1.1 Poincaré Recurrence

Let $(X, \mathscr{B})$ be a measurable space and let $T : X \to X$ be a measurable transformation. Let us say that T *has a finite invariant measure* if there is a measure μ invariant under T with $\mu(X) < \infty$ (we saw many examples in the previous lecture). Just possessing a finite invariant measure has already very important dynamical consequences. We will see in this class Poincaré Recurrence and, in §3.7, the Birkhoff Ergodic Theorem. Both assume only that there is a finite measure preserved by the transformation $T : X \to X$.

Notation 1.1.1. *If $(X, \mathscr{B}, \mu)$ is a measure space, we say that a property hold for μ-almost every point and write* for $\mu - a.e$ point *if the set of $x \in X$ for which it fails has measure zero. Similarly, if $B \subset X$ is a subset, we say that property hold for μ-almost every point $x \in B$ if the set of points in B for which it fails has measure zero.*

If μ is the Lebesgue measure or if the measure is clear from the context and there is no-ambiguity, we will simply say that the property holds for almost every *point and write that it holds for* a.e. $x \in X$.

Definition 1.1.1. Let $B \subset X$ be a subset. We say that a point $x \in B$ *returns* to B if there exists $k \geq 1$ such that $T^k(x) \in B$. We say that $x \in B$ is *infinitely recurrent* with respect to B if it returns infinitely often to B, that is there exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}(x) \in B$.

Theorem 1.1.1 (**Poincaré Recurrence, weak form).** *If $(X, \mathscr{B}, \mu)$ is a measure space, T preserves μ and μ if* finite, *then for any $B \in \mathscr{B}$ with positive measure $\mu(B) > 0$, $\mu-$almost every point $x \in B$ returns to B (that is, the set of points $x \in B$ that never returns to B has measure zero).*

Before giving the formal proof, let us explain the idea behind it: if B is a set with positive measure, let us consider the preimages $T^{-n}(B)$, $n \in \mathbb{N}$. Since T is measure preserving, all the preimages have the same measure. Since the total measure of the space is finite, the sets $B, T^{-1}(B), \dots, T^{-n}(B), \dots$ cannot be all disjoint, since otherwise the measure of their union would have infinite measure. Thus, they have to intersect. Intersections give points in B which return to B (if $x \in T^{-n}(B) \cap T^{-m}(B)$ where $m > n$, then $T^n(x) \in B$ and $T^{m-n}(T^n(x)) = T^m(x) \in B$, so $T^n(x)$ returns to B). This only shows so far that *there exists* a point in B that returns. The proof strengthen this result to almost every point.

Proof of Theorem **??**. Consider the set $A \subset B$ of points $x \in B$ which *do not* return to B. Equivalently, we have to prove that $\mu(A) = 0$. Consider the preimages $\{T^{-n}(A)\}_{n\in\mathbb{N}}$. Clearly, $T^{-n}(A) \subset X$, so

$$\left(\bigcup_{n\in\mathbb{N}} T^{-n}(A)\right) \subset X \qquad \Rightarrow \qquad \mu\left(\bigcup_{n\in\mathbb{N}} T^{-n}(A)\right) \leq \mu(X) < \infty,$$

where we used that if $E \subset F$ are measurable sets, then $\mu(E) \leq \mu(F)$ (this property of a measure, which is very intuitive, can be formally derived from the definition of measure, see Exercise **??** below). Let us show that $\{T^{-n}(A)\}_{n\in\mathbb{N}}$ are pairwise *disjoint*. If not, there exists $n, m \in N$, with $n \neq m$, such that

$$T^{-n}(A) \cap T^{-m}(A) \neq \emptyset \quad \Leftrightarrow \quad \text{there exists } x \in T^{-n}A \cap T^{-m}(A).$$

Assume that $m > n$. Then

$$T^n(x) \in A, \quad \text{ and } \quad T^{m-n}(T^n x) = T^m(x) \in A,$$

but this contradicts the definition of A (all points of A do *not* return to A). Thus, $\{T^{-n}(A)\}_{n\in\mathbb{N}}$ are all disjoint. By the countable additivity property of a measure,

$$\sum_{n=1}^{\infty} \mu(T^{-n}A) = \mu\left(\bigcup_{n\in\mathbb{N}} T^{-n}(A)\right) \leq \mu(X).$$

Remark now that since T is measure preserving, $\mu(T^{-n}(A)) = \mu(A)$ for all $n \in \mathbb{N}$. Thus, we have a finite series whose terms are all equal. If $\mu(A) > 0$, this cannot happen (the series with constant terms equal to $\mu(A) > 0$ diverges), so $\mu(A) = 0$ as desired. □

In the proof we used the following property of a measure, which you can derive from the properties in the definition of a measure:

Exercise 1.1.1. Let μ be a measure on $(X, \mathscr{B})$. Show using the property of a measure that if $E, F \in \mathscr{B}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.

