

# ergodic property

## in our context

- if the lake population is ergodic
- $A \subset L$  initial sample

# ergodic property

## in our context

- if the lake population is ergodic
- $A \subset L$  initial sample
- 

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)}$$

# ergodic property

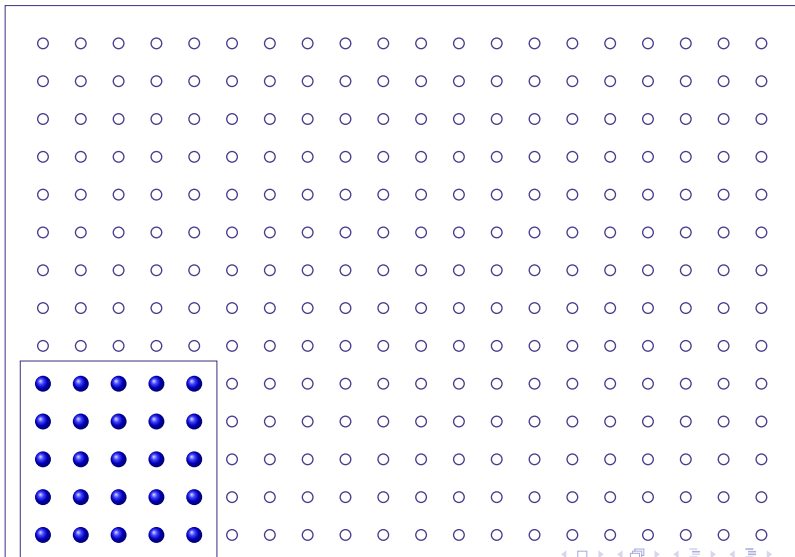
## in our context

- if the lake population is ergodic
- $A \subset L$  initial sample
- 

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$

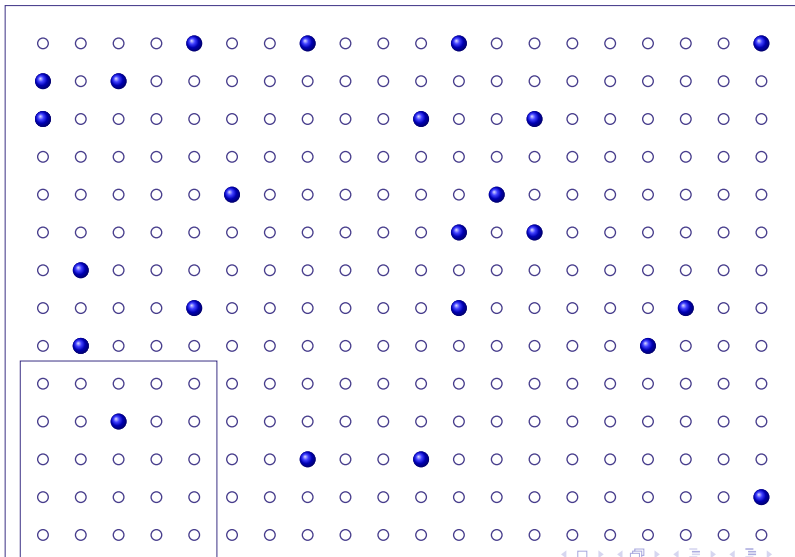
ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$



ergodicity

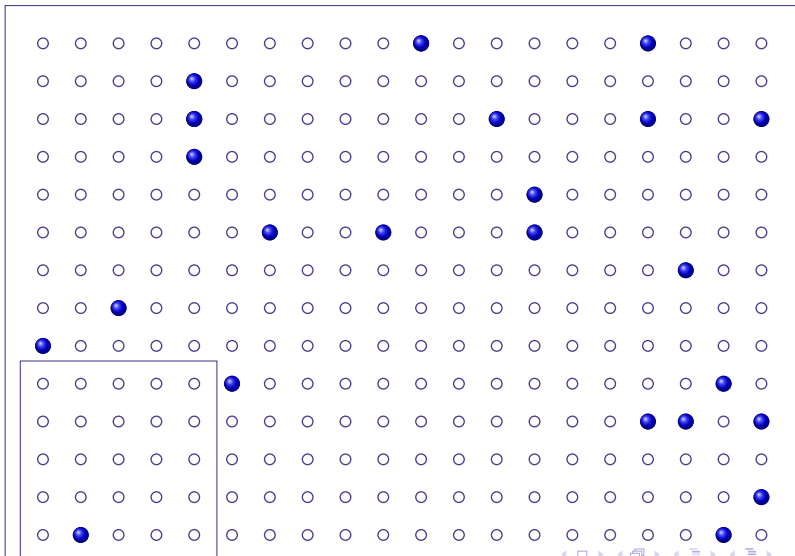
$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$





