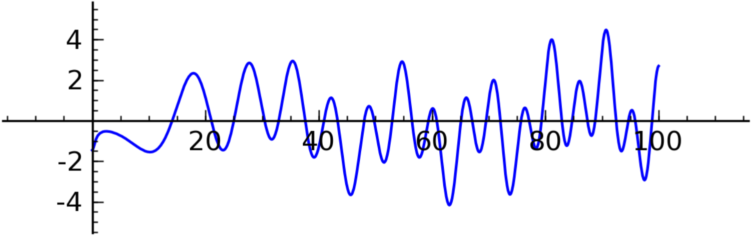
New computations of the Riemann zeta function

Jonathan Bober
(joint work with Ghaith Hiary)



The zeta function

Write $s = \sigma + it$. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \text{ for } \sigma > 1.$$

It continues to an analytic function on the whole complex plane with a simple pole at $s = 1$, and satisfies the functional equation

$$\zeta(s)\pi^{s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma\left(\frac{1-s}{2}\right).$$

From the definition and functional equation, it is relatively easy to compute $\zeta(s)$ for $\sigma > 1$ or $\sigma < 0$.

Analytic continuation and computation for $0 < \sigma < 1$

Pick some number N, and write

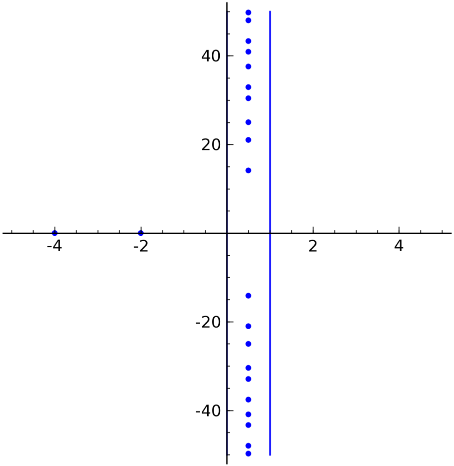
$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^\infty \frac{1}{y^s} d\lfloor y \rfloor = \sum_{n=1}^{N} \frac{1}{n^s} + s \int_N^\infty \frac{\{y\}}{y^{s+1}} + c(N)s/(1-s)$$

The integral will converge for $\sigma > 0$, so this gives some analytic continuation.

If we choose N properly, then $\zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s}$ might not be too big, and maybe we can compute the difference. This can be done using Euler-Maclaurin summation. It will require $N \asymp t$.

Analytic continuation and computation for $0 < \sigma < 1$

With Euler-Maclaurin summation it is easy to quickly make pictures like:



The Riemann-Siegel formula

Riemann knew a better method for computing in the critical strip. No one else knew it until Siegel found it in Riemann's notes 70 years later.

Define

$$Z(t) = e^{i\theta(t)}\zeta(1/2 + it)$$

where

$$\theta(t) = \arg\left(\Gamma\left(\frac{2it+1}{4}\right)\right) - \frac{\log \pi}{2}t.$$

Note that $Z(t)$ is real when t is real, and it has the same absolute value as $\zeta(1/2 + it)$. It has sign changes at simple zeros on the half line, so we'll be able to use it to locate zeros.

The Riemann-Siegel formula

When t is real, we have

$$Z(t) = 2\Re\left\{ e^{i\theta(t)} \sum_{n \le \left(\frac{t}{2\pi}\right)^{1/2}} \frac{1}{n^{\frac{1}{2}+it}} \right\} + O(t^{-1/4}).$$

($O(t^{-1/4})$ term can be computed quickly to better accuracy.)

This allows us to compute $\zeta(1/2 + it)$ to good accuracy in $O(t^{1/2})$ time.

Fast evaluation of the Riemann-Siegel formula

- After a precomputation of time $O(T^{1/2+\epsilon})$, the algorithm of Odlyzko and Schönhage can evaluate $\zeta(1/2 + it)$ for any $T < t < T^{1/2}$ in time t^{ϵ}.
- New algorithms of Ghaith Hiary allow computation at a single point in time $O(t^{1/3+\epsilon})$, or (more complicated) time $O(t^{4/13+\epsilon})$.

We've implemented the $O(t^{1/3})$ algorithm.

Past computations

Past computations

Computations by hand

	Year	Range of t	Number of zeros
Riemann	1859	$t < 26$	3
Gram	1903	$t < 65$	15
Backlund	1914	$t < 200$	79
Hutchinson	1925	$t < 300$	138
Titchmarsh, Comrie	1935–1936	$t < 1468$	1041

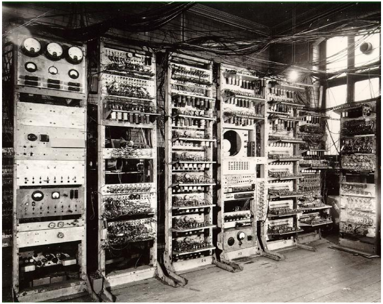
Soon, World War II intervenes.

Then, automatic electronic digital computers!

Past computations

The computer age

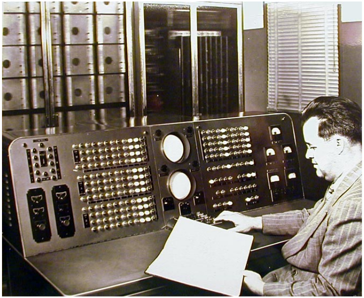
- May 1949, “Birthday of modern computing”
- June 1950, Turing checks the Riemann Hypothesis for $1414 < t < 1540$ on the Manchester University Mark I Electronic Computer, and also checks an area around 20000.



Past computations

The computer age

- 1956, D. H. Lehmer checks that the first 25000 zeros of the $\zeta(s)$ have real part $1/2$, using the National Bureau of Standards Western Automatic Computer.



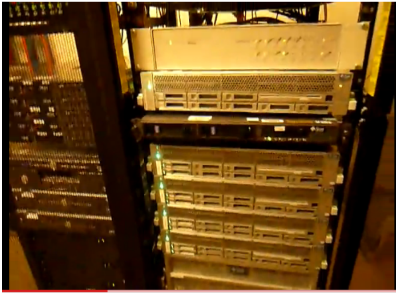
Past computations

The computer age

- 1958–1987 More computations done by Meller, Lehman, Rosser, Yohe, Schoenfeld, Brent, van de Lune, te Riele, Winter, Odlyzko
- 1988, Odlyzko—Schönhage algorithm published
- 1989, 1998, 2001, Odlyzko computes the 10^{20}th , 10^{21}st, 10^{22}nd zeros and billions more of large height
- 2001, van de Lune verifies RH for first 10000000000 (10^{10}) zeros.
- 2004, zetagrid project by S. Wedeniwski verifies RH for first 900000000000 ($9 \cdot 10^{11}$) zeros.
- 2004, X. Gourdon and P. Demichel verify RH for first 10000000000000 (10^{13}) zeros using Odlyzko—Schönhage algorithm and compute 2 billion zeros around the 10^{24}th.

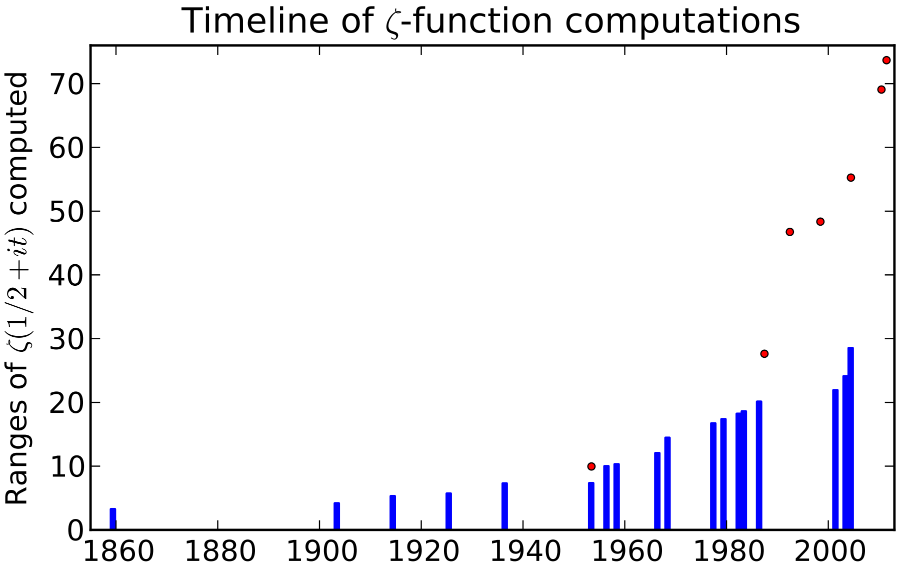
New computations

- Early September 2010: Zeros number $10^{30} - 125$ through $10^{30} + 121$ all lie on the critical line.
- Computation done on sage.math.washington.edu cluster. Thanks to NSF, Grant No. DMS-0821725 and William Stein.



New computations

- Mid-September 2010 through now and on-going: Around 180 or so spot checks of Riemann Hypothesis between 10^{26} and 10^{32}.
- Today: finished computation of range around 10^{33}rd zero.
- Computation done on riemann.math.uwaterloo.ca cluster. Thanks to Mike Rubinstein, Waterloo, and CFI, OIT, SGI.

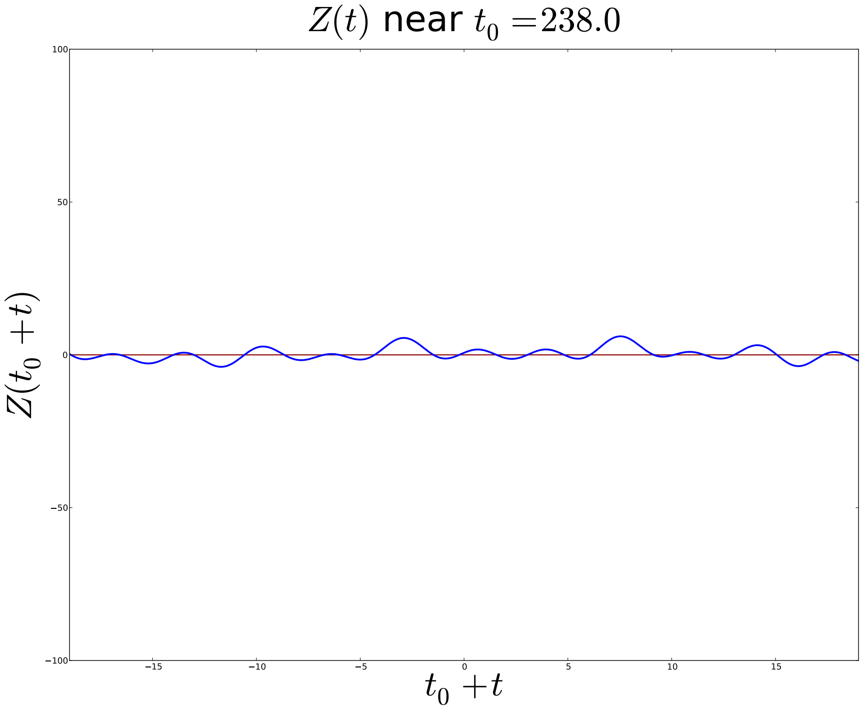

Timeline of ζ-function computations
Ranges of $\zeta(1/2+it)$ computed
0
10
20
30
40
50
60
70
1860
1880
1900
1920
1940
1960
1980
2000

End of history lesson.

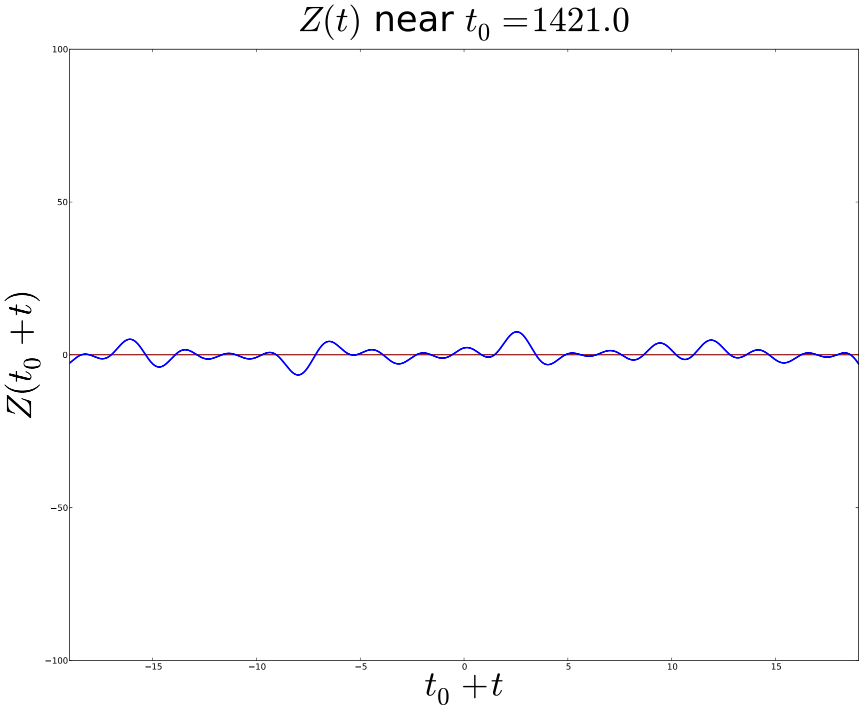
Pictures!

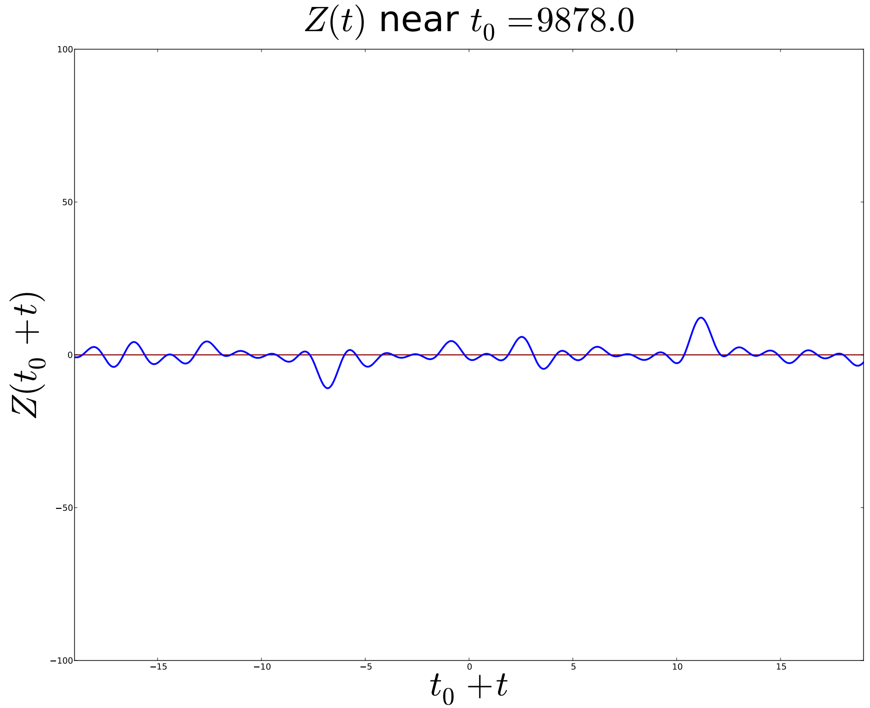
The next 32 slides show plots of $Z(t)$ around zero 10^n for $n = 2, 3, \ldots, 33$.

