1.1   Metric Spaces and Basic Topology notions

In this section we briefly overview some basic notions about metric spaces and topology.

A *metric space* (X, d) is a space X with a *distance function* $d : X \times X \to \mathbb{R}^+$ (also called *metric*, from which the name metric space), that is a function which assigns to each pair of points $x, y \in X$ a real number $d(x, y)$ (their *distance*) and has the following properties:

Definition 1.1.1. A distance d is a function $d : X \times X \to \mathbb{R}^+$ such that

1. If $d(x, y) = 0$ then $x = y$;

2. For each $x, y \in X$ we have $d(x, y) = d(y, x)$ (symmetry);

3. The triangle inequality holds, that is for all $x, y, z \in X$

$$d(x, z) \leq d(x, y) + d(y, z).$$

Examples of metric spaces and distances are the following. The first three are classical examples, while the following two are useful in dynamical systems.

Example 1.1.1.   1. $X = \mathbb{R}$ or $X = [0, 1]$ with

$$d(x, y) = |x - y|.$$

2. $X = \mathbb{R}^2$ or $X = [0, 1] \times [0, 1]$ with the Euclidean distance: if $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$, their distance is

$$d(\underline{x}, \underline{y}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3. $X = S^1$ with the arc lenght distance $d(z_1, z_2)$ defined in §1.2.

4. $\Sigma^+ = \{0, 1\}^{\mathbb{N}^+}$, the shift space of one-sided sequences, is a metric space with the following distance:

$$d((a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty) = \sum_{i=1}^\infty \frac{|a_i - b_i|}{2^i}.$$

In particular two points $(a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty \in \Sigma^+$ are close if and only if the first block of digits agree: for example, if $a_k = b_k$ for $1 \leq k \leq n$, then the distance is less than $1/2^n$.

5. Let (X, d) be any metric space and $f : X \to X$. Then for each $n \in \mathbb{N}^+$ we can define a new distance, d_n, given by

$$d_n(x, y) = \max_{k=0,\ldots,n-1} d(f^k(x), f^k(y)).$$

Two points x, y are close in the d_n metric if their orbits up to time n stay close. We will use this distance to defined topological entropy in §2.3.

Exercise 1.1.1. Check that the distances in the previous Examples satisfy the properties in Definition 1.1.1.

In a metric space one can talk about convergence and continuity as in $\mathbb{R}^n$. Let (X, d) be a metric space. Given $x \in X$ and $\epsilon > 0$, let $B_d(x, \epsilon)$ be the ball of radius ϵ around the point x defined using the distance d, that is

$$B_d(x, \epsilon) = \{y \in X \text{ such that } d(x, y) < \epsilon\}.$$

If there is no ambiguity about the distance, we will often write simply $B(x, \epsilon)$, dropping the pedex d. We can use balls to define convergence:

Definition 1.1.2. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $\overline{x}$ and we write $\lim_{n\to\infty} x_n = \overline{x}$ if for any $\epsilon > 0$ there exists $N > 0$ such that $x_n \in B_d(\overline{x}, \epsilon)$ for all $n \geq N$.

We can use the distance to define the notion of open and closed sets.

Definition 1.1.3. A set $U \subset X$ of a metric space (X, d) is *open* if for any $x \in U$ there exists an $\epsilon > 0$ such that
$$B_d(x, \epsilon) \subset U.$$
A set $C \subset X$ is *closed* if its complement $X\backslash C$ is open.

Example 1.1.2. If $X \subset \mathbb{R}$ and $d(x, y) = |x - y|$, the intervals (a, b) are open sets and the intervals $[a, b]$ are closed sets. Also intervals of the form (a, ∞) or (∞, b) are open and intervals of the form $[a, \infty)$ or $(\infty, b]$ are closed. Intervals of the form $[a, b)$ or $(a, b]$ are neither open nor closed.

Exercise 1.1.2. Prove that a ball $B_d(x, \epsilon)$ is open (use the triangle inequality).

Open and closed sets in a metric space enjoy the following property:[1]

Lemma 1.1.1.   *(1) Countable unions of open sets are open: if $U_1, U_2, \ldots, U_n, \ldots$ are open sets, than $\cup_{k\in\mathbb{N}} U_k$ is an open set;*

(2) *Finite intersections of open sets are open: if $U_1, U_2, \ldots, U_N$ are open sets, than $\cap_{k=1}^{N} U_k$ is an open set.*

Exercise 1.1.3. Prove the lemma using the Definition 1.1.3 above.

