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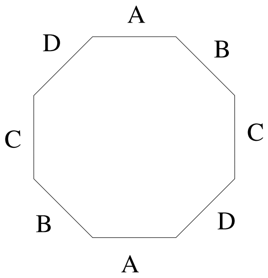

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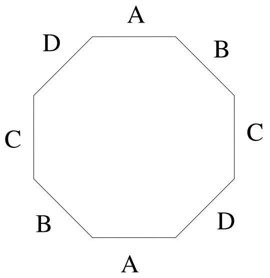

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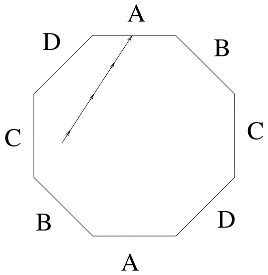

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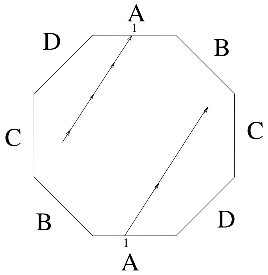

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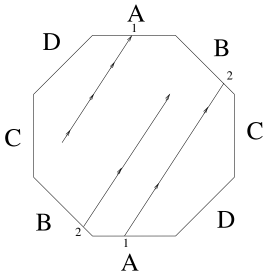

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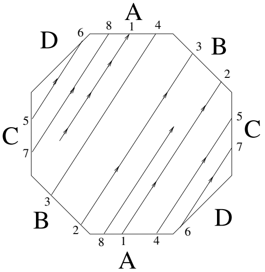

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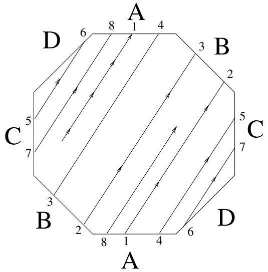

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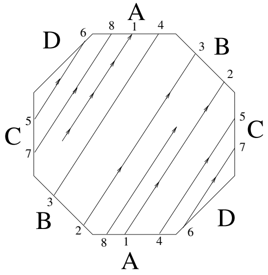

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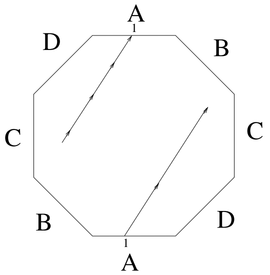

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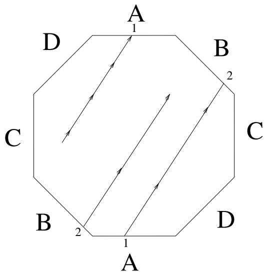

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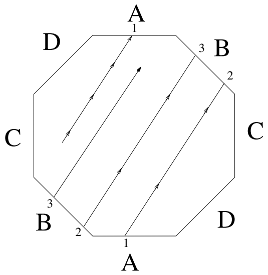

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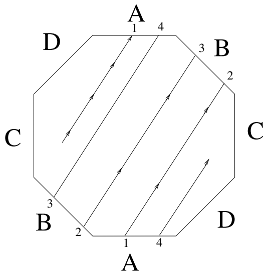

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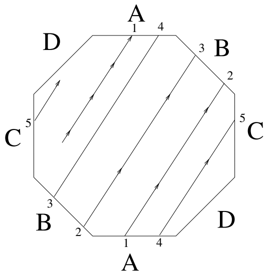

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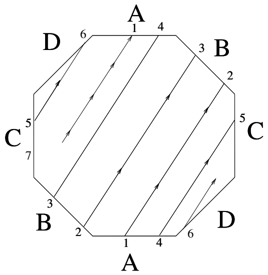

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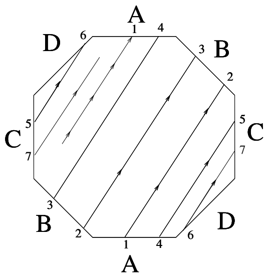

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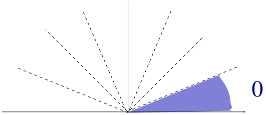
Using isometries of the octagon, we can reduce to consider $\theta \in \left[0, \frac{\pi}{8}\right]$.



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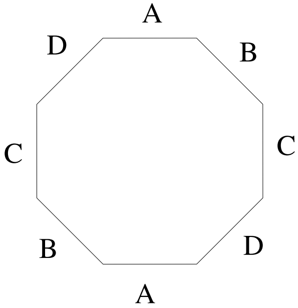

Each octagon cutting sequence of a trajectory in direction $0 \leq \theta \leq \pi/8$ determines a path in the diagram in Figure.