If $E \subset F$ is a strict inclusion, does it imply that $\mu(E) < \mu(F)$? Justify.

We can prove actually more and show that almost every point is *infinitely recurrent*:

Theorem 1.1.2 (Poincaré Recurrence, strong form). *If $(X, \mathscr{B}, \mu)$ is a measure space, T preserves μ and μ if* finite*, then for any $B \in \mathscr{B}$ with positive measure $\mu(B) > 0$, $\mu-$almost every point $x \in B$ is* infinitely recurrent *to B (that is, the set of points $x \in B$ that returns to B only finitely many times has measure zero).*

Proof. Let A be the set of points in B that *do not* return to B infinitely many times. We want to prove that $\mu(A) = 0$. The points in A are the points which return only finitely many (possibly zero) times. Thus, if $x \in A$, for all n sufficiently large $T^n(x)$ is outside B, that is

$$A = \{x \in B \text{ such that there exists } k \geq 1 \text{ such that } T^n(x) \notin B \text{ for all } n \geq k\}.$$

Consider the set

$$A_0 = \{x \in B \text{ such that } T^n(x) \notin B \text{ for all } n > 0\},$$

and the sets $A_k = T^{-k}(A_0)$ for $k \geq 1$. Then, if $x \in T^{-k}(A_0)$, $T^k(x) \in A_0$, so $T^k(x) \in B$ and $T^n(T^k(x)) = T^{k+n}(x) \notin B$ for all $n > 0$. Thus

$$A_k = \{x \text{ such that } T^k(x) \in B \text{ and } T^n(x) \notin B \text{ for all } n > k\}.$$

One can then write

$$A = B \cap \bigcup_{k=0}^{+\infty} A_k. \tag{1.1}$$

[Indeed, if $x \in A$, let k be the largest integer such that $T^k(x) \in B$, which is well defined since there are only finitely many such integers by definition of A. Then $x \in A_k$. Conversely, if $x \in B \cap A_k$ for some k, $T^n(x) \notin B$ for all $n > k$, so x can return only finitely many times and belongs to A.]

From (**??**) and Exercise (**??**), we have

$$A \subset \bigcup_{k=0}^{+\infty} A_k \qquad \Rightarrow \qquad \mu(A) \leq \mu\left(\bigcup_{k=0}^{+\infty} A_k\right).$$

Thus, to prove that $\mu(A) = 0$ it is enough to prove that the union $\cup_k A_k$ has measure zero.

Reasoning as in the proof of the weak version of Poincaré Recurrence, since we showed that $\{A_k\}_{k\in\mathbb{N}}$ are pairwise disjoint, we have

$$\sum_{k=0}^{\infty}\mu(A_k)=\mu\left(\bigcup_{k=0}^{+\infty}A_k\right)\le\mu(X)<+\infty.$$

Furthermore, $\mu(A_k)=\mu(A_0)$ for all $k\in\mathbb{N}$ since $A_k=T^{-k}(A_0)$ and T is measure preserving. Thus, the only possibility to have a convergent series with non-negative equal terms is $\mu(A_0)=\mu(A_k)=0$ for all $k\in\mathbb{N}$. But then

$$\sum_{k=0}^{\infty}\mu(A_k)=0 \qquad\Rightarrow\qquad \mu(A)\le\mu\left(\bigcup_{k=0}^{+\infty}A_k\right)=\sum_{k=0}^{\infty}\mu(A_k)=0,$$

so $\mu(A)=0$ and almost every point in B is infinitely recurrent. □

Remarks

1. Notice that recurrence is different than density. If $x\in B$ is periodic of period n, for example, it *does* return to B infinitely often even if its orbit is *not* dense, since $T^{kn}(x)=x\in B$ for any $k\in\mathbb{N}$. For example, consider a rational rotation R_α where $\alpha=p/q$. Then all orbits are periodic, so no points is dense. On the other hand, R_α preserves the Lebesgue measure and the conclusion of Poincaré Recurrence Theorem holds. Given any measurable set B, any point of B is infinitely recurrent.

2. If μ is *not* finite, Poincaré Recurrence Theorem does not hold. Consider for example $X=\mathbb{R}$ with the Borel $\sigma-$algebra and the Lebesgue measure λ. Let $T(x)=x+1$ be the translation by 1. Then T preserves λ, but no point $x\in\mathbb{R}$ is recurrent: all points tend to infinity under iterates of T.

