Beyond Sturmian: a characterization of octagon cutting sequences

Corinna Ulcigrai

(based on joint work with John Smillie
Cornell University)

ICTP, Trieste, 23 July 2018

Linear trajectories and cutting sequences

Consider a regular polygon, for simplicity with $2n$ sides.
As an example, in the talk we will consider a *regular octagon*.
Glue opposite sides.
Label pairs of sides by $\{A, B, C, D\}$.

Let φ_t^θ be the *linear flow* in direction θ: trajectories which do not hit singularities are straight lines in direction θ.

Definition (Cutting sequence)

The *cutting sequence* in $\{A, B, C, D\}^{\mathbb{Z}}$ that codes a bi-infinite linear trajectory of φ_t^θ consists of the sequence of labels of the sides hit by the trajectory.

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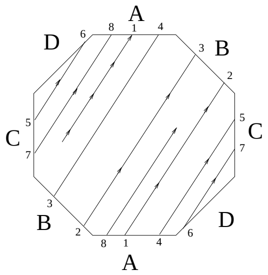

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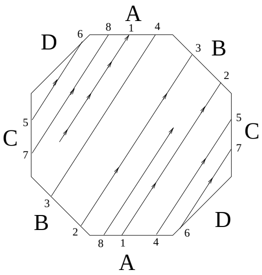

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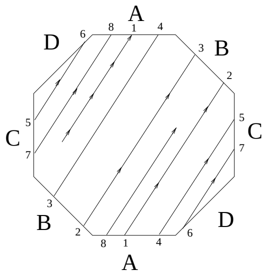

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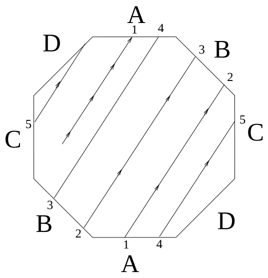

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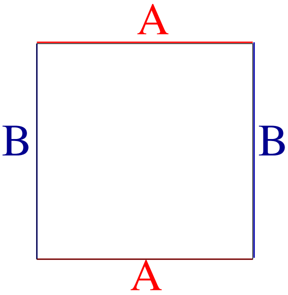
A classical case: Sturmian sequences

Consider the special case in which the polygon is a square.



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Sturmian sequences are characterized by having the smallest possible *complexity* among non-periodic sequences.

(Let $P_w(n)$ the number of words of lenght n which appear in the sequence w: $P_w(n) = n$ iff w is periodic. Sturmian sequences satisfy $P_w(n) = n + 1$.)

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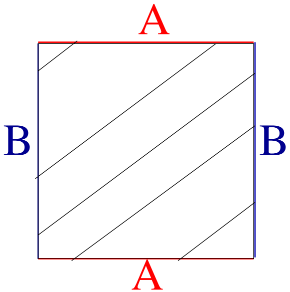

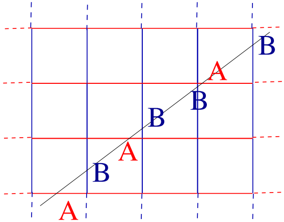

In this case the cutting sequence correspond to the sequence of horizontal (letter A) and vertical (letter B) sides crossed by a line in direction θ in a *square grid*..

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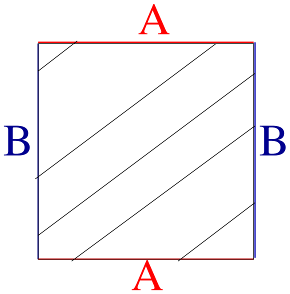

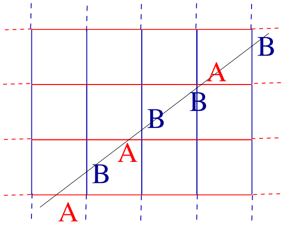

Square cutting sequences are *Sturmian sequences*. They were studied since Hedlund e Morse.