($n = 2, \ldots 11$ computed with Mike Rubinstein's lcalc)

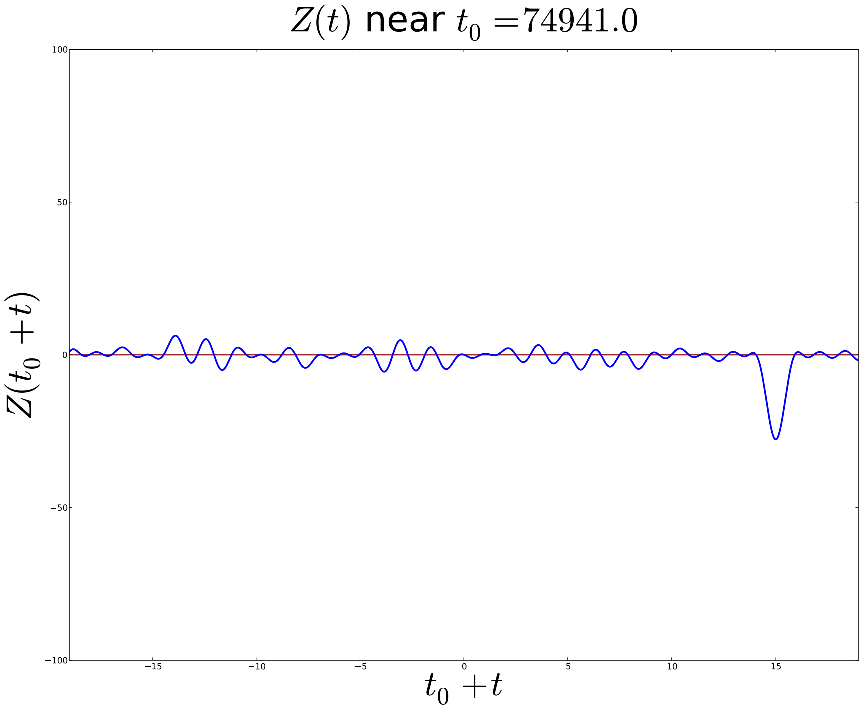


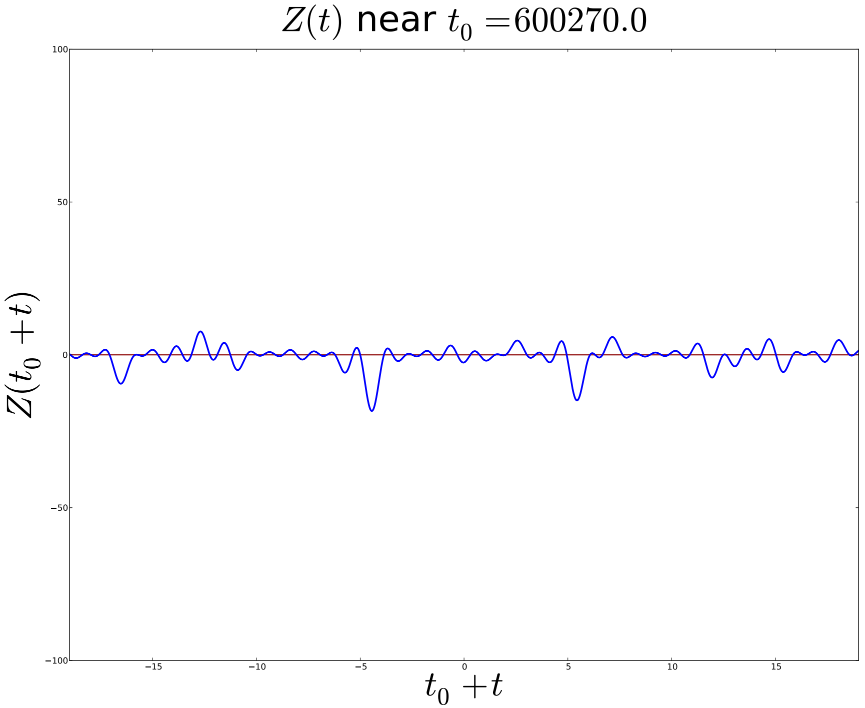
$Z(t)$ near $t_0 = 238.0$
100
50
0
−50
−100
$Z(t_0 + t)$
−15
−10
−5
0
5
10
15
$t_0 + t$

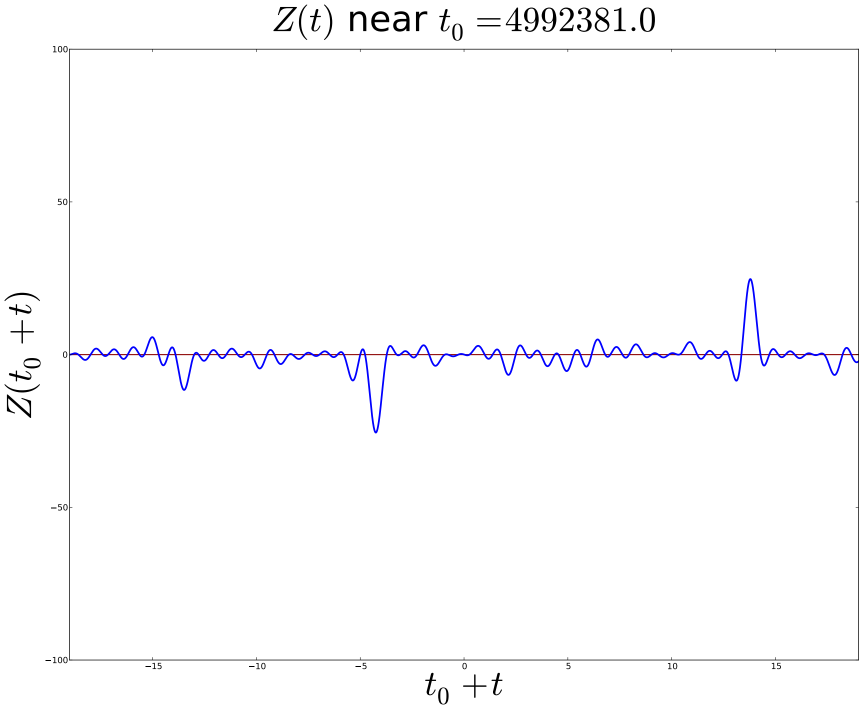

$Z(t)$ near $t_0 = 1421.0$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$



$Z(t)$ near $t_0 = 9878.0$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

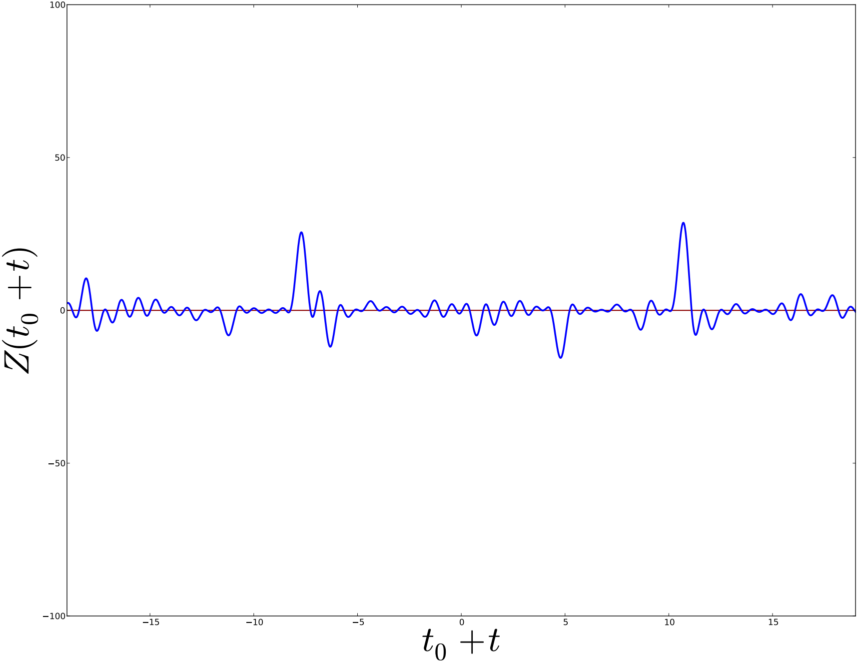

$Z(t)$ near $t_0 = 74941.0$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

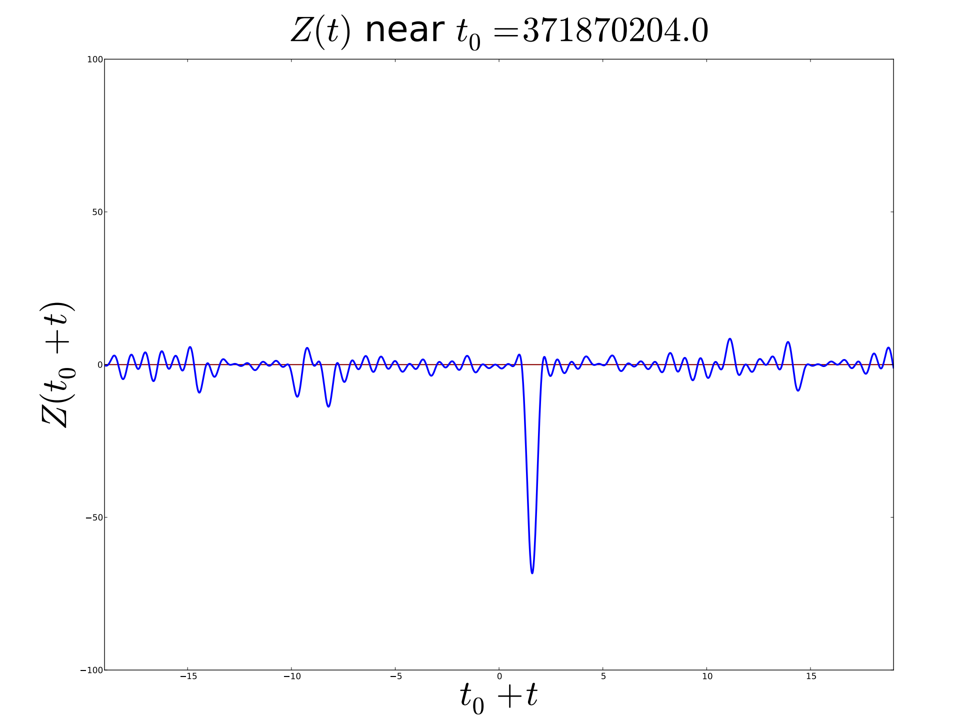

$Z(t)$ near $t_0 = 600270.0$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

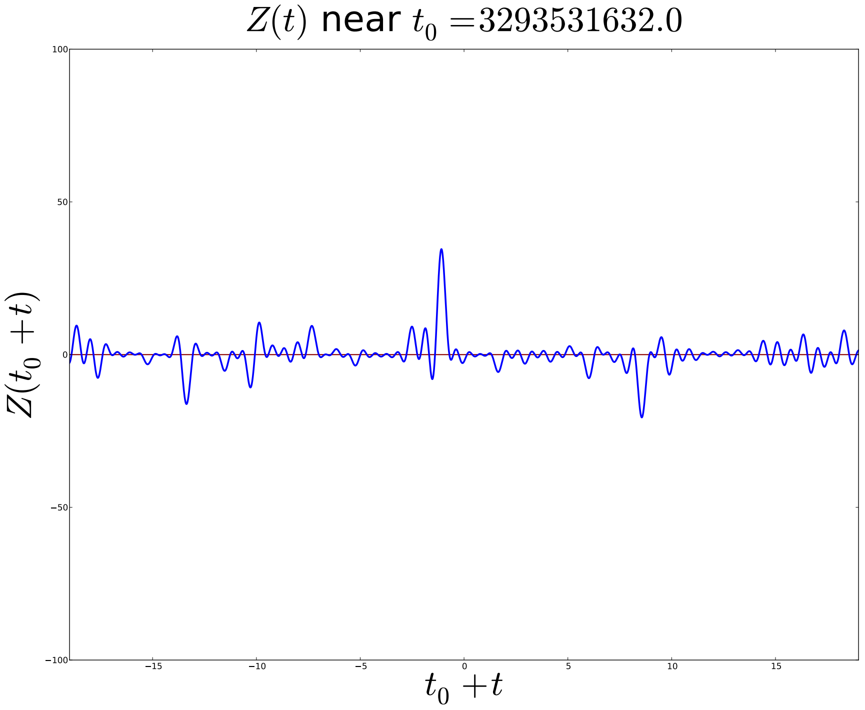


$Z(t)$ near $t_0 = 4992381.0$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