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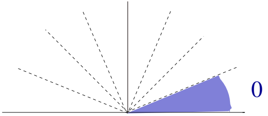

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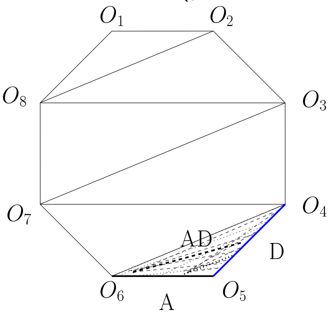

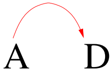

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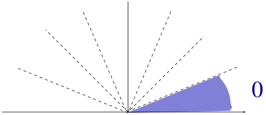

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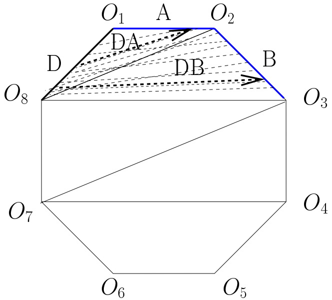

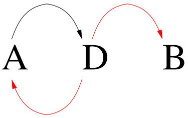

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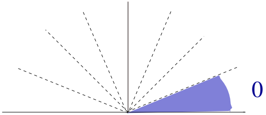

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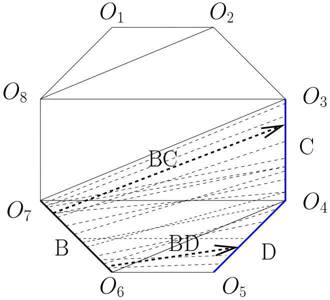

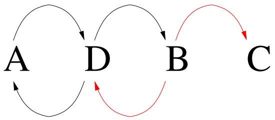

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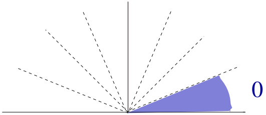

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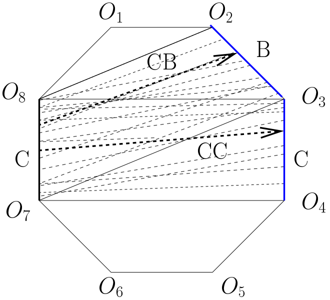

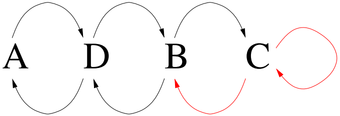

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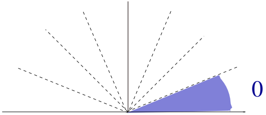

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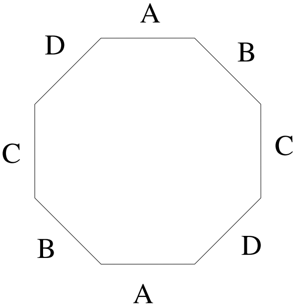

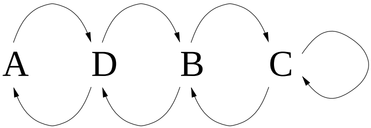

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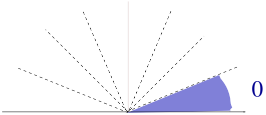

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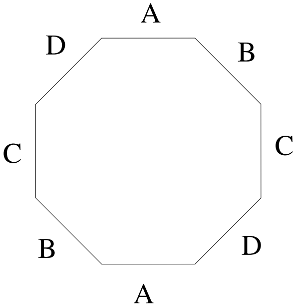

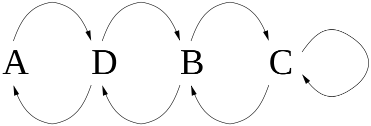

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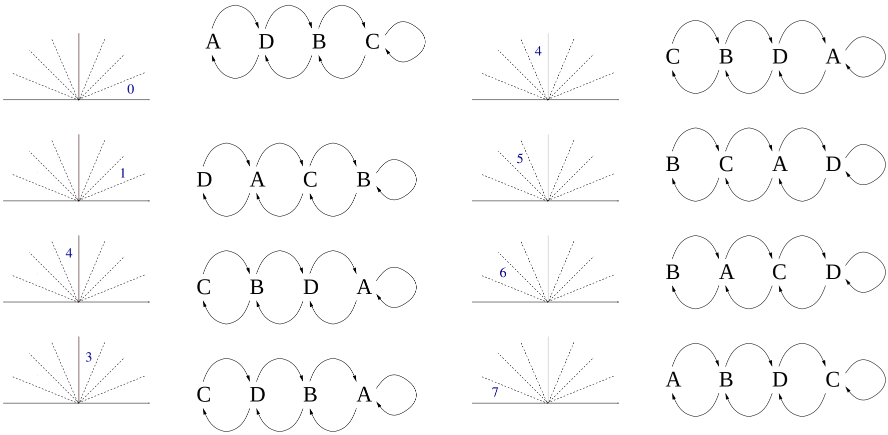
Def: A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ is *admissible* if it gives an infinite path on one of the 8 possible diagrams (8 types):