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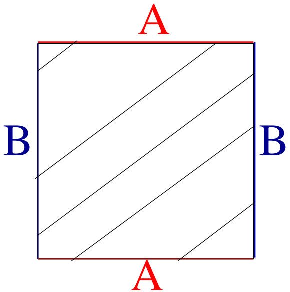

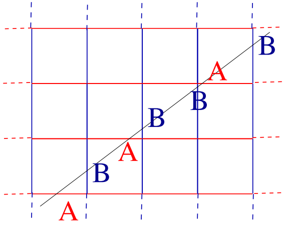

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Cutting sequences give a symbolic coding of the following systems:

- Translation surfaces
- Poligonal Billiards
- Interval exchange transformations

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  Glueing opposite sides one gets a surface of genus 2, with a flat metric with a singularity (it's a translation surface); φ_t^θ is the geodesic flow with respect to the flat metric;
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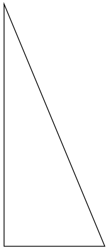

For example: A billiard with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$ and $\frac{3\pi}{8}$(motion of a particle with elastic reflections at sides) by a procedure called *unfolding* is equivalent to the flow φ_t^θ in the octagon.

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Characterization of cutting sequences

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Describe explicitely the symbolic sequences which are cutting sequences of trajectories.

In particular, answer the following questions:

D1) Which sequences in $\{A, B, C, D\}^{\mathbb{Z}}$ are cutting sequences?

D2) Given a cutting sequence, can one recover the direction of the trajectory?
Given a *finite* piece of a cutting sequence,can one recover a sector of possible directions?

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  - Connection with Continued Fractions;
  - Sketch of proof for the square;

- Regular polygons with $2n$ lati:

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The octagon: isometries and sectors

Let D_8 be the isometries group of the octagon.
The letters $\{A, B, C, D\}$ are invariant with respect to a central symmetry. The other elements induce permutations of $\{A, B, C, D\}$ *for example:*

$A \mapsto C, C \mapsto A, B \mapsto B, D \mapsto D$.

Let us assume that the direction of the trajectory is $\theta \in \left[0, \frac{\pi}{8}\right]$. A fundamental domain for D_8 is $\Sigma_0 := \left[0, \frac{\pi}{8}\right]$. Thus, acting by an element of D_8, up to a permutation of the letters $\{A, B, C, D\}$, we can consider $\theta \in \Sigma_0 := \left[0, \frac{\pi}{8}\right]$.

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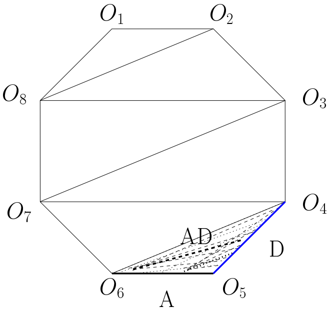

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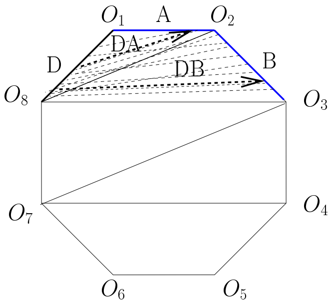

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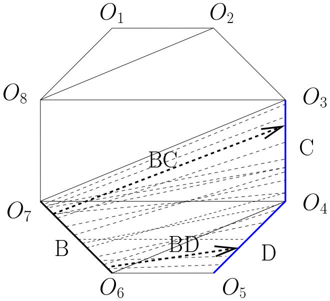

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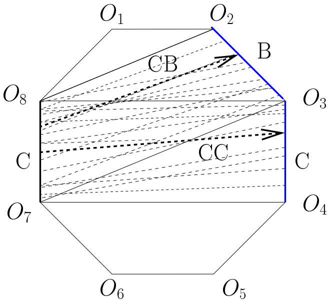

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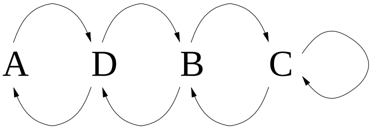