Exercise 1.1.4. Given an example in $X = \mathbb{R}$ of a countable collection of open sets whose interesection is *not* open.

By using De Morgan Law, it follows that open sets have the following properties (remark that the role of intersections and unions is reversed):

Corollary 1.1.1.   *(1) Countable intersections of closed sets are closed: if $C_1, C_2, \ldots, C_n, \ldots$ are open sets, than $\cap_{k\in\mathbb{N}} C_k$ is a closed set;*

(2) *Finite unions of closed sets are closed: if $C_1, C_2, \ldots, C_N$ are open sets, than $\cup_{k=1}^{N} C_k$ is a closed set.*

Exercise 1.1.5. Prove the Corollary from Lemma 1.1.1

Exercise 1.1.6. Given an example in $X = \mathbb{R}$ of a countable collection of closed sets whose union is *not* closed.

Definition 1.1.4. A subset $Y \subset X$ is *dense* if for any non-empty open set $U \subset X$ there is a point $y \in Y$ such that $y \in U$.

One can check that this definition of dense set reduces to the usual definition of dense set for a subset $Y \subset \mathbb{R}$, that is, for each $y \in Y$ and $\epsilon > 0$ there exists $y \in Y$ such that $|x - y| < \epsilon$.

Definition 1.1.5. A metric space (X, d) is called *separable* if it contains a countable dense subset.

---

[1]It is possible to define open sets as an abstract collection of the subsets of X which satisfy certain properties. In this case, Property (1) in the Lemma is taken as an axiom. A collection $\mathscr{U}$ of subsets of X such that $\emptyset, X \in \mathscr{U}$ and Property (1) is satisfied is called a topology. In this case, sets in $\mathscr{U}$ are called open sets and complement of sets in $\mathscr{U}$ are called closed sets. A topological space $(X, \mathscr{U})$ is a space X with a topology $\mathscr{U}$.

Example 1.1.3. If $X = \mathbb{R}^n$ with the Euclidean distance, X is separable since the set $\mathbb{Q}^n$ given by all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ whose coordinates x_i are rational numbers is dense and it is countable.

Let (X, d_X) and (Y, d_Y) be a metric space. We will now consider properties of functions $f : X \to Y$.

Definition 1.1.6. A function $f : X \to Y$ is an *isometry* if it preserves the distances, that is

$$d_Y(f(x), f(y)) = d_X(x, y) \quad \forall \ x, y \in X.$$

We already saw an example of isometry:

Example 1.1.4. If $X = Y = S^1$ is the circle with the arc lenght distance $d = d_X = d_Y$, then $f = R_\alpha$ the rotation by $2\pi\alpha$ is an isometry.

Definition 1.1.7. A function $f : X \to Y$ is *continuous* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$f(B_{d_X}(x, \delta)) \subset B_{d_Y}(y, \epsilon).$$

Exercise 1.1.7. Check that if $X, Y \subset \mathbb{R}$ and $d_X(x_1, x_2) = |x_1 - x_2|$, $d_Y(y_1, y_2) = |y_1 - y_2|$ this gives the usual ϵ, δ definition of continuity of a real function.

Exercise 1.1.8. Let $X = Y = S^1$ and $d_X = d_Y = d$ be the arc lenght distance. Prove that

(a) The rotation $R_\alpha : S^1 \to S^1$ by $2\pi\alpha$ is continuous;

(b) The doubling map $f : S^1 \to S^1$ given in this coordinates by $f(e^{2\pi i\theta}) = (e^{2\pi i2\theta})$ is continuous.

It is enough to know which are the open sets in X and Y to define the notion of continuity:[2]

Lemma 1.1.2. *A function $f : X \to Y$ is continuous if and only if for each open set $U \subset Y$ the preimage $f^{-1}(U)$ is an open set of X.*

Proof. Assume that f is continuous. Let $U \subset Y$ is open and let us show that $f^{-1}(U)$ is open. We have to show that for each $x \in f^{-1}(U)$ there is an open ball contained in $f^{-1}(U)$. Let $y = f(x)$. Clearly $y \in U$ since $x \in f^{-1}(U)$. By definition of open set there exists $\epsilon > 0$ such that $B_{d_Y}(y, \epsilon) \subset Y$. By definition of continuity, there exists $\delta > 0$ such that

$$f(B_{d_X}(x, \delta)) \subset B_{d_Y}(y, \epsilon) \subset U,$$

thus $B_{d_X}(x, \delta) \subset f^{-1}(U)$. This shows that $f^{-1}(U)$ is open.
The other implication is left as an exercise. $\qquad\square$

Exercise 1.1.9. Prove the other implication in Lemma 1.1.2, that is show that if a function $f : X \to Y$ between two metric spaces $(X, d_X), (Y, d_Y)$ is such that for each open set $U \subset Y$ the preimage $f^{-1}(U)$ is an open set of X, then f is continuous in the sense of Definition 1.1.7.

