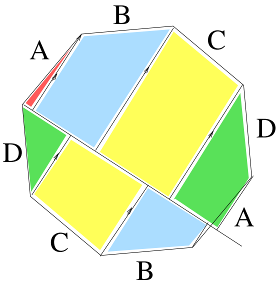
IETs as Poincaré sections

Consider a linear flow on a translation surface. Take a transverse section.



The Poincaré first return map on a section is an interval exchange transformation (IET).

[*Remark:* Cutting sequences of the linear flow are itineraries of the Poincaré section with respect to some intervals I_A, I_B, I_C, I_D.]

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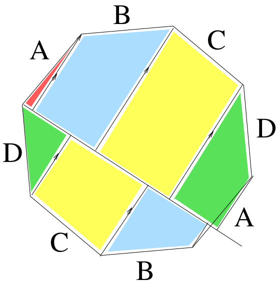

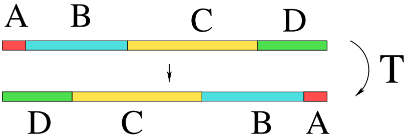

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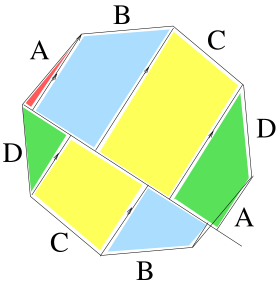

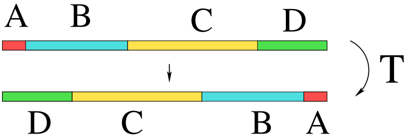

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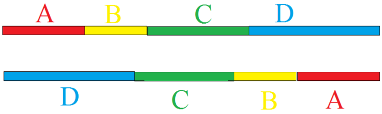
Rauzy-Veech induction: case Top



$D = \alpha_t$, $A = \alpha_b$

$\lambda_D^{(n)} > \lambda_A^{(n)}$ so α_t winner.

Induce on $[0, 1 - \lambda_{\alpha_b}] = I \setminus I_A^{(n)}$.

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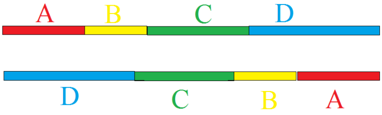

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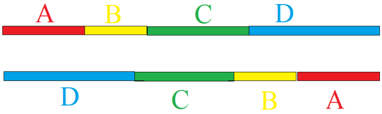

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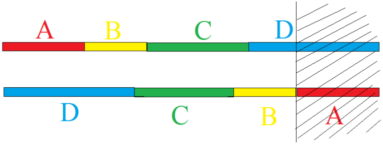

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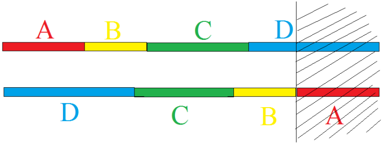

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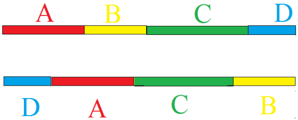

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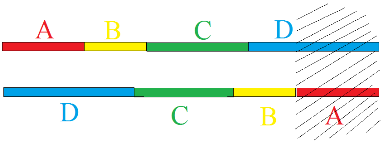

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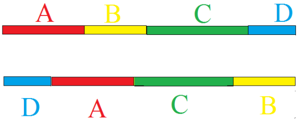

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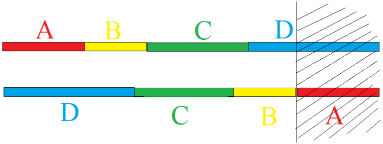

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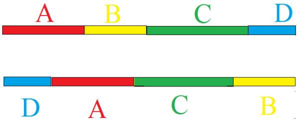

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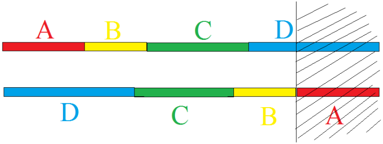

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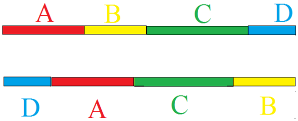

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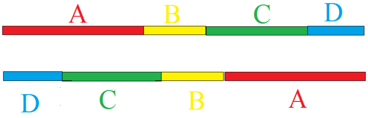

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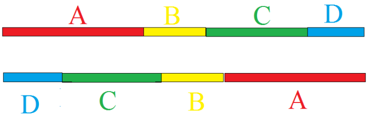

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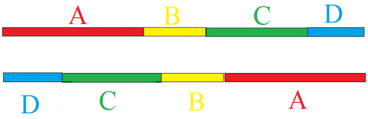

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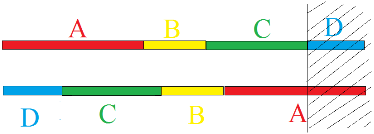

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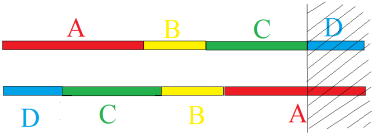

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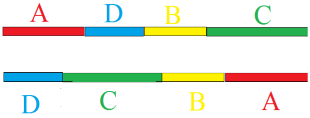

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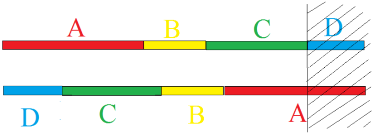

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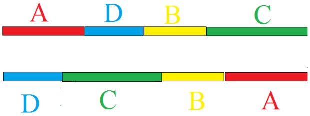

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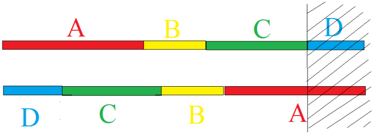

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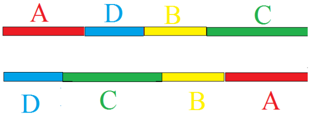

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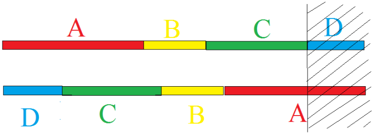

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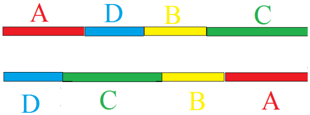

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