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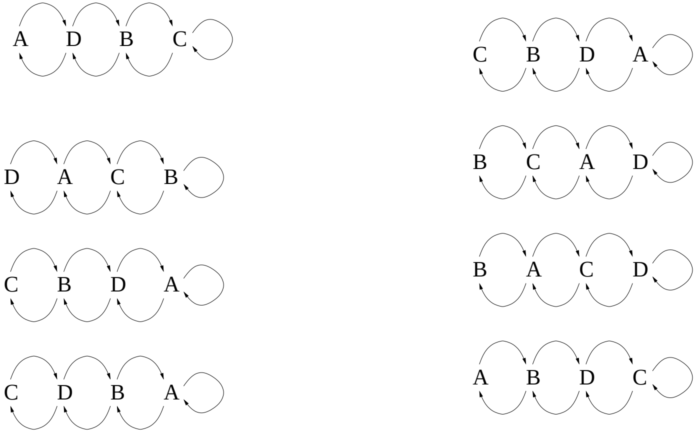
The octagon: possible transitions

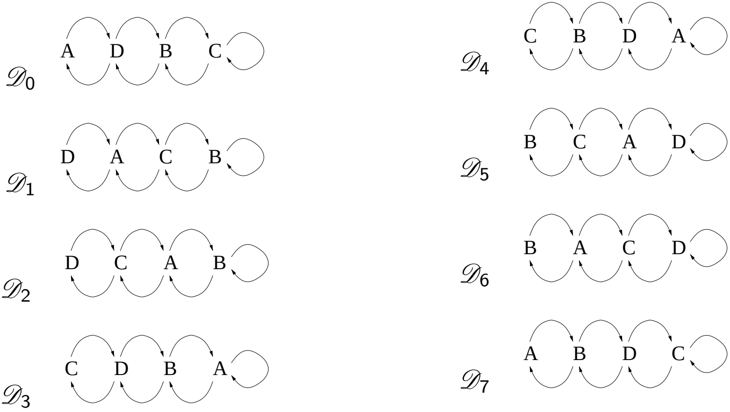
Permuting the letters we obtain the diagrams corresponding to the other sectors:



Admissible sequences

Definition

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *admissible* if it gives an infinite path on one of the following diagrams:



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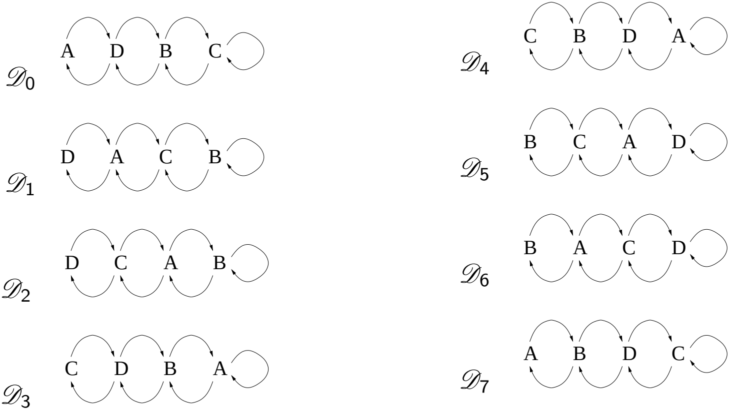

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A letter in $\{A, B, C, D\}$ is *sandwitched* if it is preceeded and followed by the same letter.

Example

In D B B C B A A D the letter C is *sandwitched* between to Bs.

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If w is an octagon cutting sequence, the derived sequence w' is obtained erasing all letters which are *NOT sandwitched*.

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If w = . . . D A D B C C B C C B D A D B C B D B D B C B D . . . ,
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Necessary condition and sequence of sectors

Theorem

An octagon cutting sequence is infinitly derivable.

The converse is not true, but we can describe exactly the condition which one needs to add.

Definition

Let w be infinitly derivable and let $w^{(n)}$ be the n^{th} derived sequence. The sequence $\{s_k\}_{k\in\mathbb{N}} \in \{0, 1, \ldots, 7\}^{\mathbb{N}}$ is a *sequence of sectors* for w if for each k, $w^{(k)}$ gives a path on the diagram $\mathscr{D}_{s_k}$.

