Exercise B.1. Let $U(\psi) = \psi \circ f$ be the Koopman operator, where $\psi \in L^2(X, \mathcal{B}, \mu)$.

(1) Prove that $U(1) = 1$, $U^*U = I$ and $\|U\| \leq 1$

(2) (*) Prove that the decomposition of $\mathcal{H} = L^2(X, \mathcal{B}, \mu)$

$$\mathcal{H} = \ker(U - I) \oplus \overline{\Im m(U - I)}$$

is orthogonal and U-invariant.

Excercise B.2. (Suggested by Amie Wilkinson). Let $f(x) = x + \frac{1}{2} \mod 1$ on the circle. For any $\psi \in L^p$, find $E(\psi|\mathcal{B}_f)$.

Exercise B.3.

(1) Prove that $\tilde{\psi} = E(\psi|\mathcal{B}_f)$ in the Mean Ergodic Theorem.

(2) Let $f(x) = 2x \mod 1$ in the circle. For any $\psi \in L^p$, find $E(\psi|\mathcal{B}_f)$