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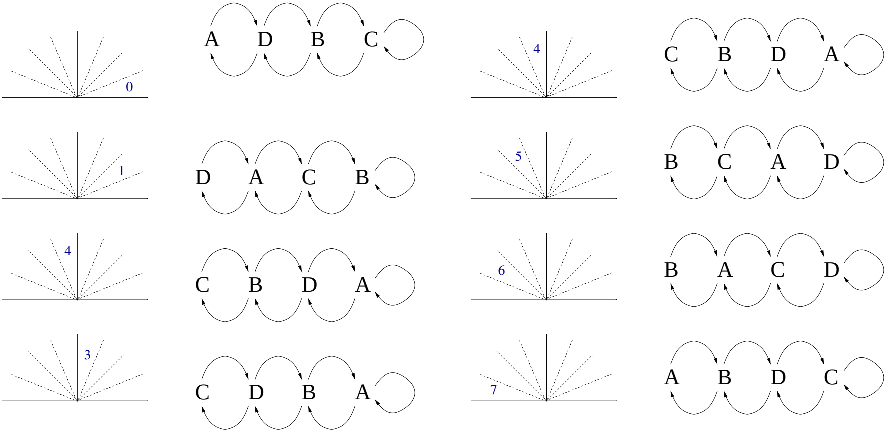

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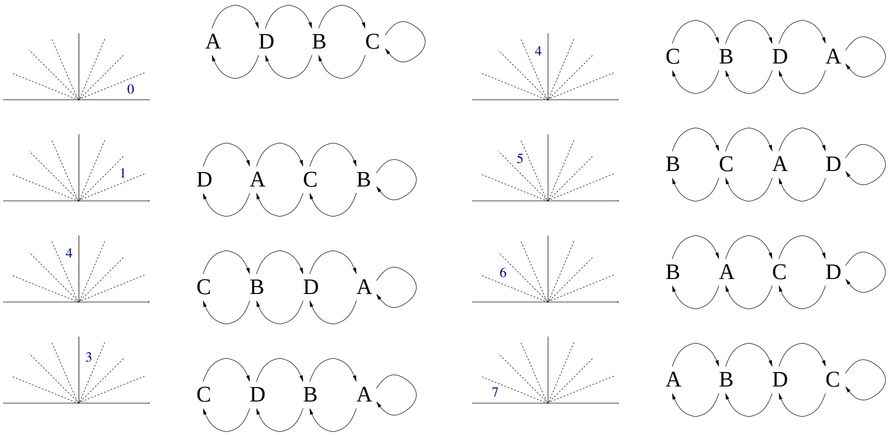

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Definition

A letter in $\{A, B, C, D\}$ is *sandwiched* if it is preceeded and followed by the same letter.

Example

In D B B C B A A D the letter C is *sandwiched* between to Bs.

Definition (Derived sequence)

If w is an octagon cutting sequence, the derived sequence w' is obtained keeping only letters which are *sandwitched*.

Example

If w = ... D A D B C C B C C B D A D B C B D B D B C B D ...,

w' = ... A

Definition (Derivable sequences)

A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ is *derivable* if it is admissible and its derivative is still admissible. The sequence w is *infinitely derivable* if each of its derivatives is derivable.

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Definition (Derived sequence)

If w is an octagon cutting sequence, the derived sequence w' is obtained keeping only letters which are *sandwitched*.

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Characterization of octagon cutting sequences

Theorem (Smillie-Ulcigrai)

An octagon cutting sequence is infinitely derivable.

- *Remark 1*: we prove a similar characterization for every regular $2n$-gon. It also holds for *double* n-gon (n odd), see Diana Davis.
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One can write substitutions σ_i, where $1 \leq i \leq 7$ s.t. w is in the closure of octagon cutting sequences iff

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[where $\mathcal{T}_{i_0}$ converts sequences in $\mathcal{A}'$ to sequences in $\mathcal{A}$ admissible in sector i_0.]

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As in the case of the square, the theorems follow if we prove that:

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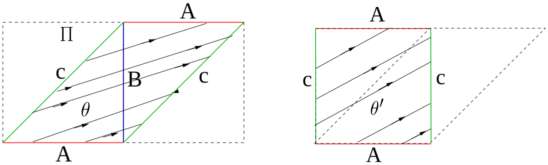

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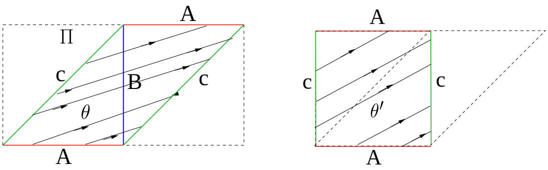

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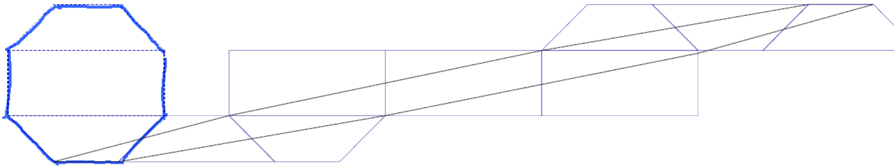
Geometric renormalization for the octagon

Let w be the cutting sequence of a trajectory in direction $0 \leq \theta \leq \pi/8$.

Let $O' = \begin{pmatrix} -1 & 2(1+\sqrt{2}) \\ 0 & 1 \end{pmatrix} O$.