Exercise 1.1.2. (a) Let $X=\mathbb{R}^2$ and let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation

$$T(x,y)=(x+y,y)\qquad\text{given by the matrix}\qquad A=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Show that the conclusion of Poincaré Recurrence Theorem fails for T.

(b) Let $X=\mathbb{T}^2$ and let $T:\mathbb{T}^2\to\mathbb{T}^2$ be the toral automorphism given by A, that is

$$T(x,y)=(x+y\mod 1,\ y\mod 1).$$

Show that in this case, for any rectangle $R=[a,b]\times[c,d]\subset\mathbb{T}^2$ *all* points $(x,y)\in R$ are infinitely recurrent to R.

[*Hint*: separate the two cases y rational and y irrational.]

Extra 1: Is Poincaré Recurrence a paradox?

Poincaré Recurrence theorem was considered for long time paradoxical. Let X be the phase space of a physical system, for example let X include all possible states of molecules in a box. The $\sigma-$algebra $\mathscr{B}$ represents the collections of *observable states* of the system and $\mu(A)$ is the probability of observing the state A. If T gives the discrete time evolution of the system, it is reasonable to expect that if the system is in equilibrium, T preserves μ, that is, the probability of observing a certain state is independent on time. Thus, we are in the set up of Poincaré Recurrence theorem. Consider now an initial state in which all the particles are in half of the box (for example imagine of having a wall which separates the box and then removing it). By Poincaré Recurrence Theorem, *almost surely*, all the molecules will return at some point in the same half of the box. This seems counter-intuitive. In reality, this is not a paradox, but simply the fact that the event will happen *almost surely* does not say anything about the *time* it will take to happen again (the *recurrence time*). Indeed, one can show that (if the transformation is ergodic, see next lecture) the average recurrence time is inversely proportional to the measure of the set to which one wants to return. Thus, since the phase space is huge and the observable corresponding to all molecules in half of the box has extremely small measure is this huge space, the time it will take will take to see again this configuration is also huge, probably longer than the age of the universe.

Extra 2: Poincaré Recurrence for incompressible transformations

Poincaré Recurrence holds not only for measure-preserving transformations, but more in general for a larger class of transformations called *incompressible.* In Exercise **??** we outline the steps of an alternative proof of the strong form of Poincaré Recurrence which holds also for incompressible transformations.

Definition 1.1.2. Let $(X, \mathscr{B}, \mu)$ be a *finite* measure space. A transformation $T : X \to X$ is called *incompressible* if for any $A \in \mathscr{B}$

$$A \subset T^{-1}(A) \quad \Rightarrow \quad \mu(T^{-1}(A)) = \mu(A).$$

Notice that here we only assume that the inclusion $A \subset T^{-1}(A)$ holds and not that A is invariant. Clearly if a transformation is measure preserving, in particular it is also incompressible.

Theorem 1.1.3 (Poincaré Recurrence for incompressible transformations). *If $(X, \mathscr{B}, \mu)$ be a* finite *measure space and $T : X \to X$ is* incompressible*, then, for any $B \in \mathscr{B}$ with positive measure, $\mu-$almost every point $x \in B$ is* infinitely recurrent *to B.*

Exercise 1.1.3. Prove and use the following steps to give a proof of Theorem **??**:

1. The set $E \subset A$ of points in $A \in \mathscr{B}$ which *are* infinitely recurrent can be written as

$$E = A \cap \bigcap_{n \in \mathbb{N}} E_n, \qquad \text{where} \quad E_n = \bigcup_{k \geq n} T^{-k} A;$$

2. The sets E_n are nested, that is $E_{n+1} \subset E_n$, and one has $\mu(\cap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n)$; [*Hint*: write $\mu(E_0 \setminus \cap_{n\in\mathbb{N}} E_n)$ as a telescopic series using the disjoint sets $E_n \setminus E_{n+1}$.]

3. Show that $T^{-1}(E_n) = E_{n+1}$. Deduce that $\lim_n \mu(E_n) = \mu(E_0)$.
   Conclude by using the remark that $A \setminus \cap_{n\in\mathbb{N}} E_n \subset E_0 \setminus \cap_{n\in\mathbb{N}} E_n$.

Extra 3: Multiple recurrence and applications to arithmetic progressions.

A stronger version of Poincaré Recurrence, known as *Multiple Recurrence*, turned out to have an elegant applications to an old problem in combinatorics, the one of finding arithmetic progressions in subsets of the integer numbers.

