ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Week 2 - Homework 1

Exercise A1 Let Λ be a unit square grid in $\mathbb{R}^2$ and let L be an irrational line of the form $\{y = ax + b\}$, where a, b are real and a is irrational. Show that the cutting sequence $w = (w_i)_{i\in\mathbb{Z}}$ of L (where $w_i = 0$ when L hits a horizontal line, or $w_i = 1$ when it hits a vertical line) coincide with the itinerary of an orbit $\mathscr{O}_{R_\alpha}(x) = \{R^i_\alpha(x),\ i \in \mathbb{Z}\}$ with respect to a suitable partition.

- If you use the horizontal section $\Sigma = [0, 1]$ that we used in class to get a rotation R_α as a Poincar'e map, you need to split the cases into $\tan\theta < 1$ and $\tan\theta > 1$ and define the coding intervals carefully. For example for $\tan\theta > 1$, one needs to take $I_0 := [0, 1-\alpha]$ and $I_{10} := [1-\alpha, 1]$ and record a 10 word for each visit to I_{10}.
- Try to find a different section Σ (i.e. another segment on the torus) which can be written as a union of two intervals I_0 and I_1 (to be determined), i.e. it is such that $R^{w_i}_\alpha(x) \in I_{w_i}$ for all $i \in \mathbb{Z}$.

Exercise A2 Let $w = (w_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence in $\{0, 1\}$. Recall that the complexity function $P = P_\omega : \mathbb{N} \to \mathbb{N}$ is defined by $P(n) :=$ number of words of lenght n which occur inside w.

(a) Show that if w is periodic, $n \mapsto P(n)$ is bounded.

(b) Let x be a Birkhoff generic point for the doubling map $f(x) = 2x \mod 1$. Let w be the itinerary of x with respect to the intervals $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$. Show that in this case the complexity P grows exponentially, i.e. $P(n) = 2^n$.

(a) Show w is periodic if and only if $n \mapsto P(n)$ is bounded.

[Actually a stronger result is true: w is periodic if and only if P is sublinear, i.e. $n \leq P(n)$ for each $n \in \mathbb{N}$.]

(c) Show that a square cutting sequence has complexity $P(n) = n + 1$ for each $n \in \mathbb{N}$.