The magic of the octagon: Veech showed that O' can be mapped back to O by *cutting and pasting* by translations.

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Lemma

The derived sequence w' coincides with the cutting sequence of the same trajectory in direction θ with respect to the sides of O'.

Let us renormalize by applying the *inverse*: $O' \mapsto O$, $\theta \mapsto \theta'$.

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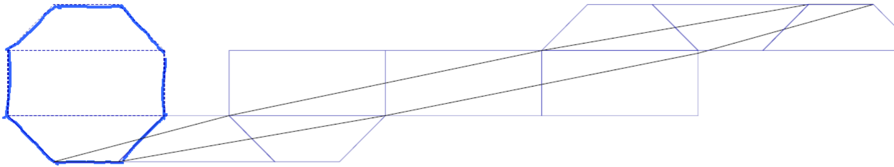

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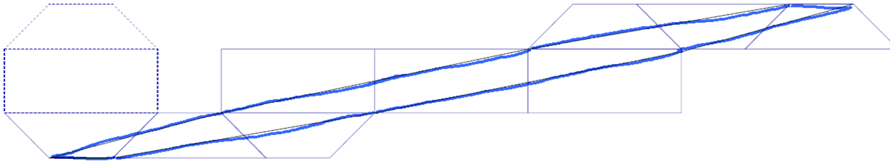

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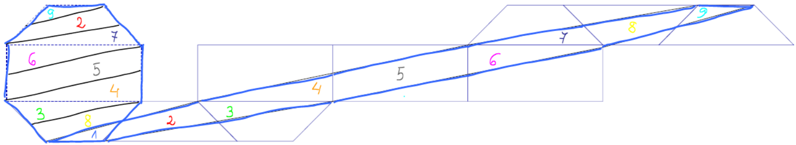

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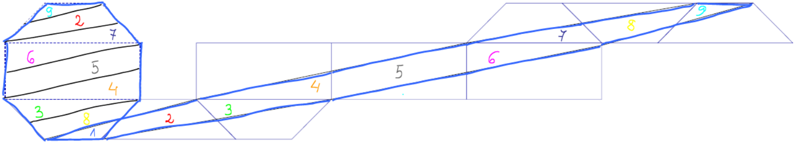

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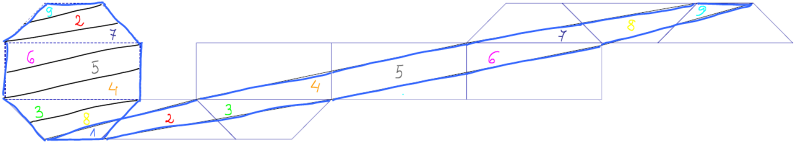

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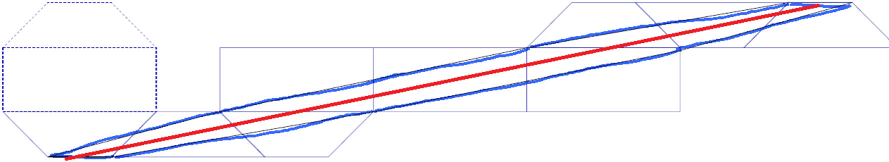

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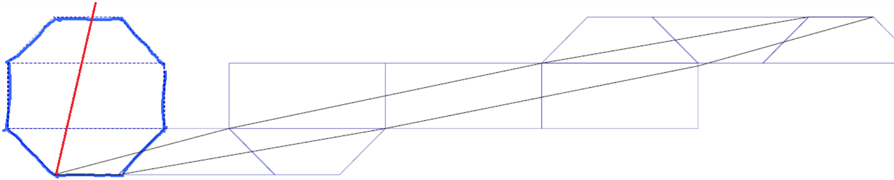

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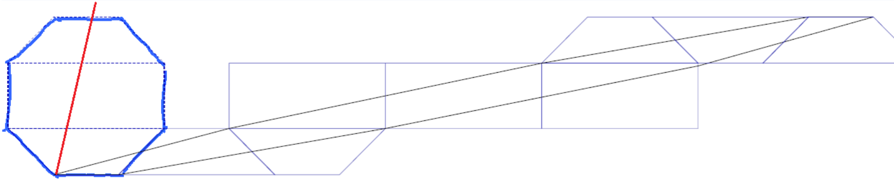

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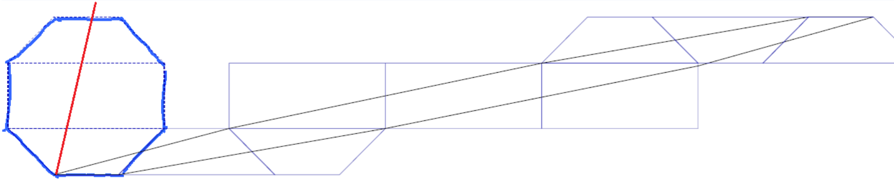

