ICTP SUMMER SCHOOL IN DYNAMICS
BIRKHOFF ERGODIC THEOREM - EXERCISES

Exercise C1. Consider the unit interval $[0,1]$ equipped with the Lebesgue measure, and let $r \geq 2$ be an integer.

(a) What does it mean for a number $x \in [0,1]$ to be *normal in base* r?
(b) Show that almost every $x \in [0,1]$ is normal in base r.
(c) Deduce that almost every $x \in [0,1]$ is simultaneously normal in all bases $r \geq 2$.

Exercise C2.* Let (X,d) be a compact metric space and let μ be a probability measure on X with *full support* (this means that for every $x \in X$, any open neighbourhood of x has positive measure). Let $T\colon X \to X$ be a μ-ergodic probability-preserving transformation. Show that, if T is an *isometry*, then for all continuous function $f\colon X \to \mathbb{R}$ and for all $x \in X$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)) \to \int_X f\,\mathrm{d}\mu \quad \text{uniformly in } x \in X;$$

i.e., T is *uniquely ergodic*.