The last metric space notion that we will use is the notion of *compact* sets. Let (X, d) be a metric space.

Definition 1.1.8. [Sequentially compact] A subset $K \subset X$ is (sequentially) *compact* if for any sequence $(x_n)_{n \in \mathbb{N}} \subset K$ there exists a convergent subsequence $(x_{n_k})_{k \in N}$ and the limit $\lim_{k \to \infty} x_{n_k} = \overline{x}$ belong to K.

---

[2]The following Lemma can be taken as definition of a continuous function when $(X, \mathscr{U})$ is a topological space.

This property is called *sequentially compactness* since the definition involves sequences. There are other notion of compactness (see compactness by covers below) which are equivalent in a metric space, so we will simply say that a set is *compact* and use the term *sequentially compact* only when we specifically want to use the above property of compact sets.

Example 1.1.5. Closed bounded intervals $[a,b] \subset \mathbb{R}$ are sequentially compact (this is known as *Heine-Borel theorem*).

Conversely, in $\mathbb{R}$, if a set is not bounded or not closed, it is not compact. The following two are *non-examples*, that is examples of spaces that are *not* compact.

Example 1.1.6. The unbounded closed interval $[0, \infty)$ is *not* sequentially compact: consider for example the sequence $(x_n)_{n \in \mathbb{N}}$ given by $x_n = n$. The sequence has no convergent subsequence.

The open interval $(0,1)$ is *not* sequentially compact: consider for example the sequence $(x_n)_{n \in \mathbb{N}}$ given by $x_n = 1/n$. We have $\lim_{n \to \infty} x_n = 0$, but $0 \notin (0,1)$.

1.1.1   Compactness by covers (level M only, Extra for level 3)

In addition to the definition of sequential compact, there is an other definitions of compactness, *compactness by open covers*, which turn out to be equivalent in a metric space. Compactness by open covers is a more general definition of compactness and can be used as a defintion of compactness in any topological space (see Extra on topological spaces if interested).

Definition 1.1.9. An open cover of $K \subset X$ is a collection $\{U_\alpha\}_\alpha$ of open sets of X such that

$$X \subset \bigcup_\alpha U_\alpha$$

(this is why we say that they *cover X*). A *finite subcover* is a finite subset $\{U_{\alpha_1}, U_{\alpha_1}, \dots, U_{\alpha_N}\} \subset \{U_\alpha\}_\alpha$ which still covers, that is such that $X \subset \cup_{i=1}^N U_{\alpha_i}$.

Definition 1.1.10. [Compact by covers] A subset $K \subset X$ is *compact by covers* if for any open cover $\{U_\alpha\}_\alpha$ there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_1}, \dots, U_{\alpha_N}\} \subset \{U_\alpha\}_\alpha$ such that $X \subset \cup_\alpha U_\alpha$.

Example 1.1.7. The open interval $(0,1)$ is *not* compact by covers: consider for example the collection

$$\mathscr{U} = \left\{ \left(\frac{1}{n+2}, \frac{1}{n} \right), \qquad n \in \mathbb{N} \right\}$$

is an open cover, but $\mathscr{U}$ does not admit a finite subcover. Indeed, a finite subset of intervals in $\mathscr{U}$ is of the form

$$\left\{ \left(\frac{1}{n_1 + 2}, \frac{1}{n_1} \right), \left(\frac{1}{n_2 + 2}, \frac{1}{n_2} \right), \qquad , \left(\frac{1}{n_k + 2}, \frac{1}{n_k} \right) \right\}$$

so that if $\overline{n} = \max_{i=1,\dots,k} n_i$, no point in $\left(0, \frac{1}{\overline{n}+2} \right)$ is covered by the finite collection.

Theorem 1.1.1. *In a metric space (X, d), a subset $K \subset X$ is sequentially compact if and only if it is compact by covers.*

Since we will work only with metric spaces, we will simply say that a set is *compact* and use equivalenty either Definition 1.1.8 or 1.1.10.