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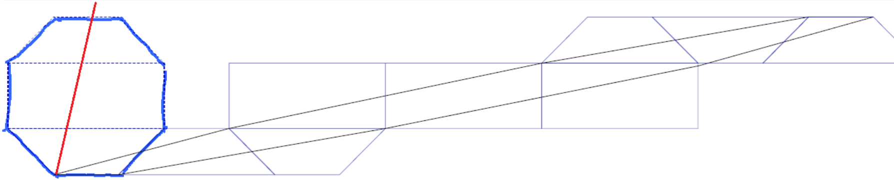

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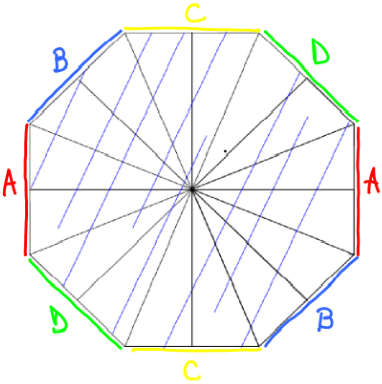

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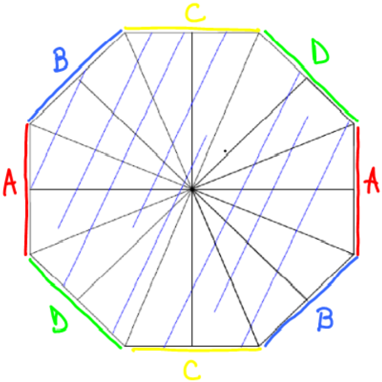

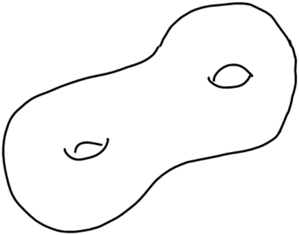

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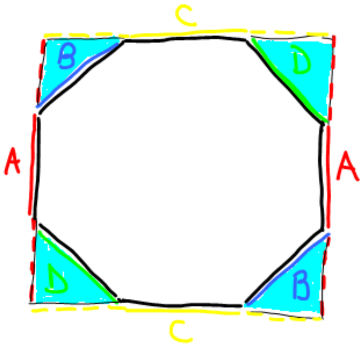

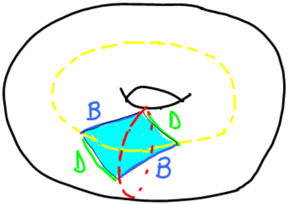

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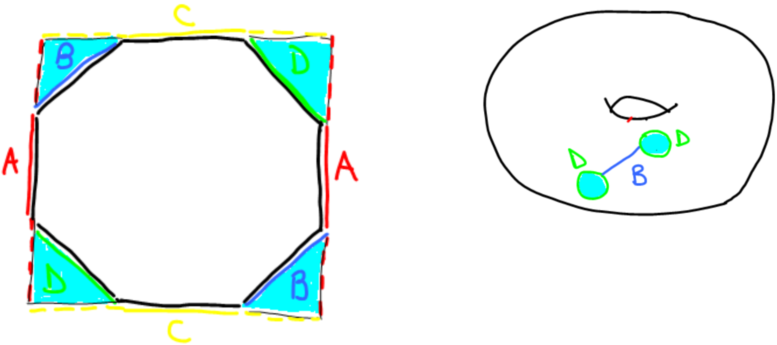

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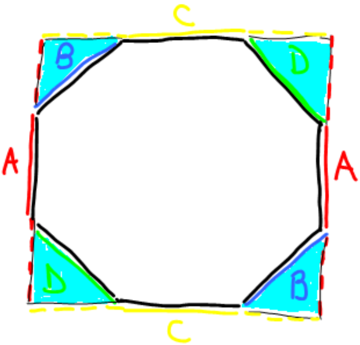

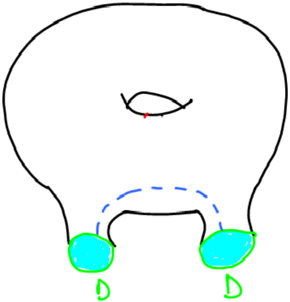

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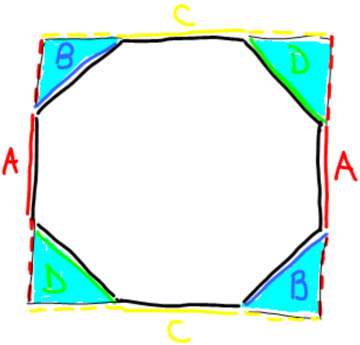

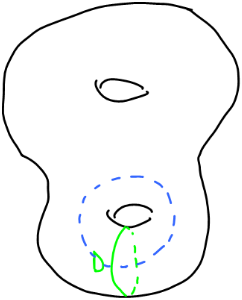

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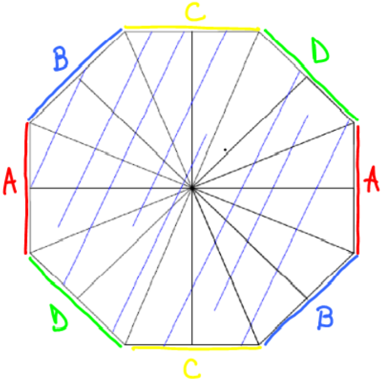