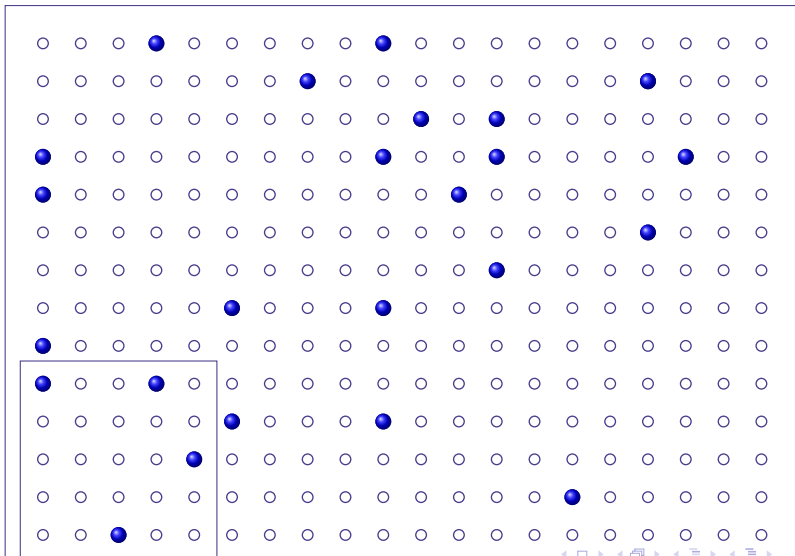
ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$



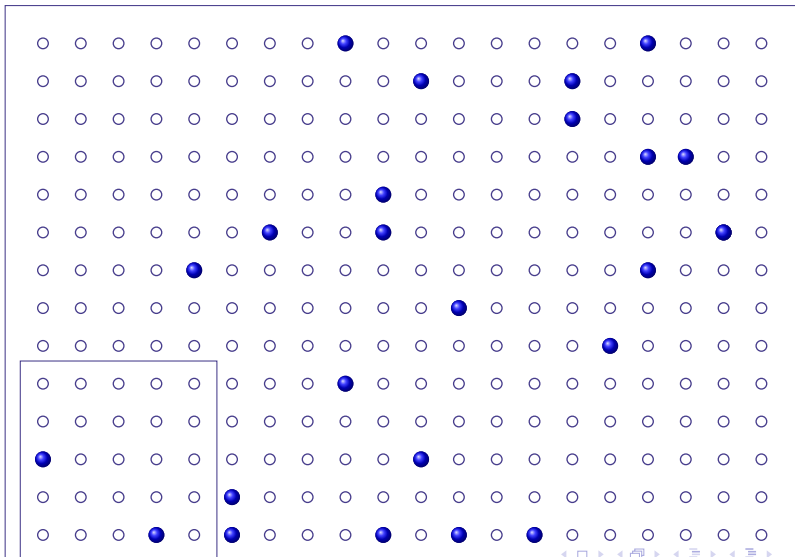
ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$



ergodicity

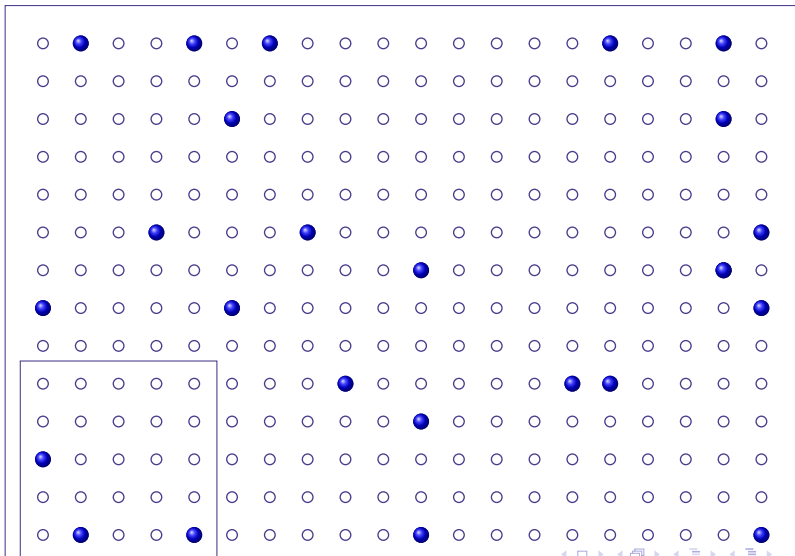
$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$