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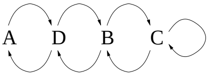

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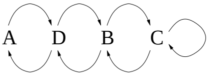

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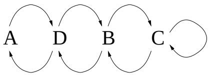

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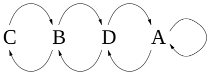

Octagon Continued Fractions

Let $F : [0, \pi] \to [\pi/8, \pi]$ the following map, that we call *Octagon Farey map*:

Definition

The octagon continued fraction expansion of θ is

$$\theta = [s_0, s_1, s_2, \ldots, s_k, \ldots]_O \quad \text{iff} \quad \{\theta\} = \cap_k F_{s_0}^{-1} F_{s_1}^{-1} \ldots F_{s_k}^{-1} [0, \pi].$$

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Let w be an octagon cutting sequence.

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As in the case of the square, the theorems follow if we prove that:

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If w is an octagon cutting sequence, also the derived sequence w' is an octagon cutting sequence.

Furthermore, if w is the cutting sequence of a trajectory in direction θ, the derived sequence w' is a cutting sequence of a trajectory in direction $\theta' = F(\theta)$, where F is the octagon Farey map.

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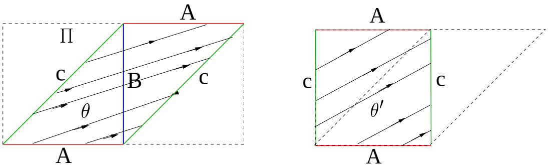

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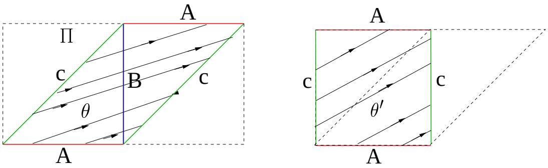
Furthermore, if w is the cutting sequence of a trajectory in direction θ, the derived sequence w' is a cutting sequence of a trajectory in direction $\theta' = F(\theta)$, where F is the octagon Farey map.

To prove it, one uses an argument in the same spirit of:



i.e. one uses renormalization in the space of affine deformations of the octagon.
Regular polygons are rich of affine automorphism.

Affine automorphism and Veech group

Let S_O be the surface obtained glueing opposite sides of the octagon by translations: S_O is an example of a *translation surface*, i.e. it has an atlas whose changes of coordinates are of the form $z \mapsto z + c$.

Definition

An automorphism $\Psi : S \mapsto S$ of a translation surface S is an affine automorphism if it is affine in each chart and $D\Psi(z)$ is independent on $z \in S$. Let

$$Aff(S) = \{\Psi, \quad \Psi \, \text{affine automorphism}\}.$$

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Example (1)

Se $\Psi \in D_8$
is an isometry of O,
clearly one has
$\Psi \in Aff(O)$.

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Example (2)

The matrix $\begin{pmatrix} 1 & 2(1+\sqrt{2}) \\ 0 & 1 \end{pmatrix} \in V(O)$.

The Veech group of the octagon

The Veech group $V(S)$ is the group of linear parts of $Aff(S)$:

$$V(S) = \{D\Psi, \quad \Psi \in Aff(S)\} \subset SL(2,\mathbb{R})^{\pm}.$$

Ex 1 $V(\mathbb{T}^2)^+ = \; < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} > \; = SL(2,\mathbb{Z}).$

Ex 2

$$V(O) = \; < \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \; \begin{pmatrix} 1 & 2(1+\sqrt{2}) \\ 0 & 1 \end{pmatrix} >$$

If S is a translation surface glued from a regular polygon, $V(S)$ is a lattice in $SL(2,\mathbb{R})^{\pm}$ (Veech)

The surfaces S for which $V(S)$ is a lattice are actively researched in Teichmüller dynamics.

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Derivation and renormalization in the octagon

Let w be the cutting sequence of a trajectory in direction $\theta \in \Sigma_0$. Let
$O' = \begin{pmatrix} -1 & 2(1+\sqrt{2}) \\ 0 & 1 \end{pmatrix} O$.

Lemma

The derived sequence w' coincides with the cutting sequence of the same trajectory in direction θ with respect to the sides of O'.