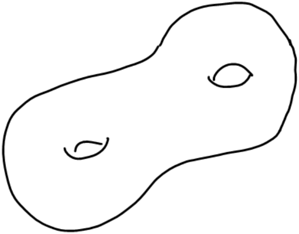

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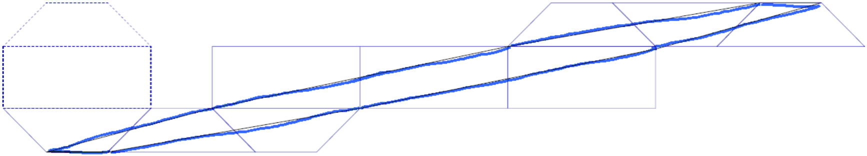

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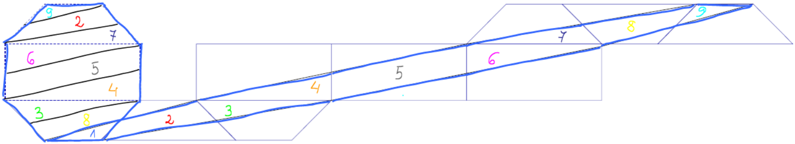

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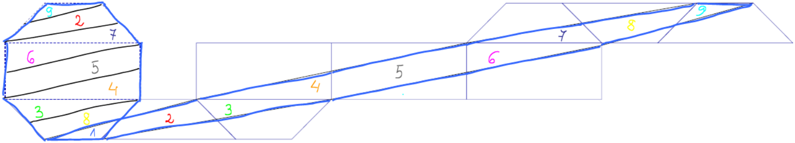

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Translation surfaces

Glueing polygons by translations we get *translation surfaces*:

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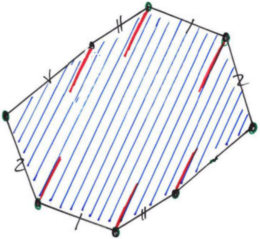

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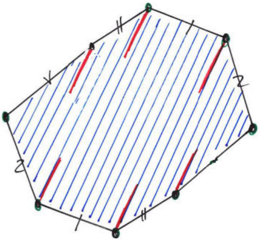

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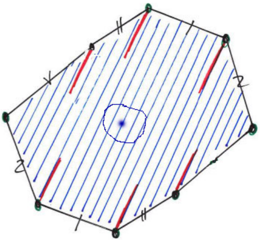

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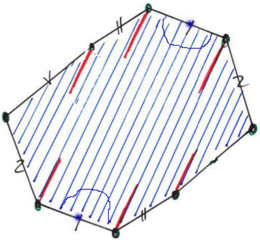

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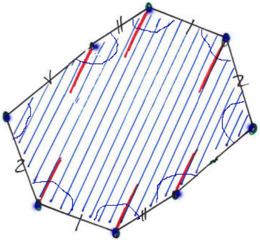

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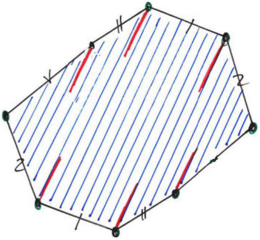

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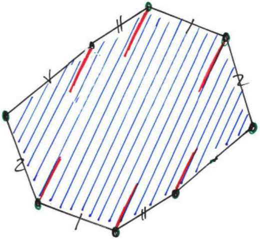

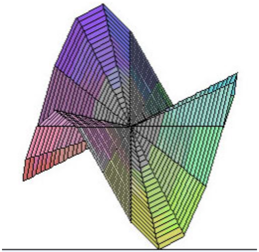

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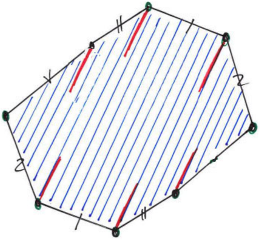

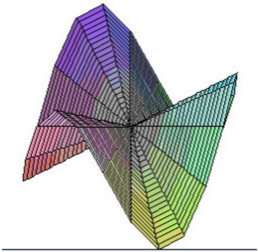

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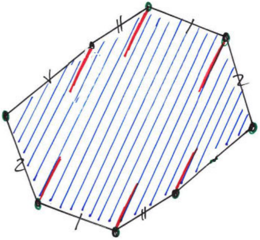

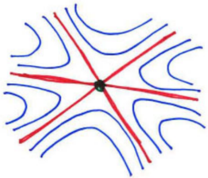

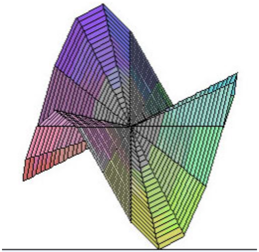

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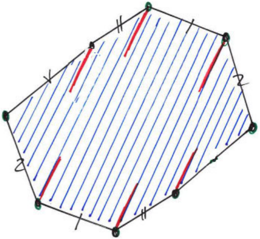