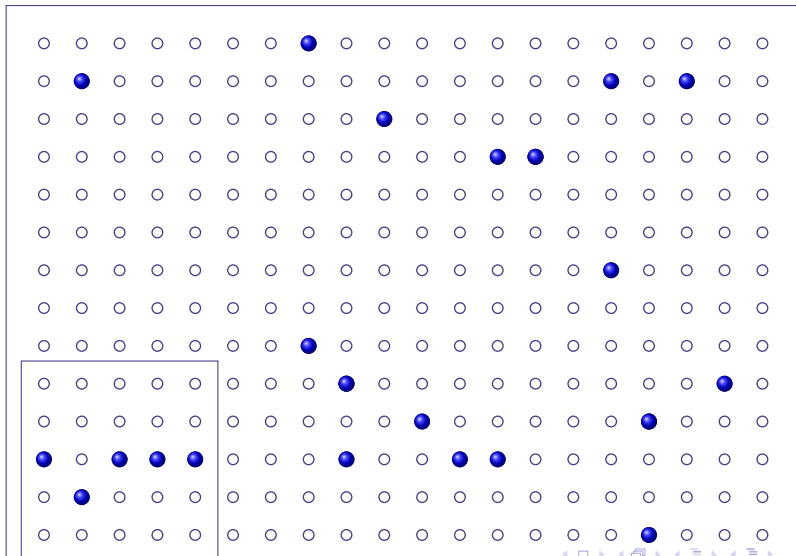
ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$



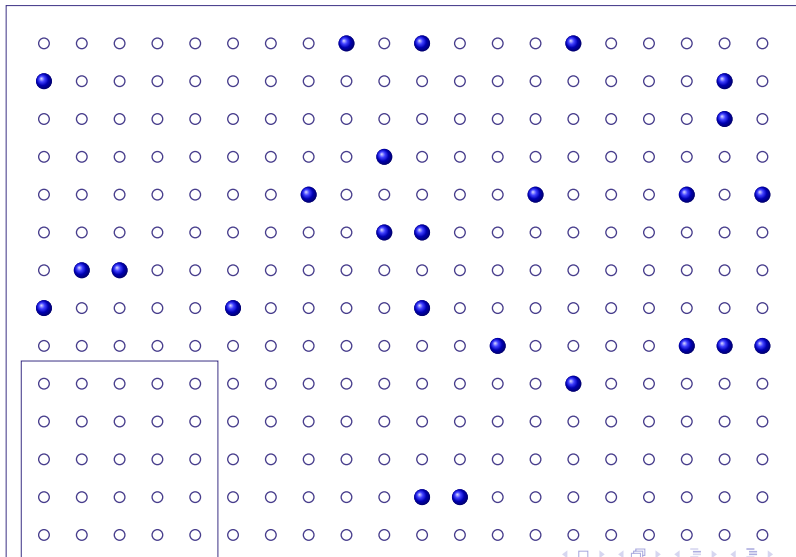
ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$



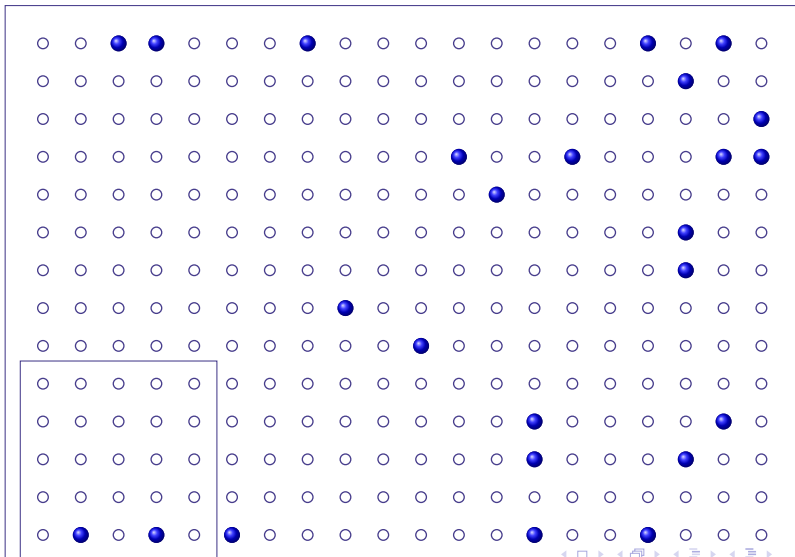
## ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A)$$



ergodicity

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{m(A \cap f^n(A))}{m(A)} \rightarrow m(A) \star$$





# exercise

exercise

# exercise

## exercise

- show that  $f$  ergodic  $\Leftrightarrow$

## exercise

## exercise

- show that  $f$  ergodic  $\Leftrightarrow$



$$m \left( \bigcup_{n \geq 0} f^n(A) \right) = 1 \quad \forall A \text{ s.t. } m(A) > 0$$

# fragility of ergodicity

irrational traslation on  $\mathbb{T}^2$

# fragility of ergodicity

rational traslation on  $\mathbb{T}^2$



# stable ergodicity

## stable ergodicity

- $f$  is stably ergodic

# stable ergodicity

## stable ergodicity

- $f$  is stably ergodic
- if

$$g \sim f \quad \Rightarrow \quad g \text{ ergodic}$$



# open problem

open problem

stably ergodic  $\Rightarrow$  mixing ?

counting fish

○○○○

stable ergodicity

mixing properties

○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

thank you

counting fish

ooooo

stable ergodicity

mixing properties

oooooooooooooooooooooooooooo

thank you

thank you!