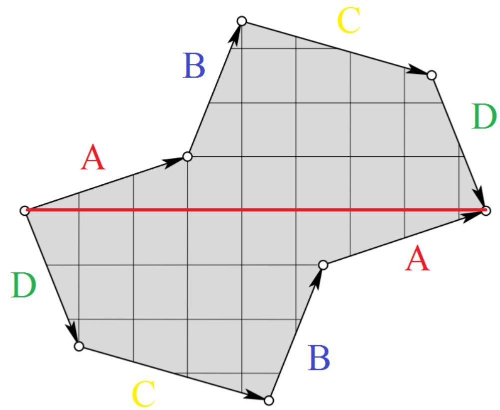
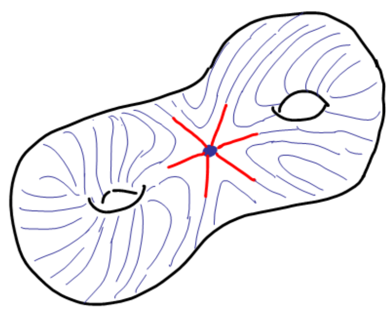
ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Week 2 - Homework 3

Exercise A1 Consider the translation surface glued from a octagon as in Figure.





(a) Consider the vertical linear flow φ^t and let Σ be the horizontal diagonal drawn in red in the above Figure. Draw the Poincaré first return map T of φ^t to Σ to see that it is a piecewise isometry of Σ.

(b) Consider now a *shorter* horizontal transversal interval Σ' contained in Σ and draw the first return map T'. How does the number of continuity intervals of T' changes as Σ' changes? What is the minimum and maximum number of continuity intervals for T'?

Exercise A2 Consider again the traslation surface S in the above figure and the vertical linear flow φ_t on it.

(a) Show that all vertices are identified to a unique point P for the surface and that the vertical flow has a saddle with 6 prongs (as in Figure). Check also that if you rotate around P on S you rotate of an angle 6π before closing up.

(b) Take now a surface glued from a (not necessarily regular) *decagon* with pairs of opposite parallel equal lenght sides, by identifying the pairs of such sides by translations. Compute its genus and show that the linear flow φ_t^θ (for a.e. choice of direction θ) has two saddles with 4 prongs each.

Exercise A3 (**Fun problem**) Consider the transformation S_α of $\mathbb{R}/\mathbb{Z}$ obtained by exchanging two intervals and flipping the first, i.e. given by

$$S_\alpha(x) = \begin{cases} x - \alpha, & \text{if } \alpha \le x < 1, \\ (\alpha - x) + 1 - \alpha, & \text{if } 0 \le x < \alpha. \end{cases}$$

Show that, for every $\alpha > 0$, S_α is periodic (i.e. all points have periodic orbits).

[This problem is known as a *cutting the pie* problem: if you cut a slice of a pie of angle $2\pi\alpha$, flip it and rotate the cake by $2\pi\alpha$ before putting it back, will the pie look the same again? It's a baby case of the study of IETs with flips, which are very different that IETs which are orientation preserving.]