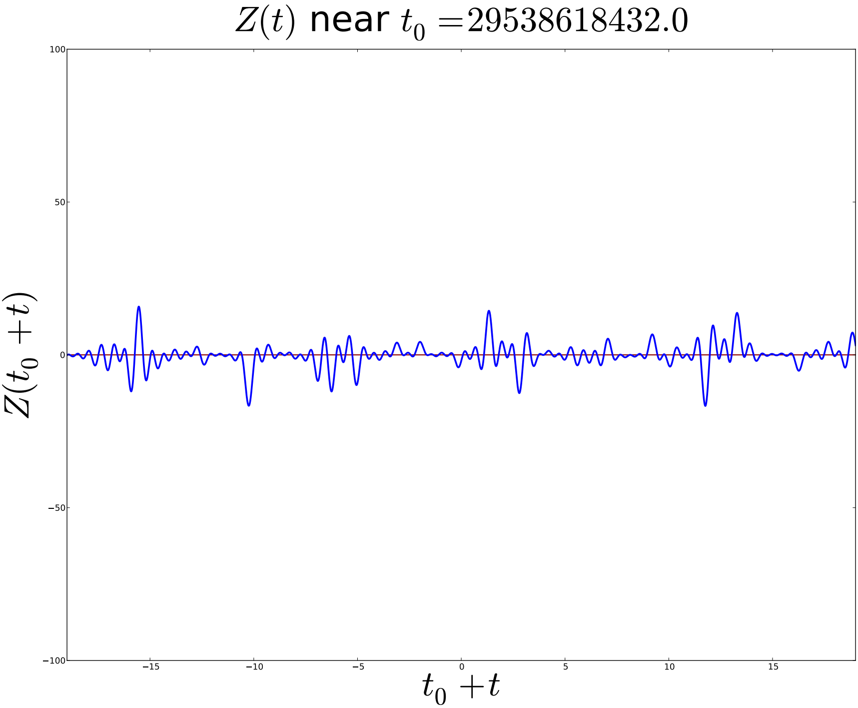
$Z(t)$ near $t_0 = 42653550.0$



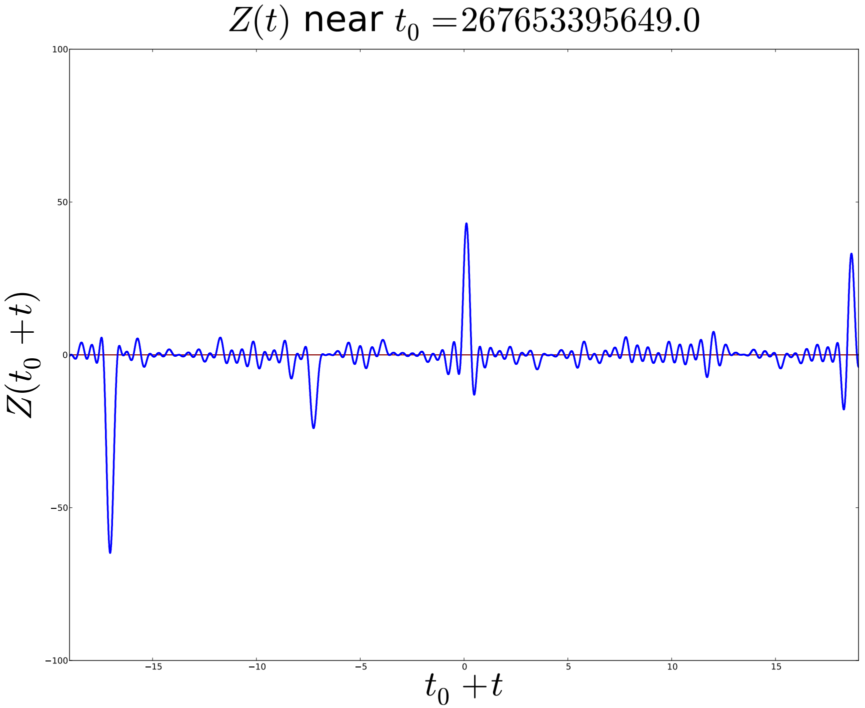

$Z(t)$ near $t_0 = 371870204.0$
$Z(t_0 + t)$
$t_0 + t$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15

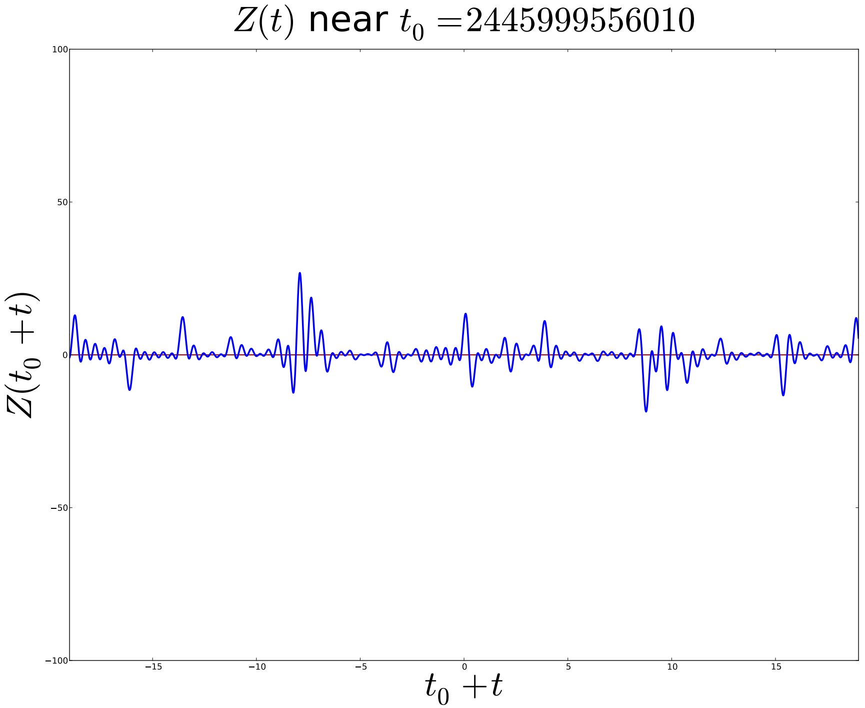


Z(t) near t0 =3293531632.0
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
Z(t0 +t)
t0 +t

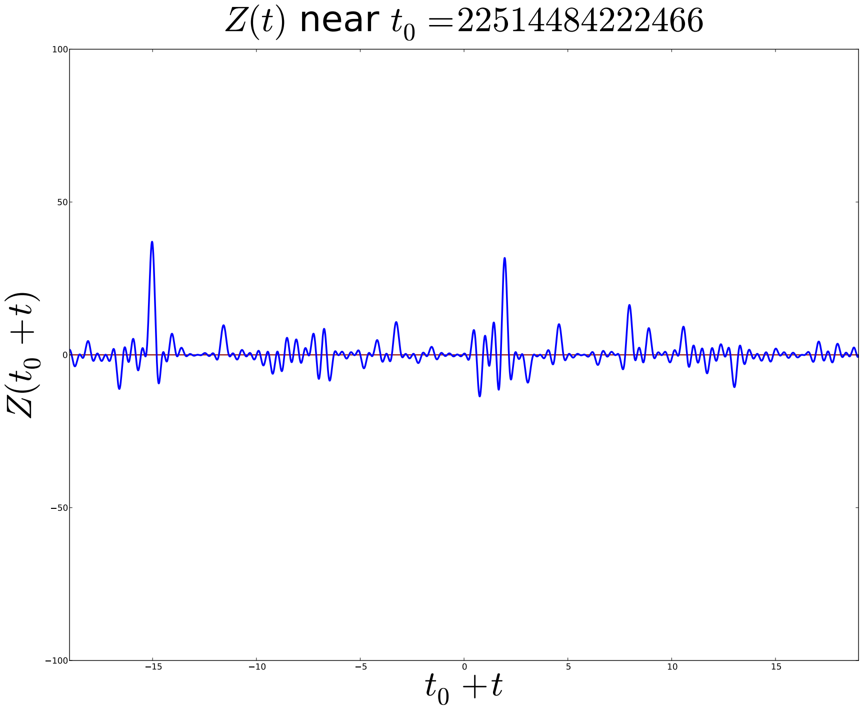


$Z(t)$ near $t_0 = 29538618432.0$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

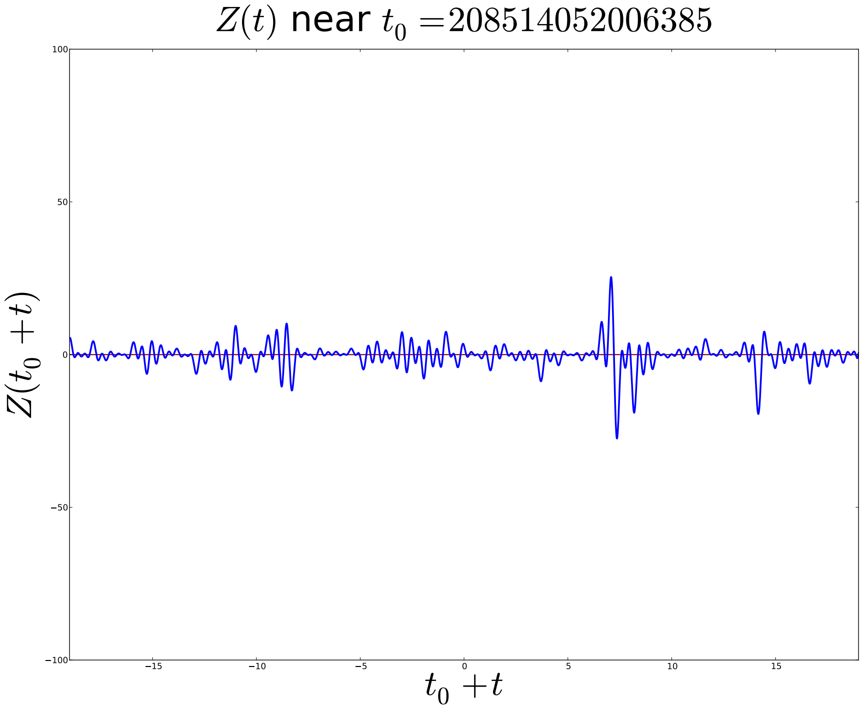

$Z(t)$ near t_0 =267653395649.0
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0+t)$
t_0+t



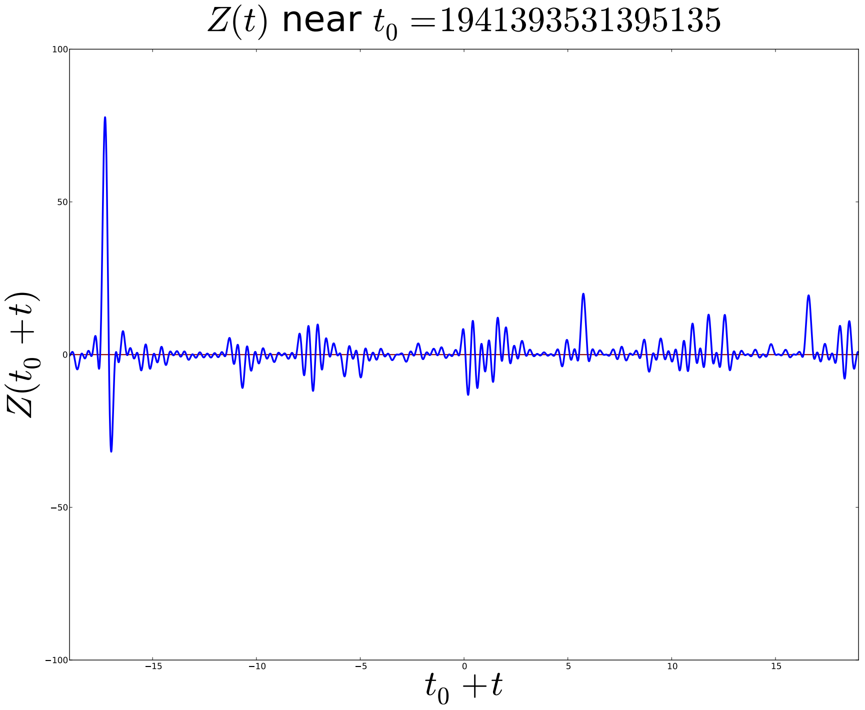
$Z(t)$ near $t_0 =$2445999556010
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$



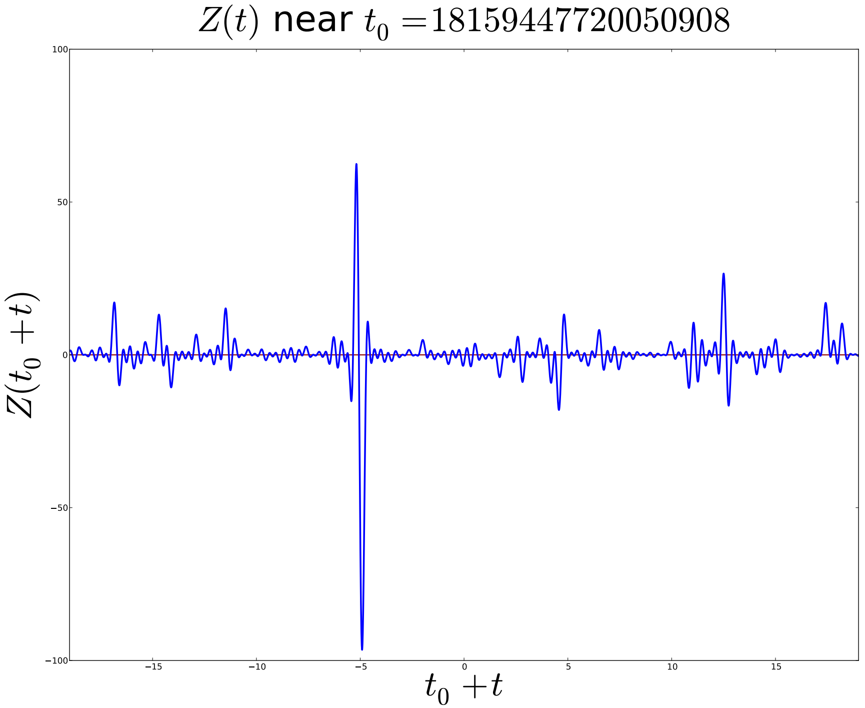
$Z(t)$ near $t_0 = 22514484222466$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

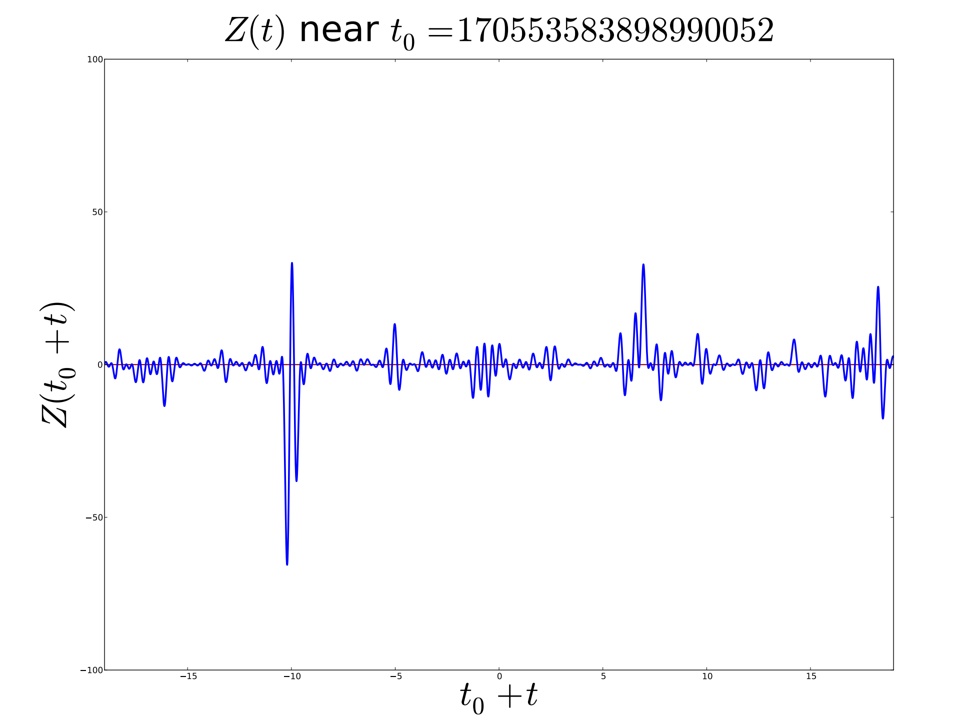


Z(t) near t0 =208514052006385
Z(t0 +t)
t0 +t
100
50
0
−50
−100
−15
−10
−5
0
5
10
15