Let us renormalize:
$O' \mapsto O$
$\theta \mapsto \theta'$

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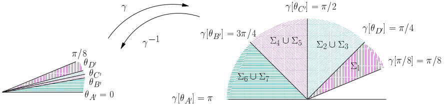

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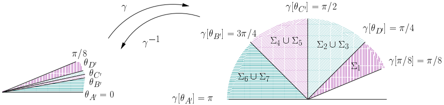

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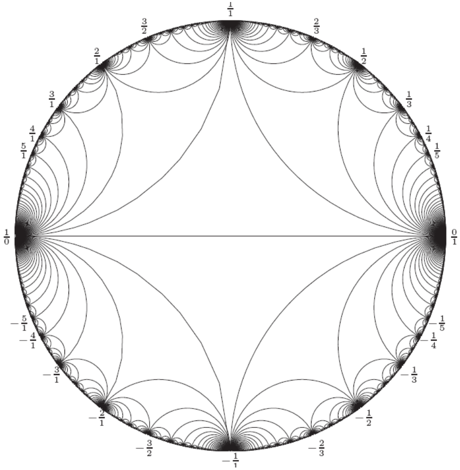
Renormalization, modular surface and continued fractions

The space of lattices is $\mathbb{H}/SL(2,\mathbb{Z})$. (moduli space of tori with a flat metric)

Farey Tessellation: $V(Q) = < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} > \mathscr{T}(0,1,\infty)$



Q square

$$S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Given θ, g_t^θ geodesics

$$\theta = \frac{1}{a_0 + \frac{1}{a_1 + \dots}}$$

g_t^θ is approximated by $S_1{}^{a_0} S_2{}^{a_1} S_1{}^{a_2} \dots$

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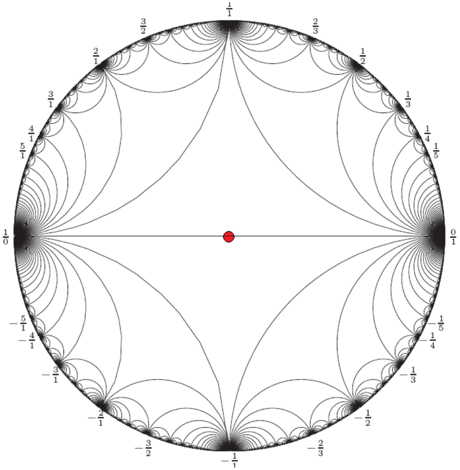

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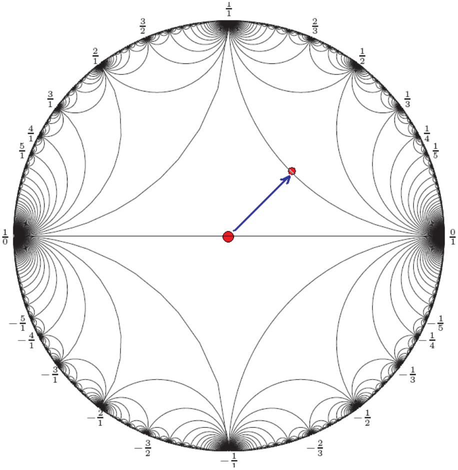

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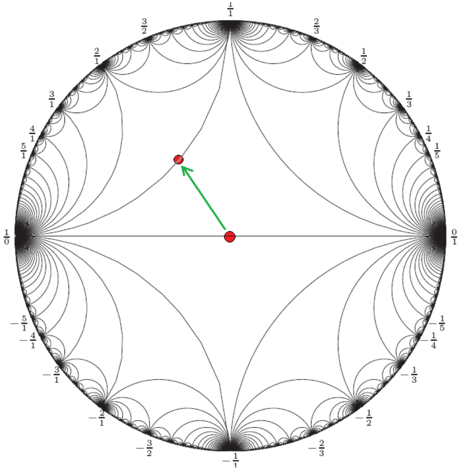

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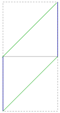

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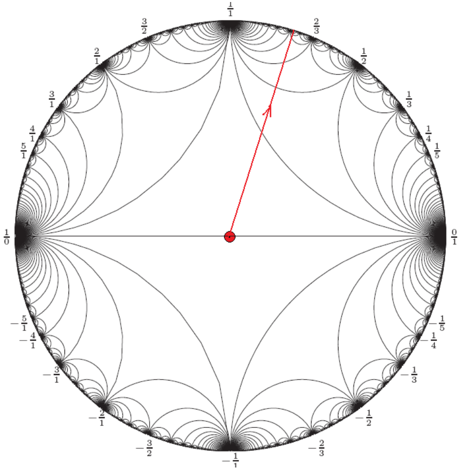