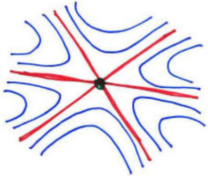

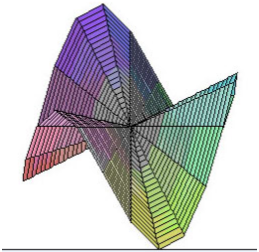

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Examples of translation surfaces S which have non trivial *affine diffeomorphisms*:

Ex 2. The regular octagon surface S_O:

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The torus and the octagon surfaces are examples of *Veech surfaces*, i.e. surfaces which are *rich* of affine diffeos.

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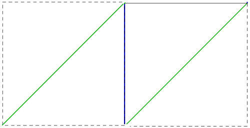

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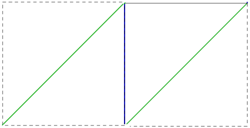

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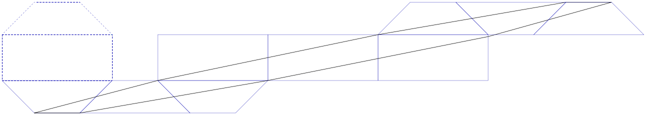

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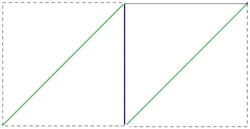

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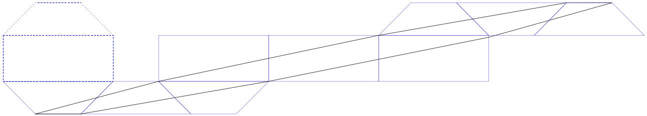

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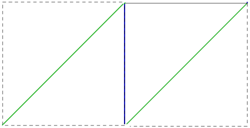

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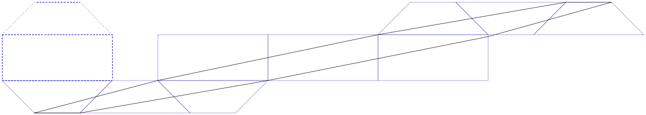

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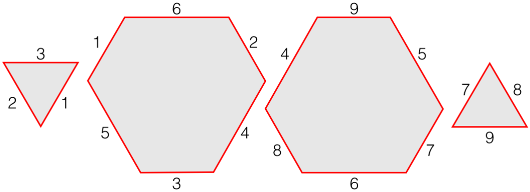
$S_{m,n}$ has m semi-regular polygons with rotational symmetry $2\pi/n$.



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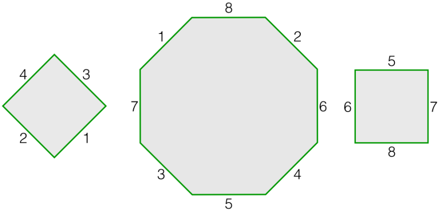

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[Polygonal presentation by Pat Hooper (in some cases by A. Wright)]

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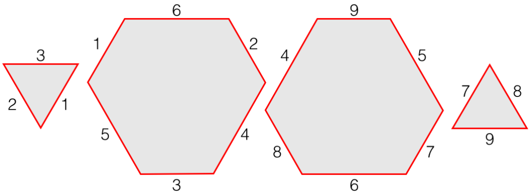

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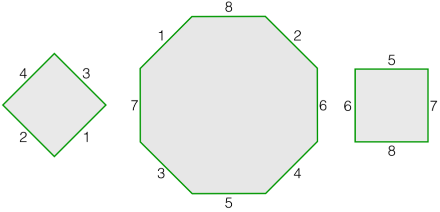

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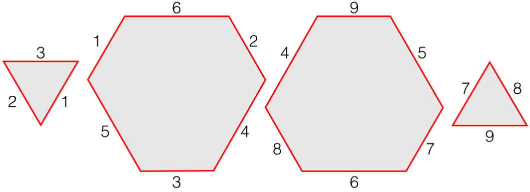

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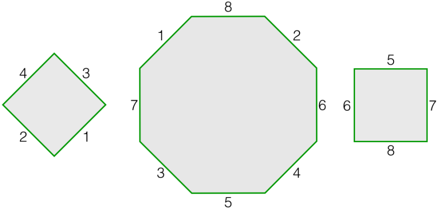
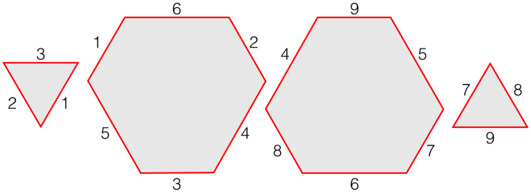


[Polygonal presentation by Pat Hooper (in some cases by A. Wright)]

Bouw-Moeller surfaces $S_{m,n}$

$S_{m,n}$ has m semi-regular polygons with rotational symmetry $2\pi/n$.