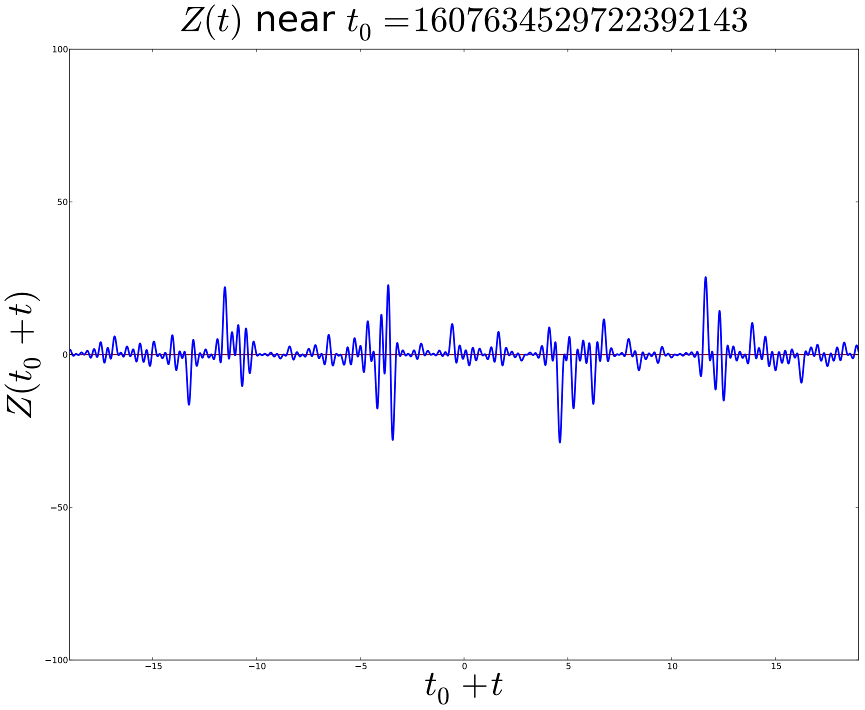


Z(t) near t0 =1941393531395135
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
Z(t0 +t)
t0 +t

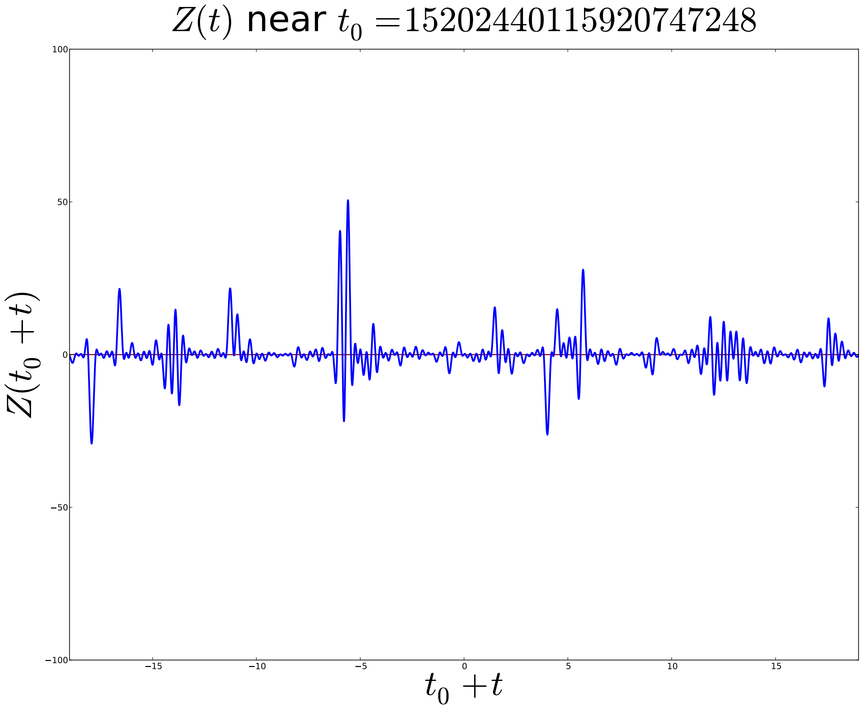

$Z(t)$ near $t_0 = 18159447720050908$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$



$Z(t)$ near $t_0 = 17055358389899005 2$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

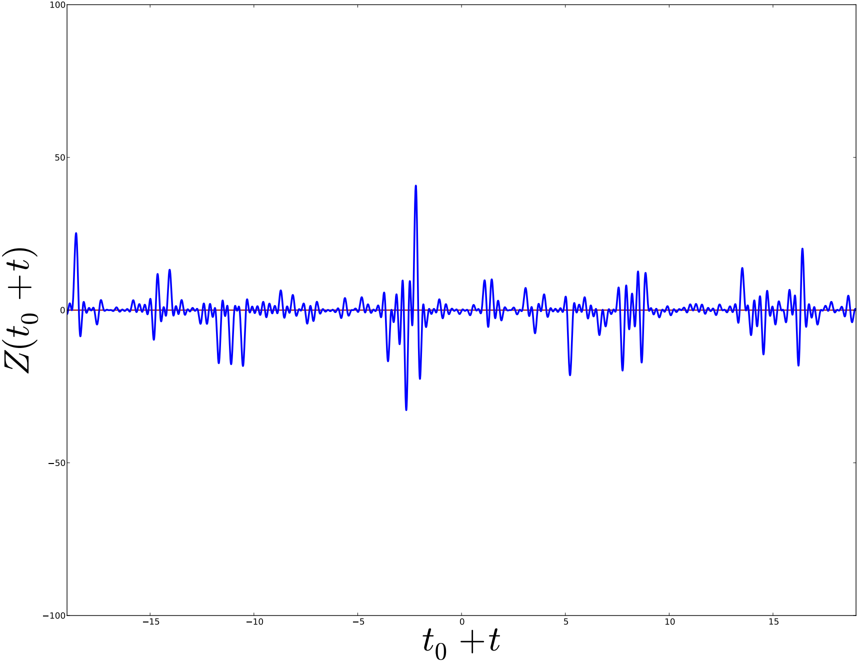


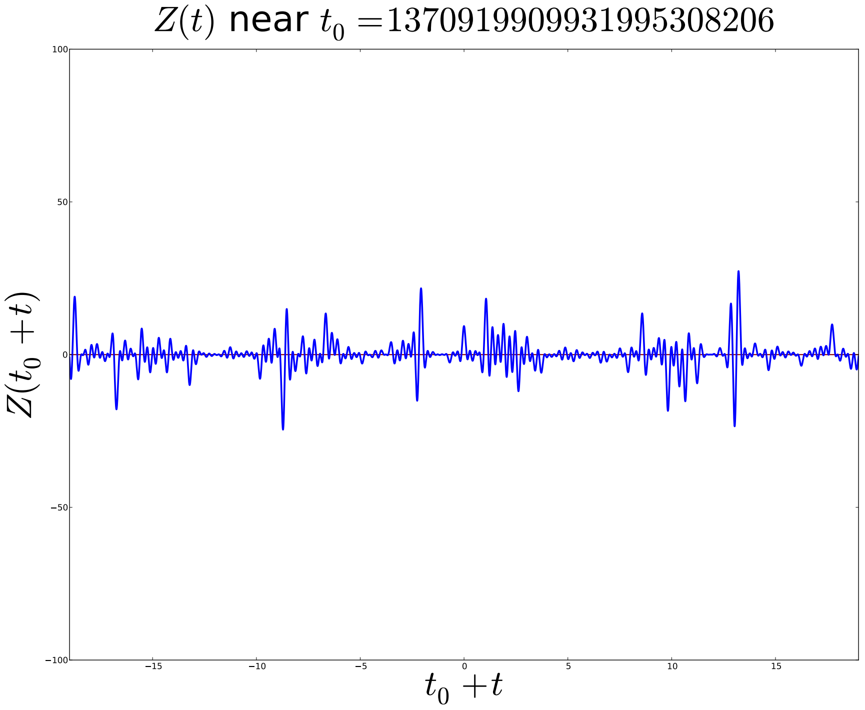
$Z(t)$ near $t_0 = 16076345297223921 43$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$



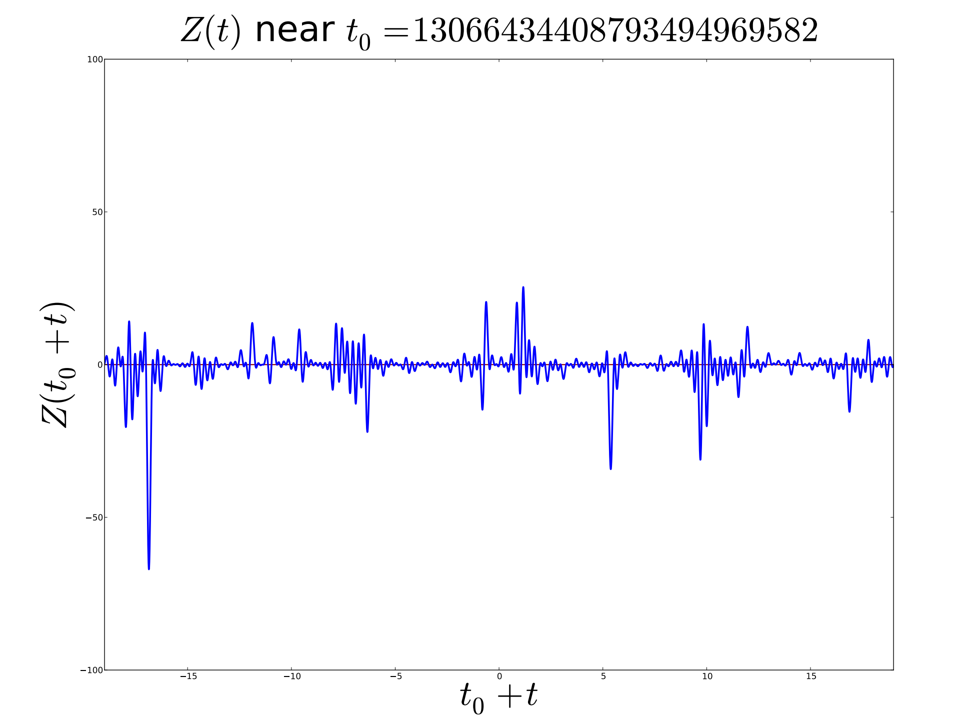
$Z(t)$ near $t_0 = 152024401159207472 48$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

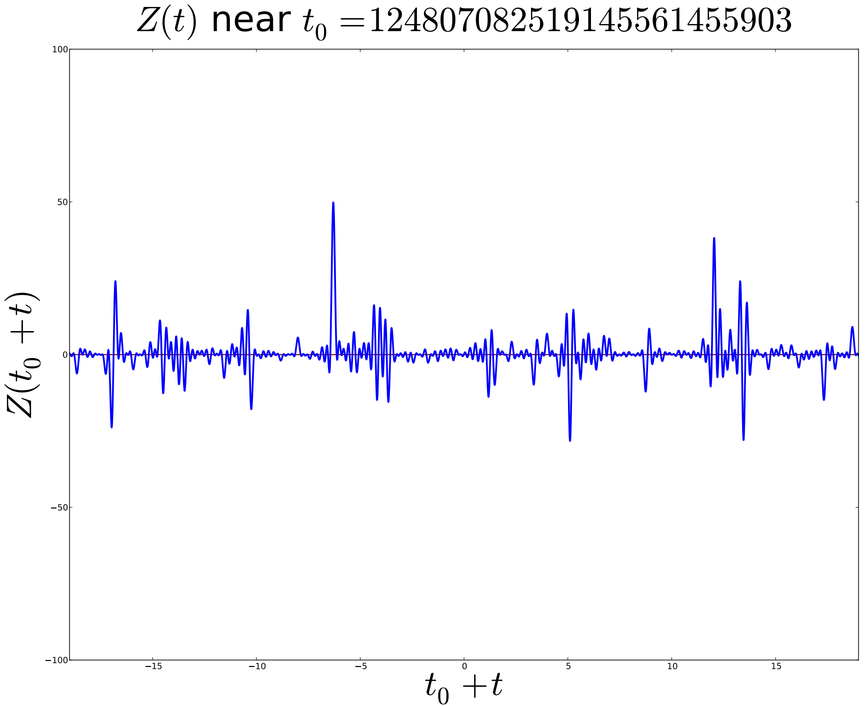
$Z(t)$ near $t_0 = 144176897509546973518$