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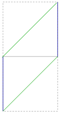

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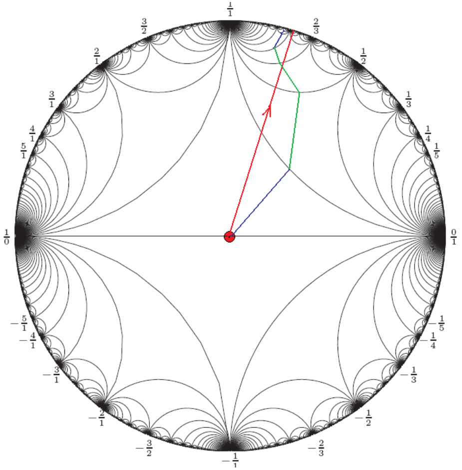

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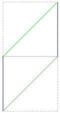

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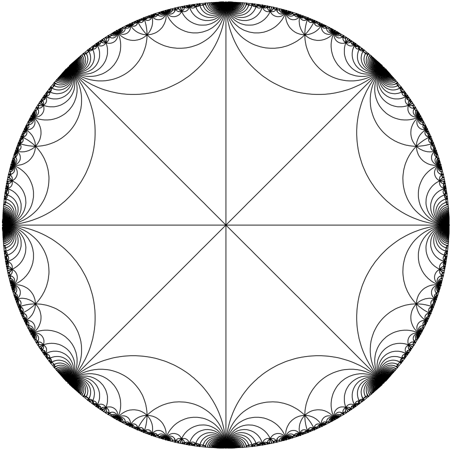
$SL(2,\mathbb{R}) \cdot O = \{A \cdot O, A \in SL(2,\mathbb{R})\}$ affine deformations of the octagon

$$\frac{\text{affine deformations}}{\text{affini automorphisms}} = \frac{SL(2,\mathbb{R})}{V(O)} \quad \text{(Teichmueller disk)}$$



$V(O) \cdot O$ centers of ideal octagons

tree of renormalization moves

Given θ, g_t^θ geodesics is approximated by a sequence of renormalization moves

the Farey map of the octagon is given by the action on $\partial\mathbb{D}$.

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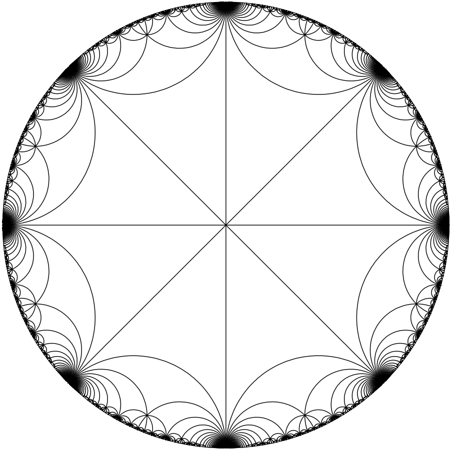

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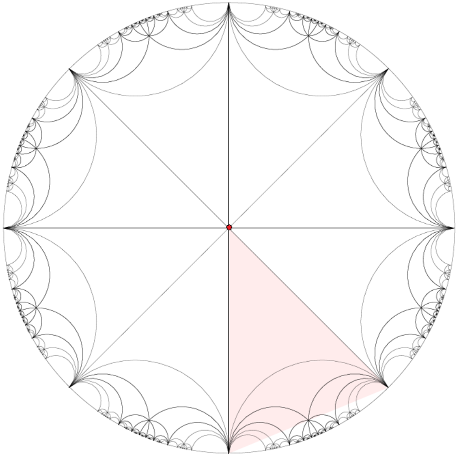