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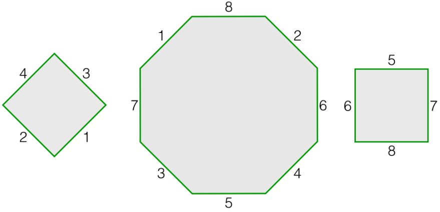

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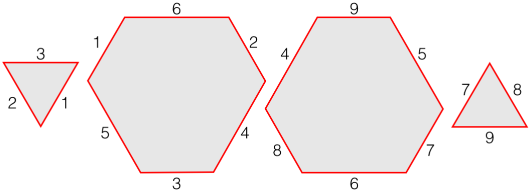

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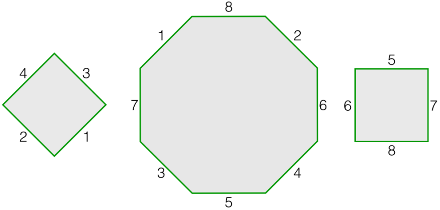

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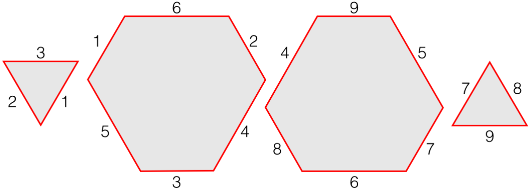

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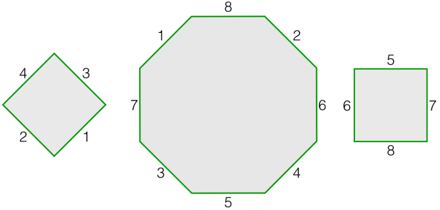

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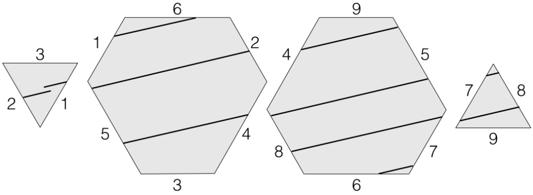
Characterization of Bouw-Moeller cutting sequences

Consider linear trajectories on the Bouw-Moeller surface $S_{m,n}$.

Labels pairs of sides by an alphabet A.



Theorem (Diana Davis- Irene Pasquinelli-U')

One can write substitutions σ_i, $1 \leq i \leq (m-1)(n-1)$, s.t. w is in the closure of cutting sequences for $S_{m,n}$ iff

$$\exists\ (i_k)_{i \in \mathbb{N}},\ s.t. \quad w \in \bigcap_n \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\mathscr{A}^{\mathbb{Z}}.$$

- This gives an $\mathcal{S}$-adic characterization and a *recognition algorithm*;

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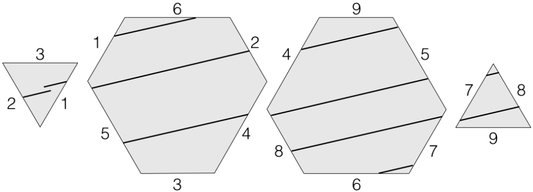

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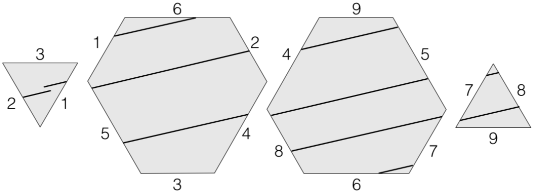

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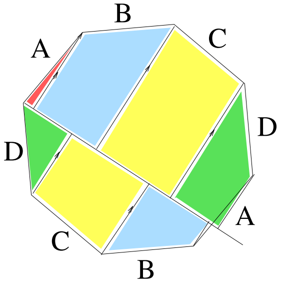
IETs as Poincaré sections

Consider a linear flow on a translation surface. Take a transverse section.



The Poincaré first return map on a section is an interval exchange transformation (IET).

[*Remark:* Linear flows on translation surfaces and IETs have entropy zero. Cutting sequences of linear trajectories and itineraries of IETs have linear complexity.]

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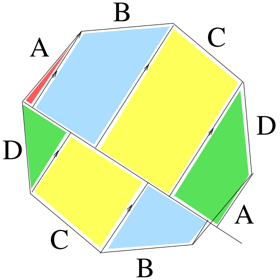

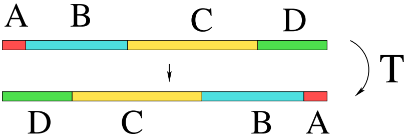

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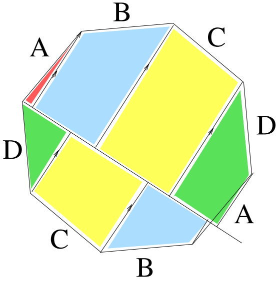

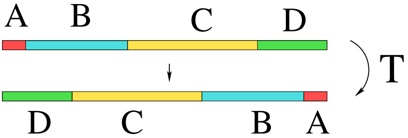

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