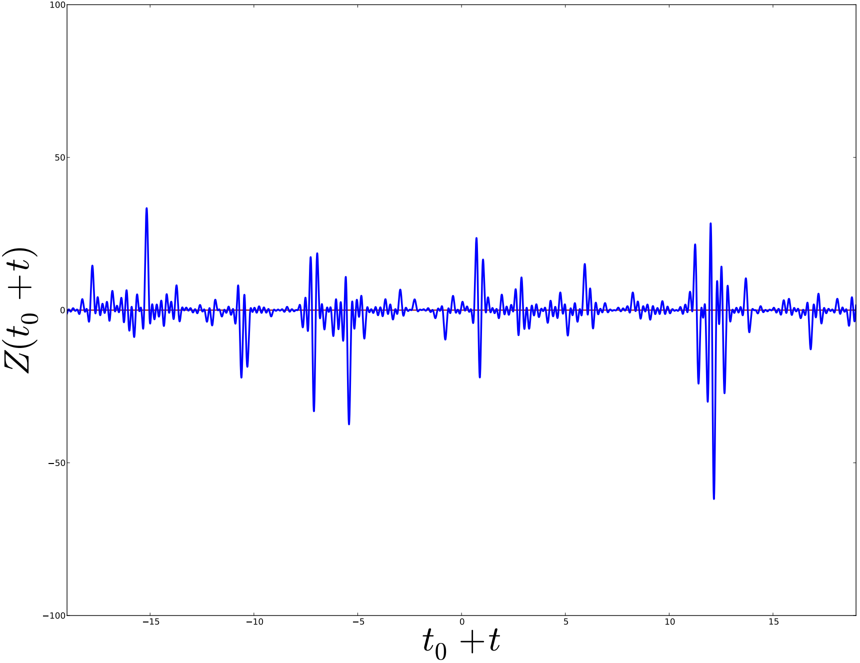


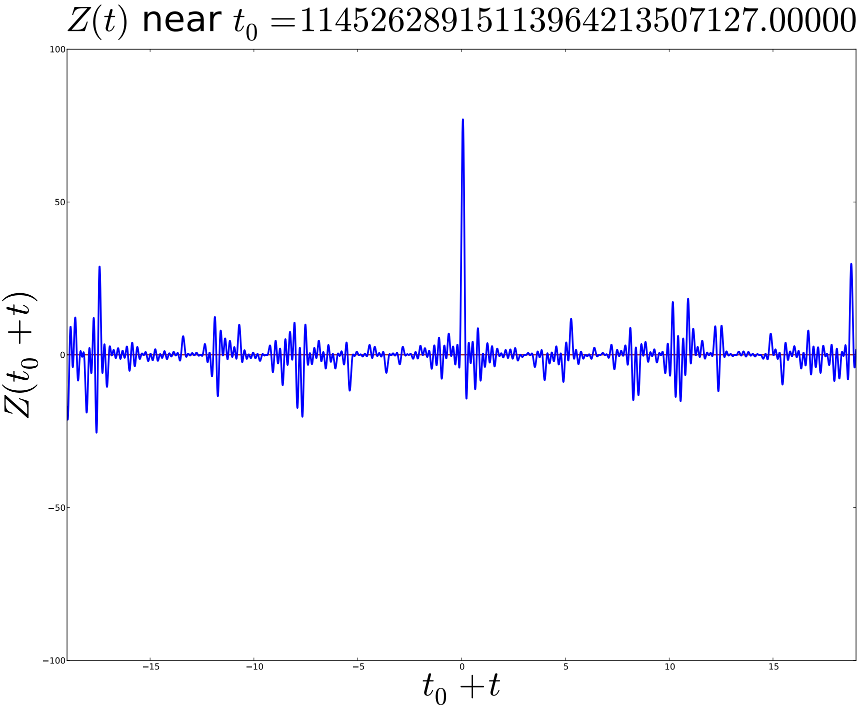
$Z(t)$ near $t_0 = 1370919909931995308206$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

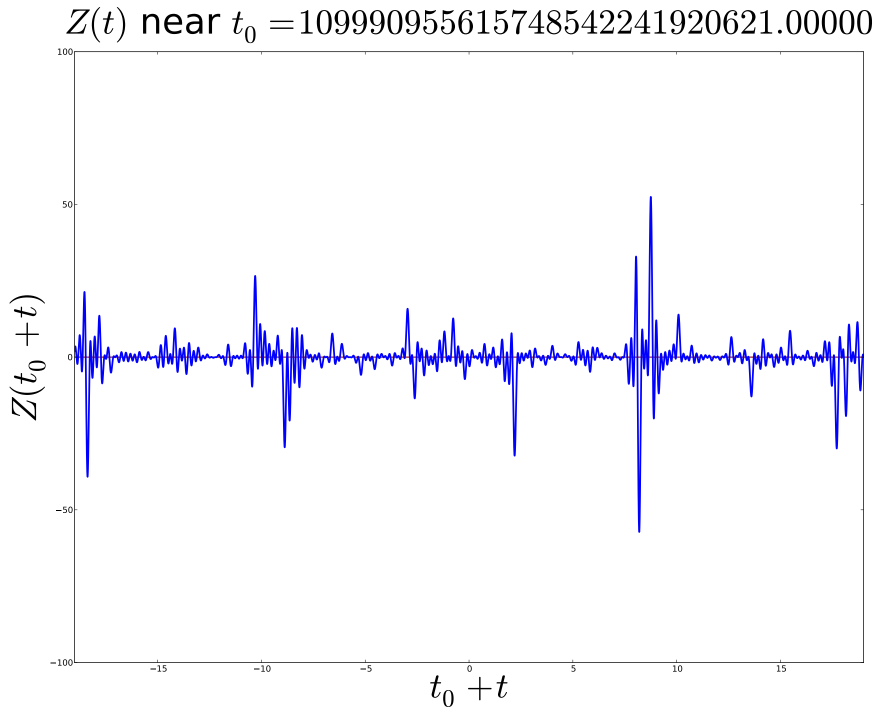

$Z(t)$ near $t_0 = 13066434408793494969582$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$


$Z(t)$ near $t_0 = 124807082519145561455903$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

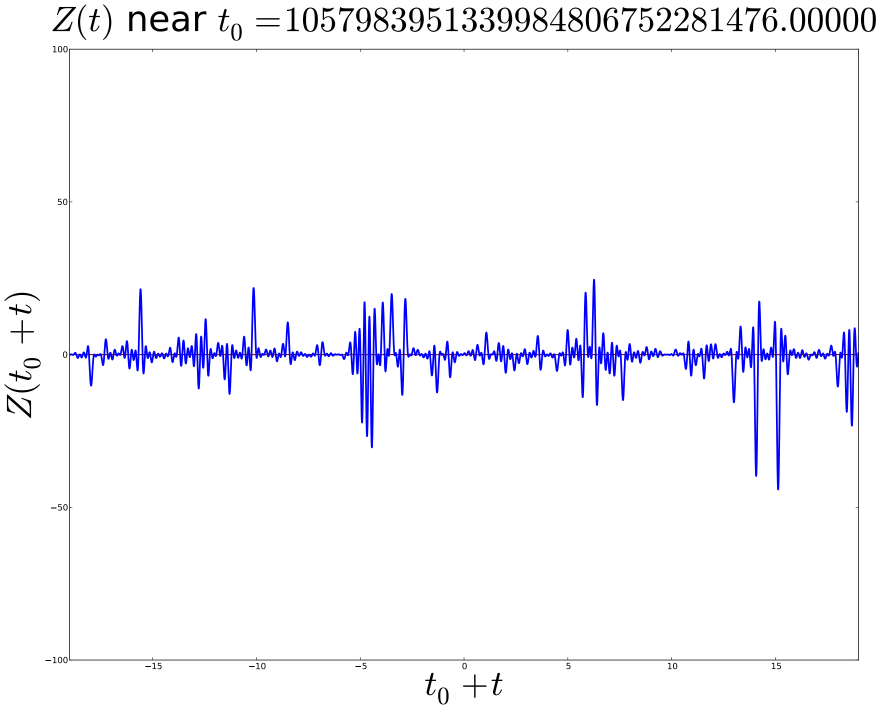
$Z(t)$ near $t_0 = 1194479330178301585147871.00000$



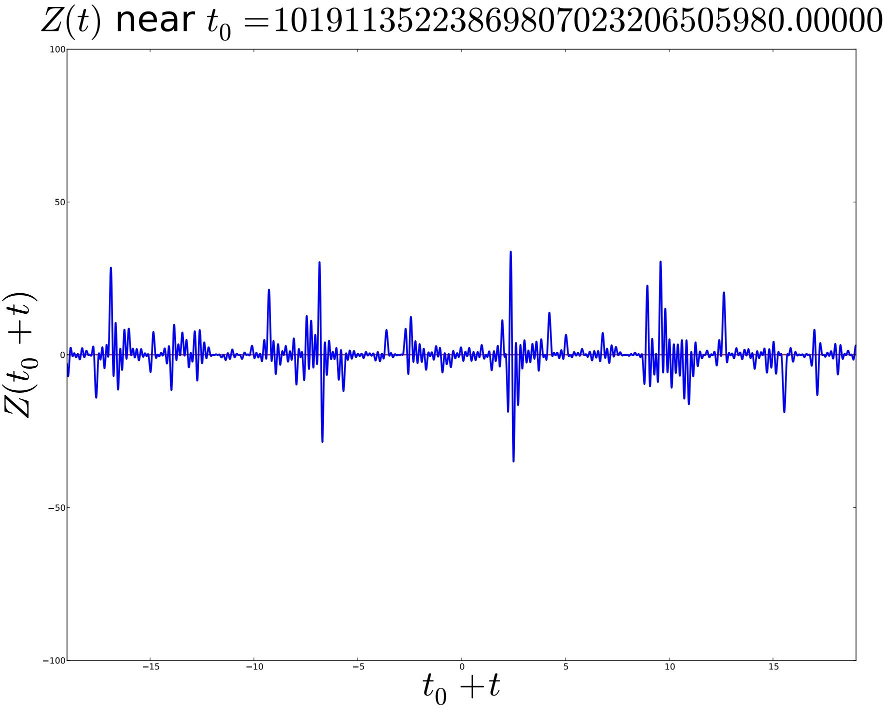

$Z(t)$ near $t_0 = 11452628915113964213507127.00000$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$



$Z(t)$ near $t_0 = 1099909556157485422419206 21.00000$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$

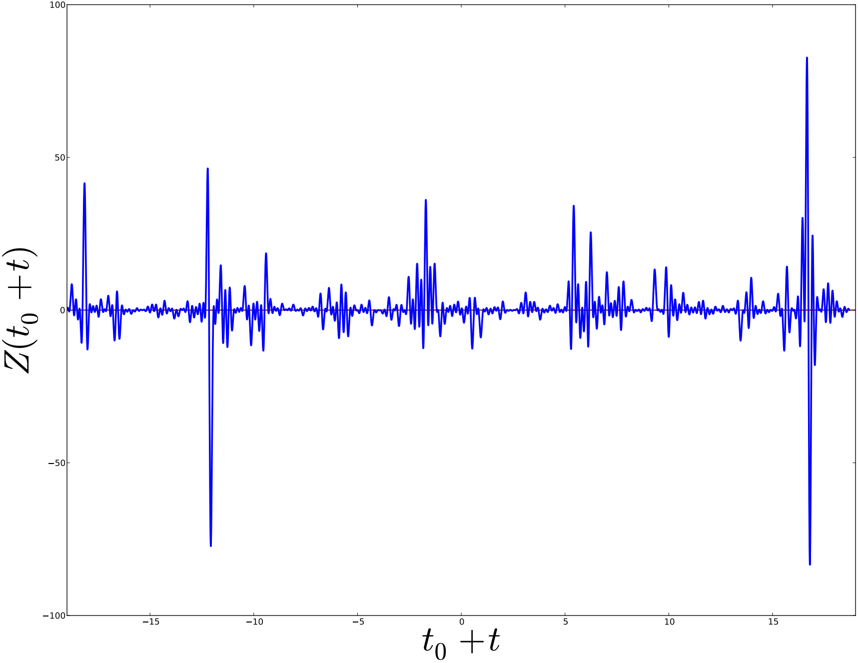


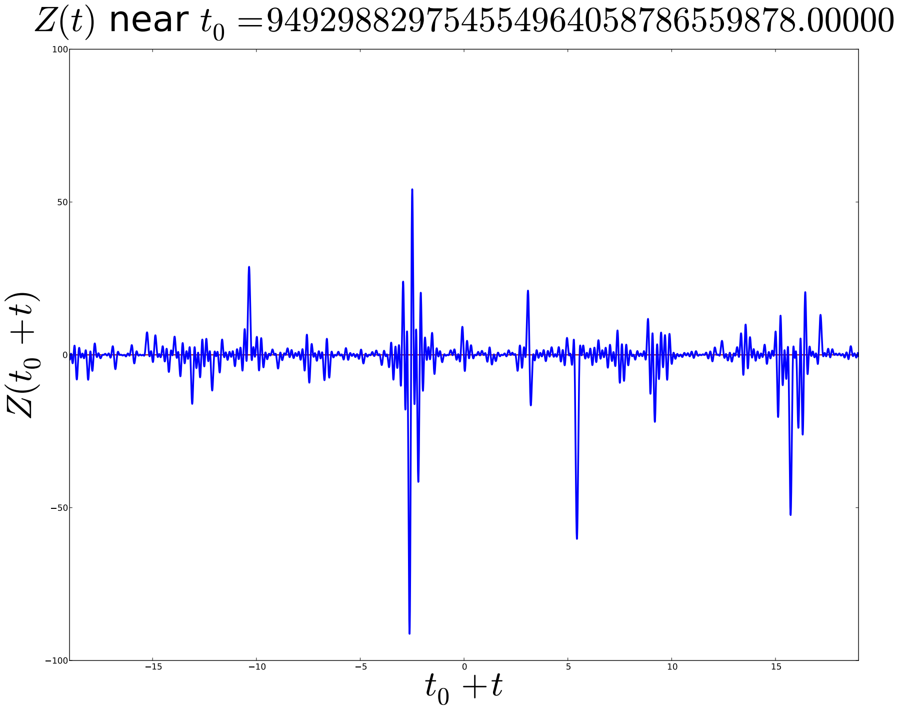
$Z(t)$ near $t_0 = 1057983951339984806752281476.00000$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$



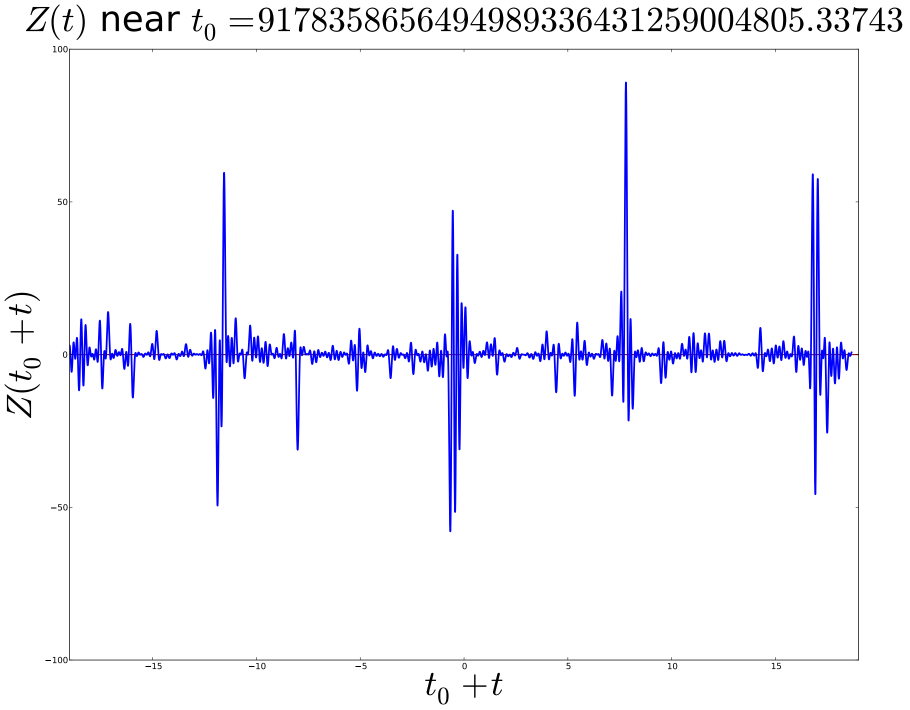
$Z(t)$ near $t_0 = 10191135223869807023206505980.00000$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0+t)$
t_0+t