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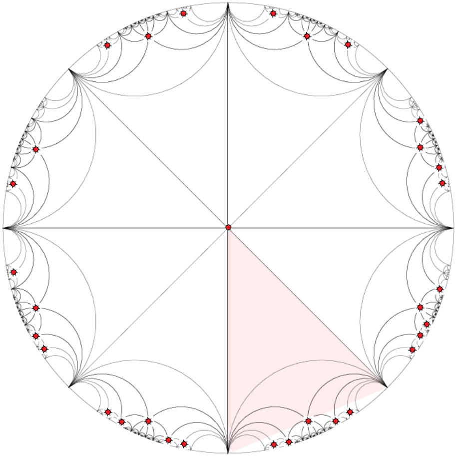

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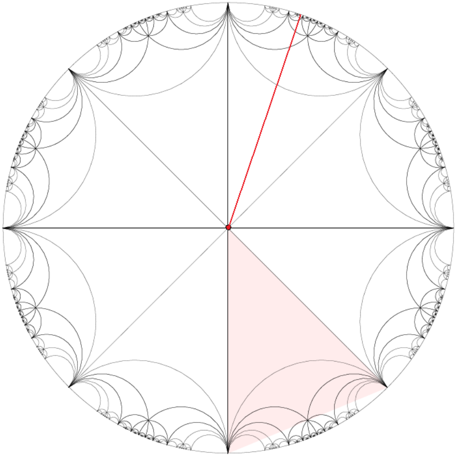

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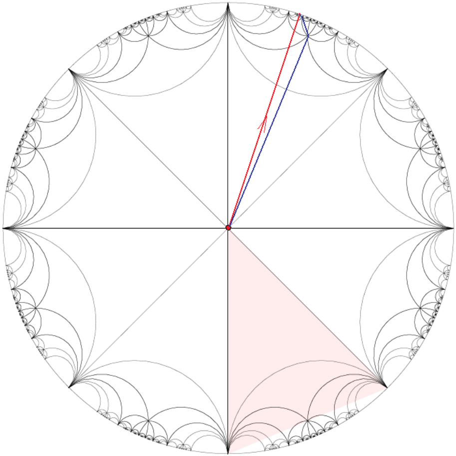

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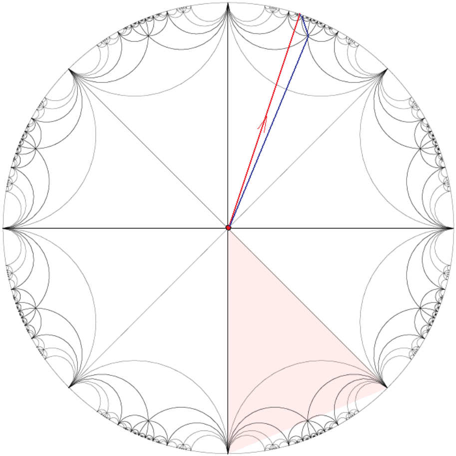

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Generation of cutting sequences

Let us define operators g_i^j that invert derivation.

The operator g_i^j *interpolates* a sequence w admissible in $\mathscr{D}_j$ producing a sequence ammissible in $\mathscr{D}_i$ and such that $(g_i^j(w))' = w$.

Definition

Let w be ammissible in $\mathscr{D}_0$. The sequence $g_0^2 w$ us obtained interpolating the letters corresponding to vertices in Figure with the words on the arrows:

Example

$w \quad = \ldots$ D C A B B B A C $\ldots$

$g_0^3 w \quad = \ldots$ D B C $\ldots$

The only sandwiched letters are the coloured ones, thus $(g_0^3 w)' = w$.

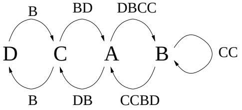
Generation of cutting sequences

Let us define operators g_i^j that invert derivation.

The operator g_i^j *interpolates* a sequence w admissible in $\mathscr{D}_j$ producing a sequence ammissible in $\mathscr{D}_i$ and such that $(g_i^j(w))' = w$.

Definition

Let w be ammissible in $\mathscr{D}_0$. The sequence $g_0^2 w$ us obtained interpolating the letters corresponding to vertices in Figure with the words on the arrows:



Example

$$w = \ldots\ D\ C\ A\ B\ B\ B\ A\ C\ \ldots$$
$$g_0^3 w = \ldots\ D\ \ B\ C\ \ldots$$

The only sandwiched letters are the coloured ones, thus $(g_0^3 w)' = w$.