Definition 1.1.3. An *arithmetic progression of length N* is a set of the form

$$\{a, a+b, a+2b, \ldots, a+(N-1)b\} = \{a+kb, \quad \text{where } a, b \in \mathbb{Z},\ b \neq 0,\ k = 0, \ldots, N-1\}. \quad (1.2)$$

For example, $5, 8, 11, 14, 17$ is an arithmetic progression of length 5 with $a = 5$ and $b = 3$.

Consider the set $\mathbb{Z}$ and imagine of coloring the integers with a finite number r of colors. Formally, consider a partition

$$\mathbb{Z} = B_1 \cup \dots B_r, \qquad \text{where the sets } B_i \subset \mathbb{Z} \text{ are disjoint.} \tag{1.3}$$

(each set represents a color). Are there arbitrarily long arithmetic progressions of numbers all of the same color?

Theorem 1.1.4 (Van der Warden)**.** *If $\{B_1, \dots, B_r\}$ is a finite partition of $\mathbb{Z}$ as in (**??**), there exists a $1 \le j \le r$ such that B_j contains arithmetic progressions of arbitrary length, that is, for any N there exists $a, b \in \mathbb{Z}$, $b \neq 0$, such that $\{a + kb\}_{k=0}^{N-1} \subset B_j$.*

This Theorem can be proved using topological dynamics[1]. A proof can be found in the book by Pollicott and Yuri. A stronger result is true. The *density* (or upper density) of a subset $\mathcal{S} \subset \mathbb{Z}$ is defined as

$$\rho(\mathcal{S}) = \limsup_{n \to \infty} \frac{Card\{\ k \in \mathcal{S}, \quad -n \le k \le n\ \}}{2n+1}.$$

Thus, we consider the proportion of integers contained in $\mathcal{S}$ in each block of the form $[-n, n] \cap \mathbb{Z}$ and take the limit (if it exists, otherwise the limsup) as n grows. A set $\mathcal{S} \subset \mathbb{Z}$ has *positive (upper) density* if $\rho(\mathcal{S}) > 0$.

Theorem 1.1.5 (Szemeredi)**.** *If $\mathcal{S} \subset \mathbb{Z}$ has* positive (upper) density*, it contains arithmetic progressions of arbitrary length, that is, for any N there exists $a, b \in \mathbb{Z}$, $b \neq 0$, such that $\{a + kb\}_{k=0}^{N-1} \subset \mathcal{S}$.*

This result was conjectured in 1936 by Erdos and Turan. The theorem was first proved by Szemeredi in 1969 for $N = 4$ and then in 1975 for any N. Szemeredi's proof is combinatorial and very complicated. A few years later, in 1977, Furstenberg gave a proof of Szemeredi's theorem by using ergodic theory. The essential ingredient in his proof, which is very elegant, is based on a stroger version of Poincaré Recurrence, known as *Multiple Recurrence*:

Theorem 1.1.6 (Multiple Recurrence)**.** *If $(X, \mathscr{B}, \mu)$ is a measure space, T preserves μ and μ if* finite*, then for any $B \in \mathscr{B}$ with positive measure $\mu(B) > 0$ and any $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that*

$$\mu\left(B \cap T^{-k}(B) \cap T^{-2k}(B) \cap \dots \cap T^{-(N-1)k}(B)\right) > 0. \tag{1.4}$$

The theorem conclusion means that there exists a positive measure set of points of B which return to B *along an arithmetic progression*: if x belongs to the intersection (**??**), then $x \in B$, $T^k(x) \in B, T^{2k}(x) \in B, \dots, T^{(N-1)k}(x) \in B$, that is, returns to B happen along an arithmetic progression of return times of length N.

From the Multiple Recurrence theorem, one can deduce Szemeredi Theorem in few lines (it is enough to set up a good space and map, which turns out to be a shift map on a shift space, find an invariant measure and translate the problem of existence of arithmetic progressions into a problem of recurrence along an arithmetic sequence of times). A reference both for this simple argument is again the book by Pollicott and Yuri, see §16.2 (in the same Chapter 16 you can find also a sketch of the full proof of the Multiple Recurrence Theorem. The proof is quite long and involved and uses more tools in ergodic theory).

[1]The original proof was given by Van der Waerden in 1927. The dynamical proof is due to Fursterberg and Weiss in 1978.

A much harder question, open until recently, was whether there are arbitrarily long arithmetic progressions such that all the elements $a + kb$ are *prime numbers*. Unfortunately Szemeredi theorem does not apply if we take as set $\mathcal{S}$ the set of prime numbers: indeed, primes have zero density. Recently, Green and Tao gave a proof that the primes contain arbitrarily large arithmetic progressions. The proof is a mixture of ergodic theory and additive combinatorics.