$Z(t)$ near $t_0 = 98297762869274424758690514842.30236$

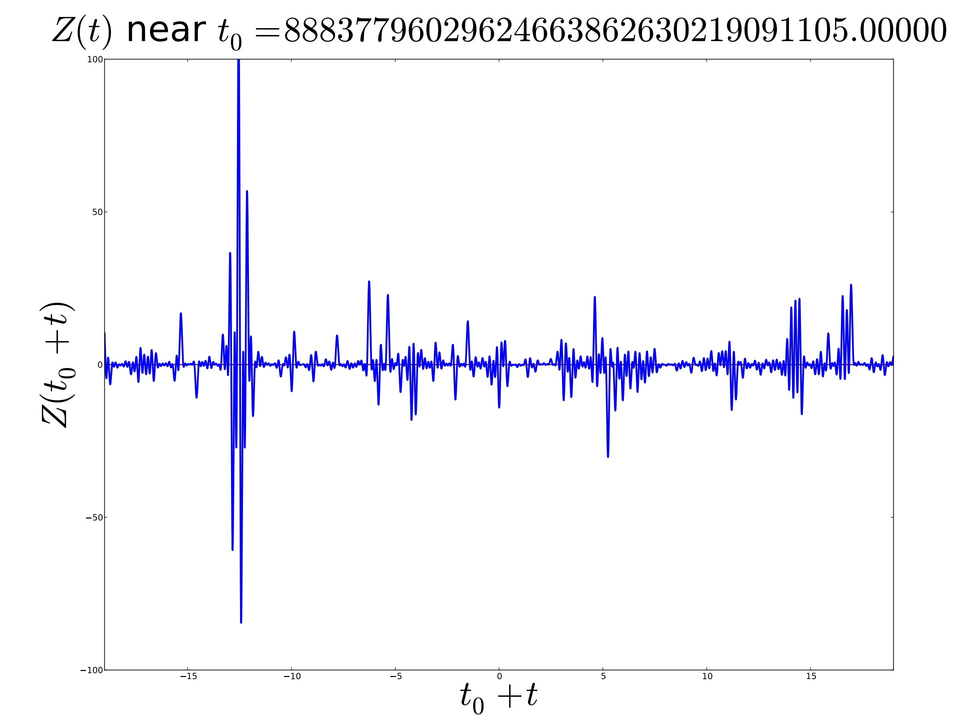


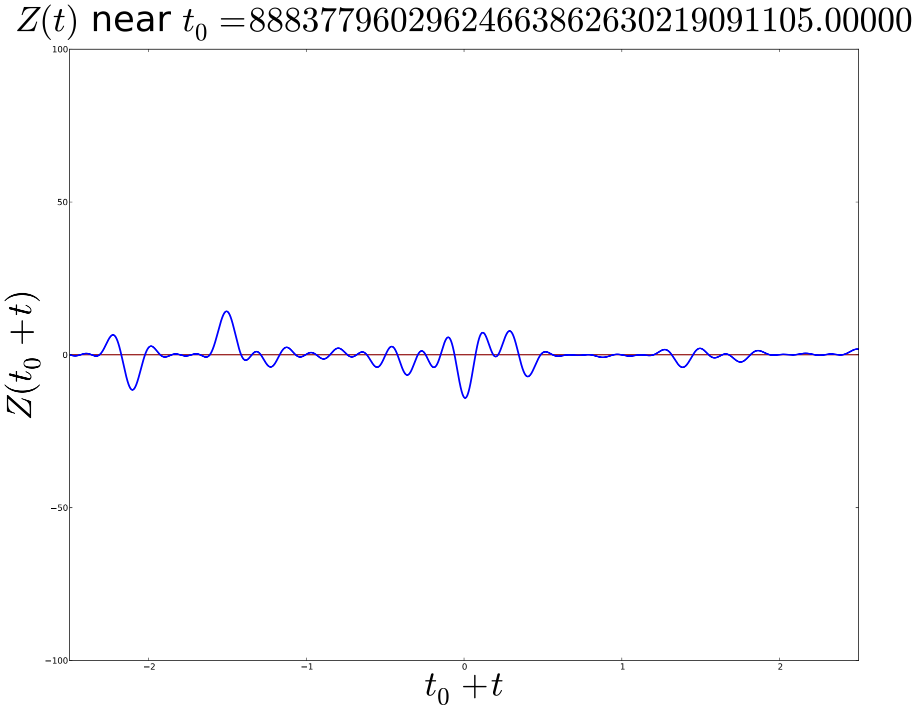


$Z(t)$ near $t_0 = 9492988297545549640587865 59878.00000$
$Z(t_0 + t)$
$t_0 + t$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15



$Z(t)$ near $t_0 = 9178358656494989336431259004805.33743$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15
$Z(t_0 + t)$
$t_0 + t$


$Z(t)$ near $t_0 = 888377960296246638626302190911 05.00000$
$Z(t_0 + t)$
$t_0 + t$
100
50
0
−50
−100
−15
−10
−5
0
5
10
15


$Z(t)$ near $t_0 = 888377960296246638626302190911 05.00000$
100
50
0
−50
−100
−2
−1
0
1
2
$Z(t_0 + t)$
$t_0 + t$

Some more details on how we compute.

Hiary's $O(t^{1/3})$ algorithm

We break the Riemann Siegel main sum into smaller pieces:

$$\sum_{n\leq N}\frac{1}{n^{1/2+it}} = \sum_{v\in V}\frac{\exp(it\log v)}{\sqrt{v}}\sum_{k=0}^{K-1}\frac{\exp(it\log(1+k/v))}{\sqrt{1+k/v}}$$

for some suitably chosen V.
Then use Taylor expansions to rewrite each block as

$$\sum_{l=0}^{\infty}\frac{C_l}{v^l}\sum_{k=0}^{K-1}k^l\exp\left(it\sum_{m=1}^{\infty}\frac{(-1)^{m+1}k^m}{mv^m}\right)$$
$$\approx \sum_{l=0}^{18}\frac{C_l}{v^l}\sum_{k=0}^{K-1}k^l\exp\left(it\sum_{m=1}^{2}\frac{(-1)^{m+1}k^m}{mv^m}\right)$$

for some coefficients C_l that depend only on l. The complexity of computing the zeta function is determined by how large we can make K without losing too much accuracy.

Hiary's $O(t^{1/3})$ algorithm

Then use Taylor expansions to rewrite each block as

$$\sum_{l=0}^{\infty} \frac{C_l}{v^l} \sum_{k=0}^{K-1} k^l \exp\left(it \sum_{m=1}^{\infty} \frac{(-1)^{m+1} k^m}{m v^m} \right)$$
$$\approx \sum_{l=0}^{18} \frac{C_l}{v^l} \sum_{k=0}^{K-1} k^l \exp\left(it \sum_{m=1}^{2} \frac{(-1)^{m+1} k^m}{m v^m} \right)$$

for some coefficients C_l that depend only on l. The complexity is determined by how large we can make K without losing too much accuracy and how fast we can compute the exponential sum.

Hiary's theta sum algorithm

The core component of our method becomes an algorithm of Ghaith Hiary to compute quadratic exponential sums.

Algorithm (Hiary, 2008)

For fixed j, the sum

$$\frac{1}{K^j}\sum_{k=0}^{K-1} k^j \exp(2\pi iak + 2\pi ibk^2)$$

can be computed in $O((\log K)^2)$ arithmetic operations.

Hiary's $O(t^{1/3})$ algorithm

When we optimize block sizes, this gives complexity $O(t^{1/3} + \epsilon)$ for the computation of $\zeta(1/2 + it)$.

Evaluating at multiple points

To evaluate $\zeta(1/2 + i(t+\delta))$. we need to compute

$$\frac{\exp(i((t+\delta)\log v)}{\sqrt{v}} \sum_{k=0}^{K-1} \frac{\exp(i(t+\delta)\log(1+k/v))}{\sqrt{1+k/v}}.$$

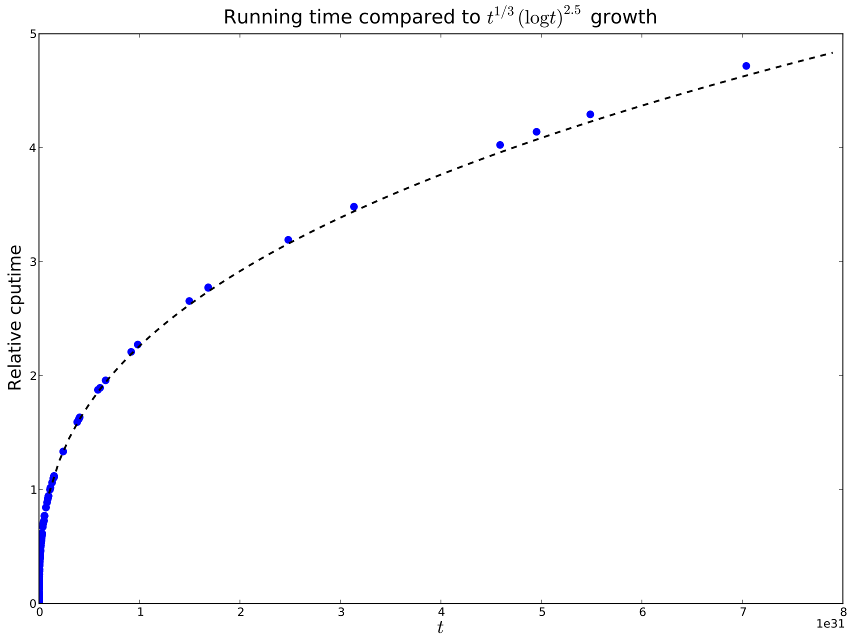
We simply pretend the sum is constant, and estimate that this is approximately

$$\exp(it\delta \log v)\frac{\exp(it\log v)}{\sqrt{v}} \sum_{k=0}^{K-1} \frac{\exp(i\log(1+k/v))}{\sqrt{1+k/v}}.$$

If δ is small enough, and K is much smaller than v, the error is small, so we can evaluate this block to decent accuracy at n evenly spaced points using n multiplications.

Evaluating at multiple points

Once we have the main sum evaluated on a grid of points, we can use band limited interpolation to recover the values of the sum at any nearby point.

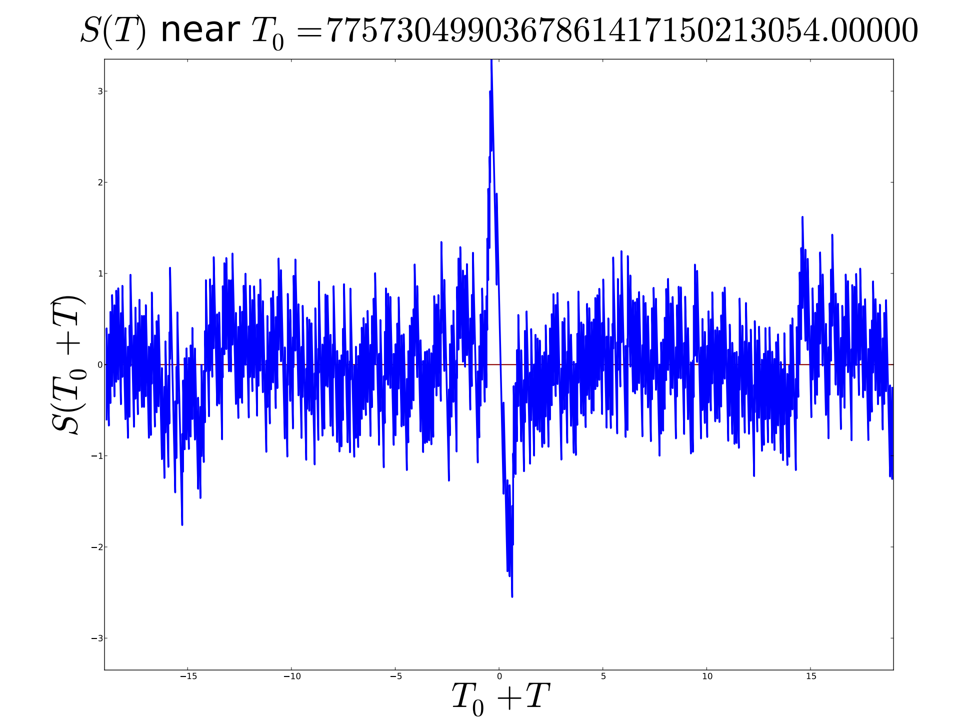

Running time compared to $t^{1/3}(\log t)^{2.5}$ growth
Relative cputime
t
1e31
0
1
2
3
4
5
6
7
8

Calculating the number of zeros in a range

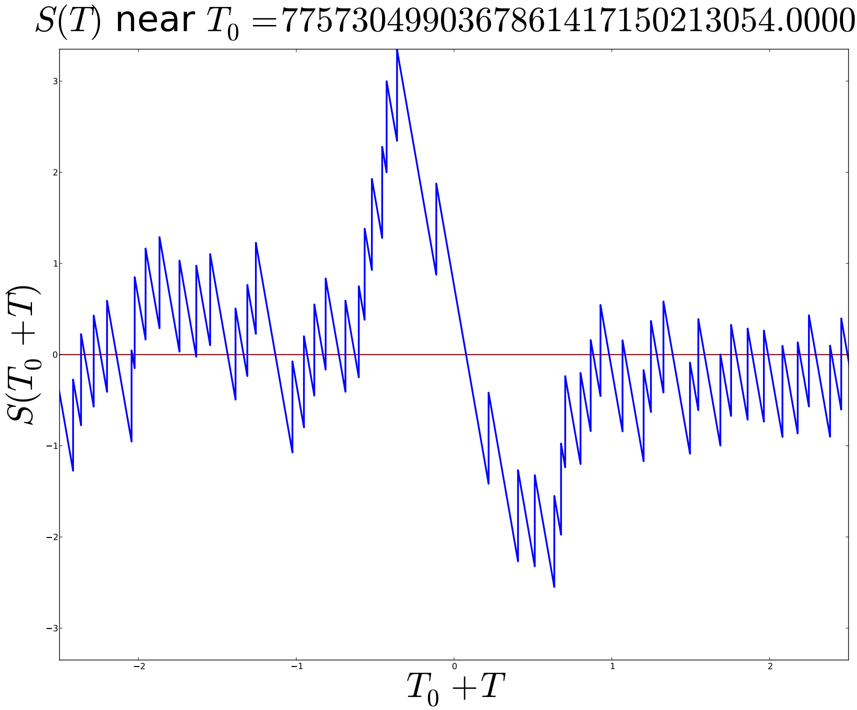
Let $N(T)$ denote the number of zeros of $\zeta(\sigma + it)$ with $0 < t < T$. To determine which zero is the 10^{30}th, or to verify that all the zeros in a range have real part $1/2$, we need to calculate $N(T)$.