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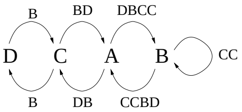

Example

$$\begin{aligned} \text{w} &= \ldots \text{ D C A B B B A C} \ldots \\ g_0^3 w &= \ldots \text{ D } \quad \text{B C} \ldots \end{aligned}$$

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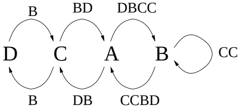

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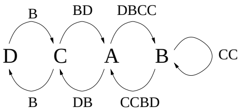

Example

$$\begin{array}{ll} \text{w} & = \ldots \text{ D } \quad \text{C} \qquad \text{A B B B A C} \ldots \\ g_0^3 w & = \ldots \text{ D B C BD A } \ldots \end{array}$$

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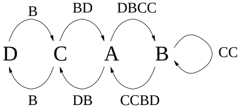

Example

$w \quad = \ldots$ D C A B B B A C ...

$g_0^3 w \quad = \ldots$ D B C BD A DBCC B ...

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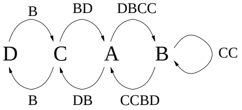

Example

$$\begin{array}{lcl} w & = & \ldots\ D\ \ C\ \ A\ \ B\ \ B\ \ B\ \ A\ \ C\ \ldots \\ g_0^3 w & = & \ldots\ D\ B\ C\ BD\ A\ DBCC\ B\ CC\ B\ CC\ B\ CCBD\ A\ DB\ C\ \ldots \end{array}$$

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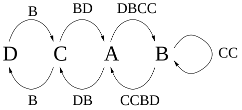
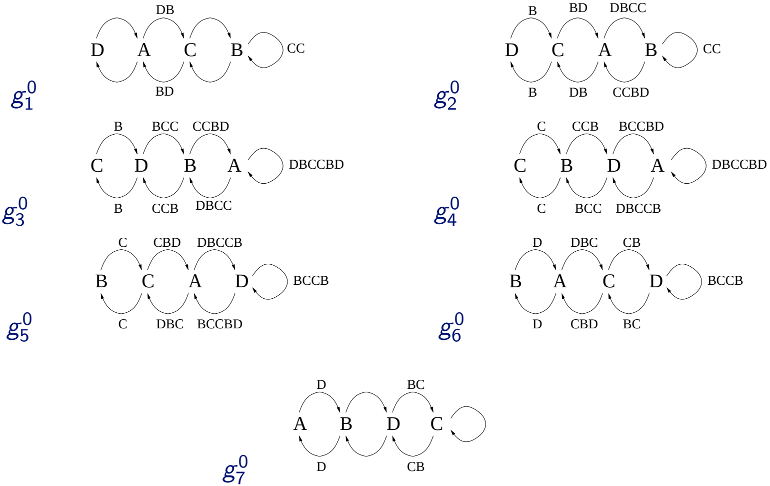


Example

w	=	...	D		C		A			B		B		B		A		C	...
$g_0^3 w$	=	...	D	B	C	BD	A	DBCC		B	CC	B	CC	B	CCBD	A	DB	C	...

The only sandwiched letters are the coloured ones, thus $(g_0^3 w)' = w$.

The interpolation operators



The other operators are obtained from these ones by permuting the letters.

Characterization of the closure of cutting sequences

Lemma

If w is a cutting sequence and $\{s_n\}_{n\in\mathbb{N}}$ a sequence of sectors, we have

$$w \in \cap_n \, g_{s_1}^{s_0} \, g_{s_2}^{s_1} \dots g_{s_n}^{s_{n-1}} \{A, B, C, D\}^{\mathbb{Z}}.$$

This condition, together with infinite derivability, is *necessary* and *sufficient* to characterize the *closure* of octagon cutting sequences (in $\{A, B, C, D\}^{\mathbb{Z}}$):

Theorem (Smillie-U)

A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ belongs to the closure of cutting sequences in the octagon iff there exists a squence $\{s_n\}_{n\in\mathbb{N}} \in \{0, \dots, 7\}^{\mathbb{N}}$ such that

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An analogous theorem holds for every regular polyon.

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