Remark 1.1.1. In $\mathbb{R}^n$, any subset $C \subset X$ which is *closed* and *bounded*, that is such that $\sup_{x,y \in C} d(x,y) < +\infty$, is compact.

Extra: Topological Spaces

We will consider only metric spaces and define all notions of topological dynamics (see next section) in the context of metric spaces. More in general, all notions of topological dynamics that we will see can be applyed in the more general setting of *topological spaces*. Metric spaces are a special example of topological spaces. In a metric space, we defined the notion of open and closed sets (see Definition 1.1.3. The collection of open sets determines what is called a *topology* on the metric space. We also saw, from the definition of open set in a metric space, that countable unions and finite intersections of open sets are again open sets (see Lemma 1.1.1) and that countable intersections and finite unions of closed sets are closed (Corollary 1.1.1). These properties of open and closed sets can be taken as axioms to characterize open and closed sets in spaces where a distance is not necessarily given. This leads to the following definitions:

Definition 1.1.11. A *topology* $\mathscr{T}$ on X is a collection $\mathscr{T} \subset \mathscr{P}(X)^3$ of subsets of X, which are known as the *open sets* of X, which satisfy the following properties:

(T1) The empty set and the whole space X belong to $\mathscr{T}$;

(T2) Countable unions of open sets are open: if $U_1, U_2, \dots, U_n, \dots$ are open sets, then $\cup_{k \in \mathbb{N}} U_k$ is an open set;

(T3) Finite intersections of open sets are open: if $U_1, U_2, \dots, U_n$ are open sets, then $\cup_{k=1}^{n} U_k$ is an open set.

Exercise 1.1.10. If (X, d) is a metric space, the collection $\mathscr{T}$ of all sets which are open in the metric space according to Definition 1.1.3, that is all the sets $U \subset X$ such that for each $x \in U$ there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$, form a topology. Indeed, X and $\emptyset$ satisfy Definition 1.1.3 trivially and hence belong to $\mathscr{T}$, proving $(T1)$. The second property $(T2)$ follows from Lemma 1.1.1 The collection of open sets in a metric space give a topology to the metric space X

Definition 1.1.12. A *topological space* $(X, \mathscr{T})$ is a space X together with a topology $\mathscr{T}$.

Example 1.1.8. [Metric space topology] A metric space (X, d) with the topology given in the egample 1.1.10 is a topological space.

   The following two are examples of trivial topologies that exist on any set X.

Example 1.1.9. [Trivial topology] Consider a space X and let $\mathscr{T}_{tr} = \{\emptyset, X\}$. One can check that $\mathscr{T}_{tr}$ satisfies $(T1), (T2), (T3)$. This topology is known as *trivial topology*. Thus, $(X, \mathscr{T}_{tr})$ is a topological space.

Example 1.1.10. [Point topology] Consider a space X and let $\mathscr{T}_{pt} = \mathscr{P}(X)$ be the collection of all subsets of X. One can check that also $\mathscr{T}_{pt}$ satisfies $(T1), (T2), (T3)$. This topology is known as *point topology*. Thus, $(X, \mathscr{T}_{pt})$ is a topological space.

   In a topological space one can define the notion of convergence or density in the same way we did with metric spaces, just using open sets instead than balls. Similarly, one can define what it means for a function to be continous, taking as definition of continuity the equivalent characterization given by Lemma 1.1.2.

Definition 1.1.13. Let $(X, \mathscr{T})$ be a topological space. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ converges to $\overline{x}$ and we write $\lim_{n \to \infty} x_n = \overline{x}$ if for any open set $U \in mathscrT$ that contains x, there exists $N > 0$ such that $x_n \in U$ for all $n \geq N$.

---

[3]The notation $\mathscr{P}(X)$ denotes the parts of X, that is the collection of all subsets of X.

Lemma 1.1.3. *A function $f : X \to Y$ between two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ is* continuous *if and only if for each open set $V \in \mathscr{T}_Y$ the preimage $f^{-1}(V)$ is an open set of X, that is $f^{-1}(V) \in \mathscr{T}_X$.*

In the next sections we will define, in the context of metric spaces, dynamical properties as topological transitivity, topological minimality and topological mixing. All this properties can be defined more in general for topological spaces. This is why they are called *topological* properties and why we talk of *topological* dynamics.