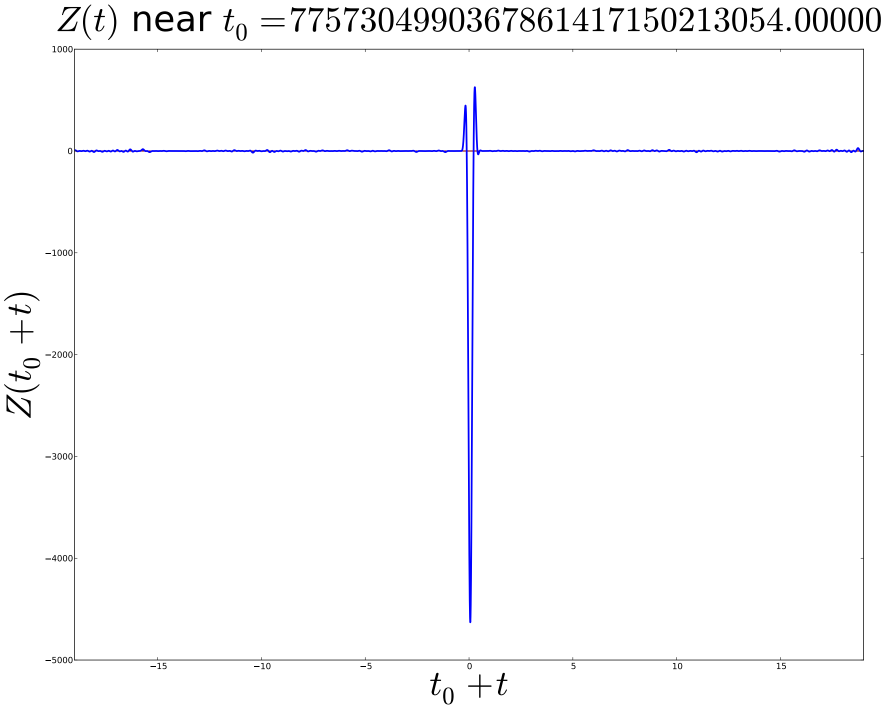
Rough outline of method:

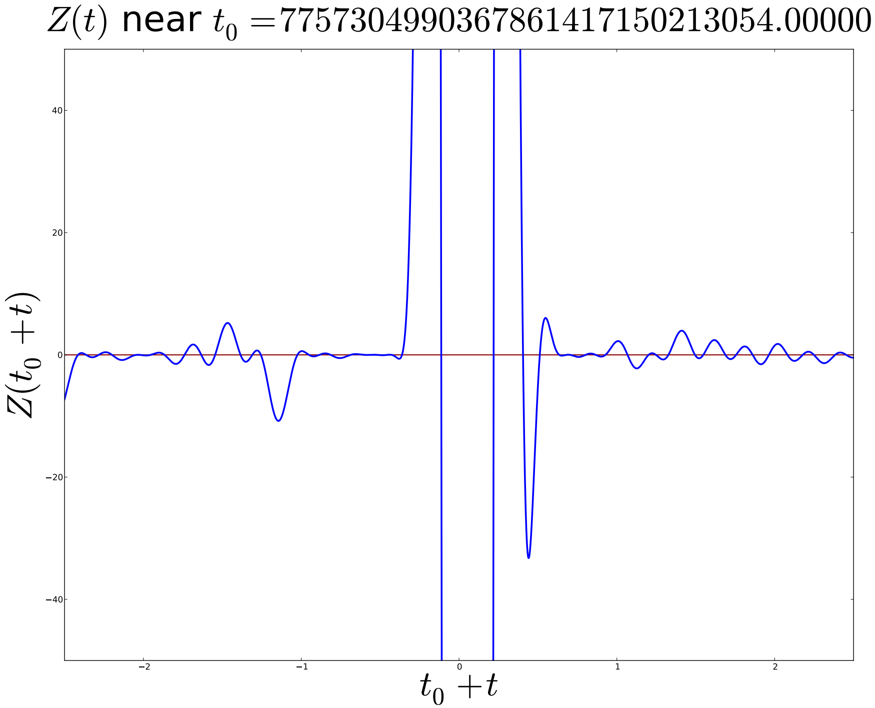
- $N(T) = N_0(T) + S(T)$, where $N_0(T)$ is easy to compute and understand, and $S(T)$ is not.
- But $S(T)$ is usually small, and 0 on average.
- Turing's method: Pick a spot where $S(T)$ is an even integer. Compute the zeta function nearby and if you find the right number of zeros, then $-2 < S(T) < 2$. (Otherwise there would be a contradiction with the fact that $S(T)$ is zero on average.)
- Recent improved bounds on average of $S(T)$, due to T. Trudgian, are useful here.



S(T) near T_0 =7757304990367861417150213054.00000
S(T_0 + T)
T_0 + T
3
2
1
0
−1
−2
−3
−15
−10
−5
0
5
10
15


$S(T)$ near $T_0 = 775730499036786141715021305 4.00000$
$S(T_0 + T)$
$T_0 + T$
3
2
1
0
-1
-2
-3
−2
−1
0
1
2


$Z(t)$ near $t_0 = 7757304990367861417150213054.00000$
1000
0
−1000
−2000
−3000
−4000
−5000
$Z(t_0 + t)$
−15
−10
−5
0
5
10
15
$t_0 + t$



$Z(t)$ near $t_0 = 7757304990367861417150213054.00000$
40
20
0
−20
−40
−2
−1
0
1
2
$Z(t_0 + t)$
$t_0 + t$

Finding large values of $\zeta(s)$

Odlyzko describes a method for locating large values of $\zeta(s)$ which is based on finding values of t such that $p^{it} \approx 1$ for many values of t. Using this method we have so far found values of the zeta function as large as 14000.

Finding large values of $\zeta(s)$

We choose some parameters m, r, and k, and then apply the LLL algorithm to the matrix

$$\begin{pmatrix} \left\lfloor p_1^{1/4} 2^{m-r} \log p_1 \right\rfloor & \left\lfloor p_2^{1/4} 2^{m-r} \log p_2 \right\rfloor & \cdots & \left\lfloor p_k^{1/4} 2^{m-r} \log p_k \right\rfloor & 1 \\ \left\lfloor 2\pi p_1^{1/4} 2^m \right\rfloor & 0 & \cdots & 0 & 0 \\ 0 & \left\lfloor 2\pi p_2^{1/4} 2^m \right\rfloor & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \left\lfloor 2\pi p_k^{1/4} 2^m \right\rfloor & 0 \end{pmatrix}$$

A vector in the reduced basis will look like

$$(M \left\lfloor p_1^{1/4} 2^{m-r} \log p_1 \right\rfloor - m_1 \left\lfloor 2\pi p_1^{1/4} 2^m \right\rfloor, \ldots, M \left\lfloor p_k^{1/4} 2^{m-r} \log p_k \right\rfloor - m_k \left\lfloor 2\pi p_k^{1/4} 2^m \right\rfloor, M)$$

Finding large values of $\zeta(s)$

A vector in the reduced basis will look like

$$\left(M\left\lfloor p_1^{1/4}2^{m-r}\log p_1\right\rfloor - m_1\left\lfloor 2\pi p_1^{1/4}2^m\right\rfloor, \ldots, M\left\lfloor p_k^{1/4}2^{m-r}\log p_k\right\rfloor - m_k\left\lfloor 2\pi p_k^{1/4}2^m\right\rfloor, M\right)$$

For this to be small, we expect that all of

$$M2^{-r}\log p_k - 2\pi m_k$$

will be relatively small, and so we expect that

$$\zeta(1/2 + iM2^{-r})$$

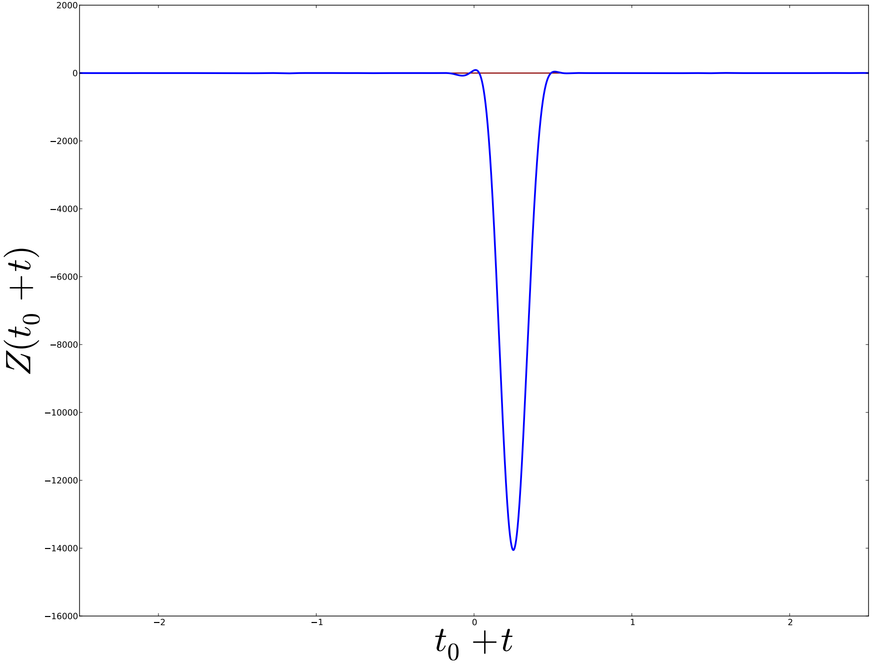
will be large.

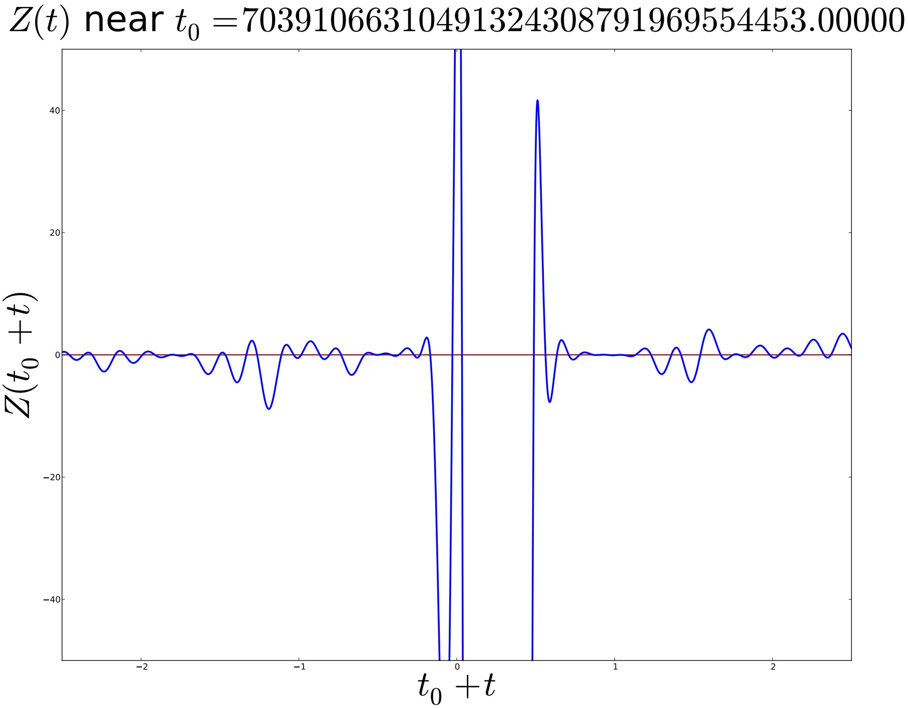
In practice, choose k, r, m, and a reduction parameter δ randomly in some range, and repeat many times.

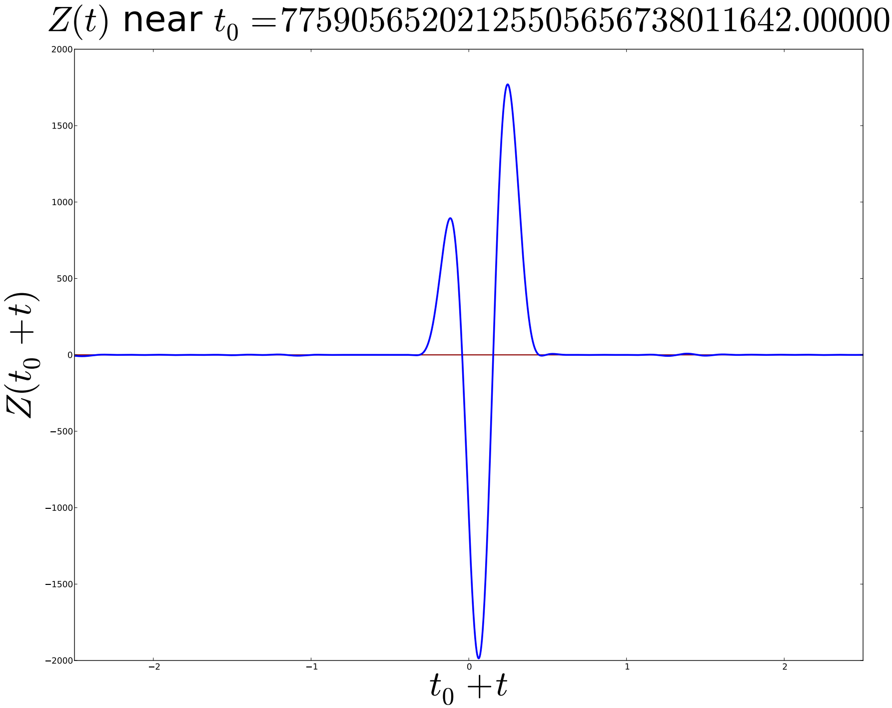
Pictures!

