Introduction to Ergodic Theory

Lecture I – Crash course in measure theory

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Why do we care about measure theory?

- Dynamical system $T : X \to X$
- What can we say about typical orbits?
- Push forward on measures $T_*\lambda(A) = \lambda(T^{-1}A)$ for $A \subset X$?

The goal?

Associate to every subset $A \subset \mathbb{R}^n$ a non-negative number $\lambda(A)$ with the following reasonable properties

- $\lambda\left((0,1)^n\right) = 1$
- $\lambda(\bigcup_k A_k) = \sum_k \lambda(A_k)$ when A_k are pairwise disjoint
- $\lambda(A) \leq \lambda(B)$ when $A \subseteq B$
- $\lambda(x + A) = \lambda(A)$

Why isn't this possible? What's the best that can be done? What happens when we drop some requirements?

Definition

A collection $\mathcal{A}$ of subsets of a space X is called an *algebra*a σ*-algebra* of subsets if

- $\emptyset \in \mathcal{A}$
- $\mathcal{A}$ is closed under complements, i.e., $A^c = X \setminus A \in \mathcal{A}$ whenever $A \in \mathcal{A}$
- $\mathcal{A}$ is closed under finite unions, i.e., $\bigcup_{k=1}^{N} A_k \in \mathcal{A}$ whenever $A_1, \ldots, A_N \in \mathcal{A}$ $\mathcal{A}$ is closed under countable unions, i.e., $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$ whenever $A_1, A_2 \ldots \in \mathcal{A}$

Definition

If $\mathcal{S}$ a collection of subsets of X, we denote by $\sigma(\mathcal{S})$ the smallest σ-algebra which contains $\mathcal{S}$.

Example

Let $X = \mathbb{R}$ and let $\mathcal{A}$ denote the collection of all finite unions of subintervals

Example

Let $X = \mathbb{R}$ and let $\mathcal{A}$ denote the collection of all subsets of $\mathbb{R}$

Definition

Let X be any topological space. The *Borel σ-algebra* is defined to be the smallest σ-algebra which contains all open subsets of X

Definition

A *measurable space* $(X, \mathcal{A})$ is a space X together with a σ-algebra $\mathcal{A}$ of subsets

Definition

Let $(X, \mathcal{A})$ be a measurable space. A *measure* μ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$
- If $A_1, A_2 \ldots \in \mathcal{A}$ is a countable collection of pairwise disjoint measurable sets then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

Definition

If $\mu(X) < \infty$ the measure is said to be *finite*. If $\mu(X) = 1$ the measure is said to be a *probability measure*.

Example

Let $X = \mathbb{R}$. The *delta*-measure at a point $a \in \mathbb{R}$ is defined as

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}$$

Translation invariant?

Definition

A measure space is said to be *complete* if every subset of any zero measure set is measurable

Definition

If $A \subset \mathbb{R}$ we define *outer measure* to be the quantity

$$\lambda^*(A) := \inf \left\{ \sum_{k=1}^{\infty} (b_k - a_k) : \{I_k = (a_k, b_k)\}_k \text{ is a set of intervals which covers } A \right\}$$

Definition

A set $A \subset \mathbb{R}$ is said to be *Lebesgue measurable* if, for every $E \subset \mathbb{R}$,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda * (E \setminus A)$$

Definition

For any Lebesgue measurable set $A \subset \mathbb{R}$ we define the lebesgue measure $\lambda(A) = \lambda^*(A)$

Exercise A

1. Let $X = \mathbb{N}$ and let $\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ is finite}\}$. Define

$$\mu(A) = \begin{cases} 1 & \text{if } A^c \text{ is finite} \\ 0 & \text{if } A \text{ is finite} \end{cases}.$$

   Is this function additive? Is it countable additive?

2. Show that the collection of Lebesgue measurable sets is a σ-algebra

Theorem (Caratéodory extension)

Let $\mathcal{A}$ be an algebra of subsets of X. If $\mu^* : \mathcal{A} \to [0,1]$ satisfies

- $\mu^*(\emptyset) = 0$, $\mu^*(X) < \infty$
- *If $A_1, A_2 \ldots \in \mathcal{A}$ is a countable collection of pairwise disjoint measurable sets and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$ then*

$$\mu^* \left(\bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mu * (A_k).$$

Then there exists a unique measure $\mu : \mathcal{A} \to [0,\infty)$ on $\sigma(\mathcal{A})$ the σ-algebra generated by $\mathcal{A}$ which extends μ^*.

Exercise B

1. Show that there are subsets of $\mathbb{R}$ which are not Lebesgue measurable (hint: consider an irrational circle rotation, choose a single point on each distinct orbit)
2. Show that there are Lebesgue measurable sets which are not Borel measurable (hint: recall the Cantor function, modify it to make it invertible, consider some preimage)