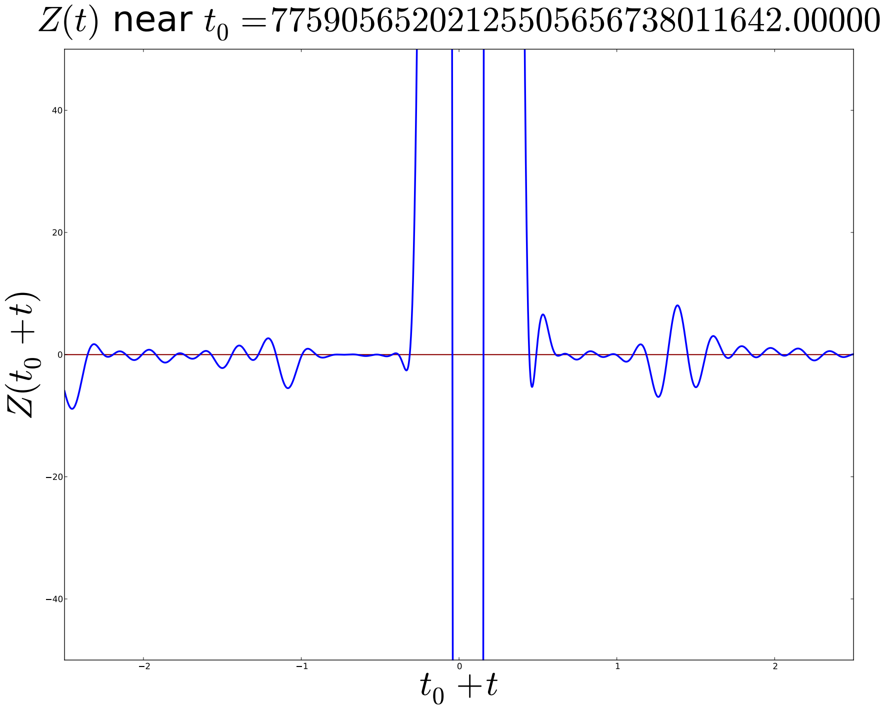
More examples

$Z(t)$ near $t_0 = 70391066310491324308791969554453.00000$

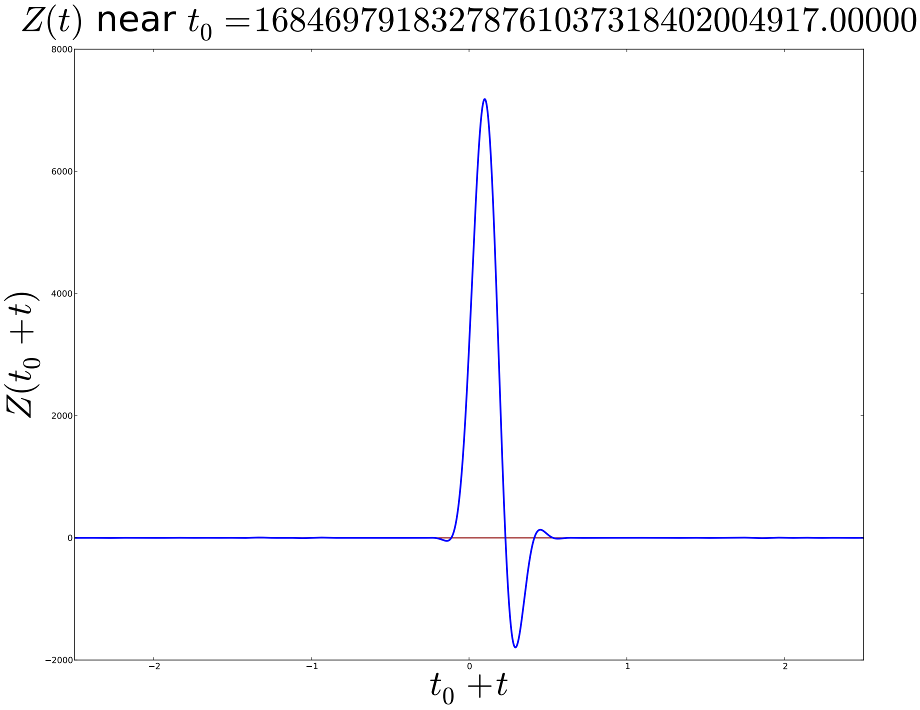



$Z(t)$ near $t_0 = 7039106631049132430879196955445 3.00000$
40
20
0
−20
−40
−2
−1
0
1
2
$Z(t_0+t)$
t_0+t

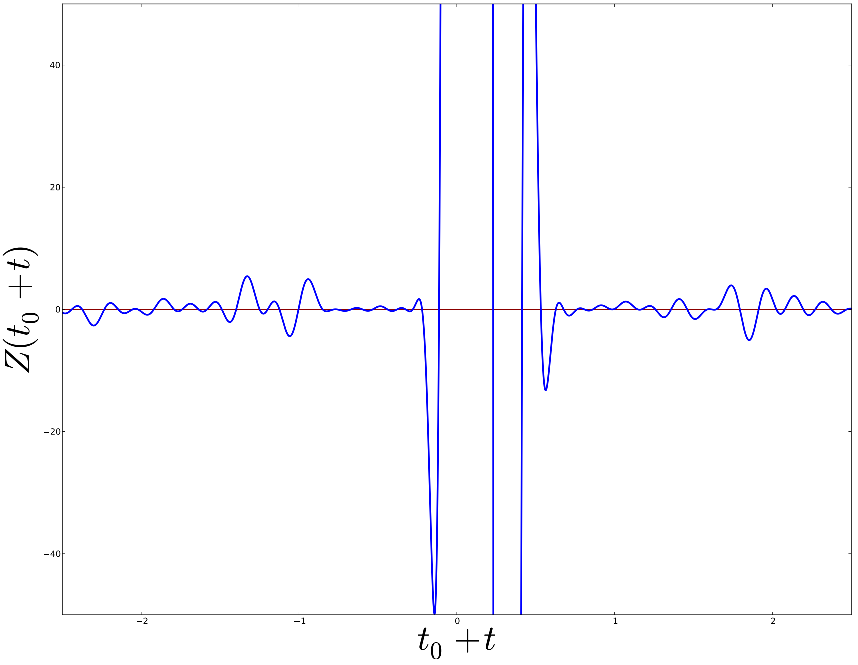

$Z(t)$ near $t_0 = 7759056520212550565673801164 2.00000$
2000
1500
1000
500
0
−500
−1000
−1500
−2000
$Z(t_0 + t)$
−2
−1
0
1
2
$t_0 + t$



$Z(t)$ near $t_0 = 7759056520212550565673801164 2.00000$
40
20
0
−20
−40
−2
−1
0
1
2
$Z(t_0 + t)$
$t_0 + t$


$Z(t)$ near $t_0 = 16846979183278761037318402004917.00000$
8000
6000
4000
2000
0
−2000
−2
−1
0
1
2
$Z(t_0 + t)$
$t_0 + t$

$Z(t)$ near $t_0 = 16846979183278761037318402004917.00000$



Growth of the zeta function?

Theorem (Chandee and Soundararajan)

Assuming the Riemann Hypothesis,

$$\log\left(\left|\zeta\left(\frac{1}{2}+it\right)\right|+1\right) \le \frac{\log 2}{2}\frac{\log t}{\log\log t} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^2}\right)$$

Conjecture (Farmer, Gonek, and Hughes)

$$\log\left(\left|\zeta\left(\frac{1}{2}+it\right)\right|+1\right) \le \sqrt{\left(\frac{1}{2}+o(1)\right)\log t \log\log t}$$

Theorem (Soundararajan)

$$\max_{t\le T}\log\left(\left|\zeta\left(\frac{1}{2}+it\right)\right|+1\right) \ge (1+o(1))\sqrt{\frac{\log T}{\log\log T}}$$

Growth of the zeta function?

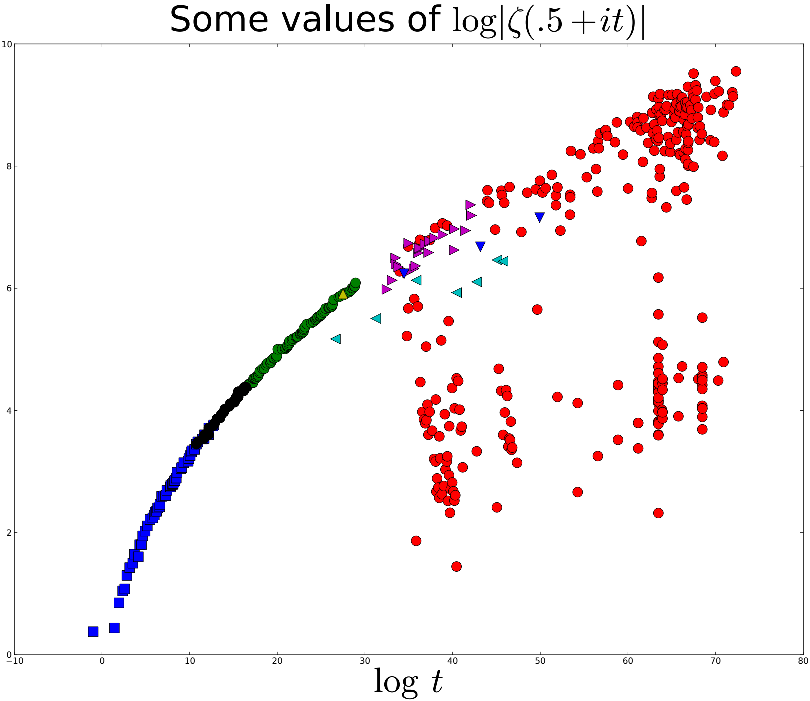
Data from various sources:

1. T. Kotnik ("Computational estimation of the of order of $\zeta\ldots$") computed all "increasingly large extrema" of $Z(t)$ for $t < 10^6$.
2. Kotnik also did a search up to 10^{13} using a heuristic method to narrow the search space.
3. Odlyzko ("The 10^{20}th zero and ...") reports the largest values he found near the 10^nth zero, for $n = 12, 14, 16, 18, 19, 20$, and also large values found using the search procedure described earlier.
4. Gourdon lists largest value that he came across below 10^{13}th zero (not necessarily largest in that range, "missed" by Kotnik).

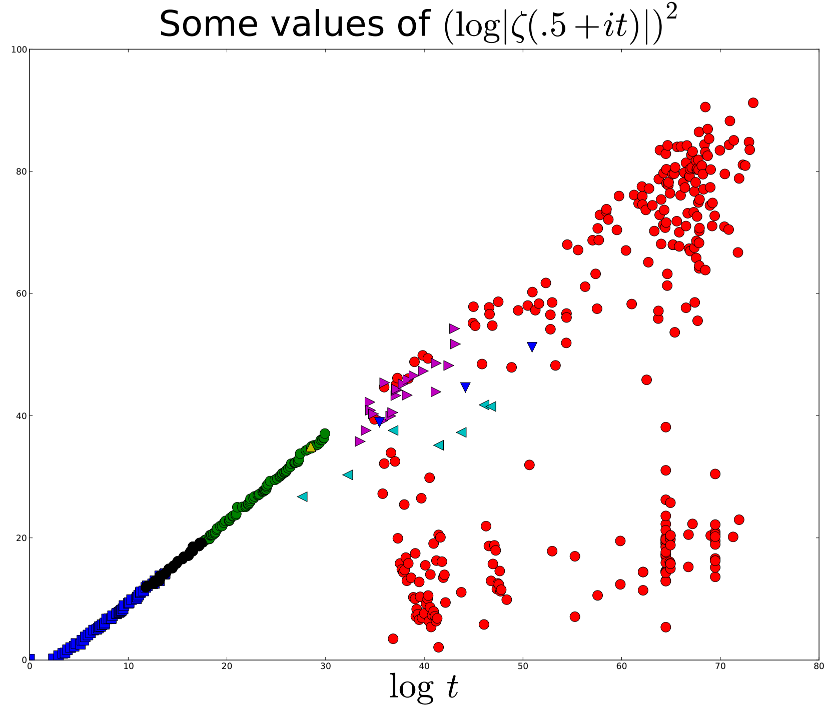
Growth of the zeta function?

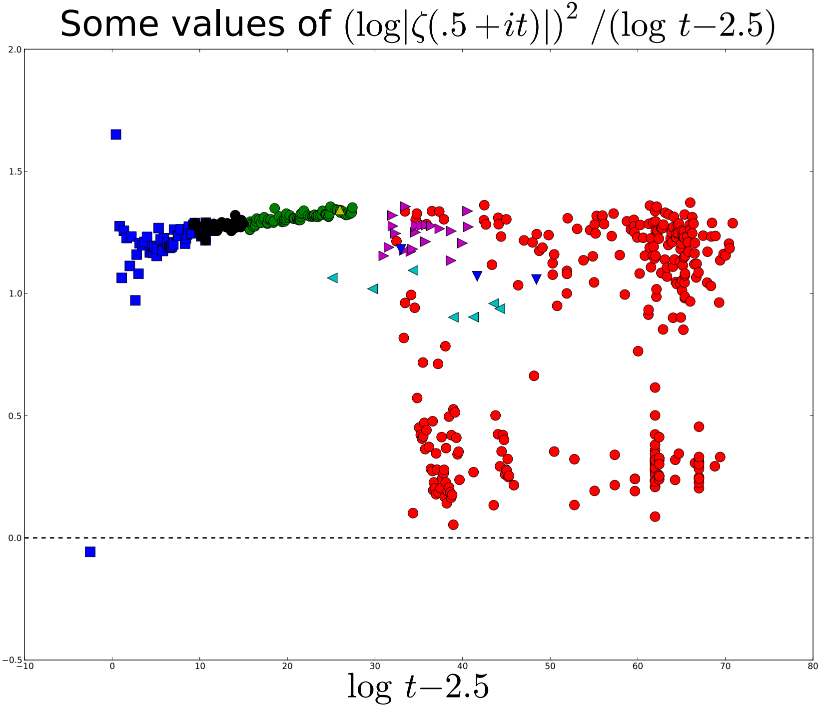
Data from various sources:

5. From G. Hiary: Largest values in big ranges around 10^n, $n = 16, 20, 23$, complete data up to 10^8.
6. Large values computed by us using method described earlier.

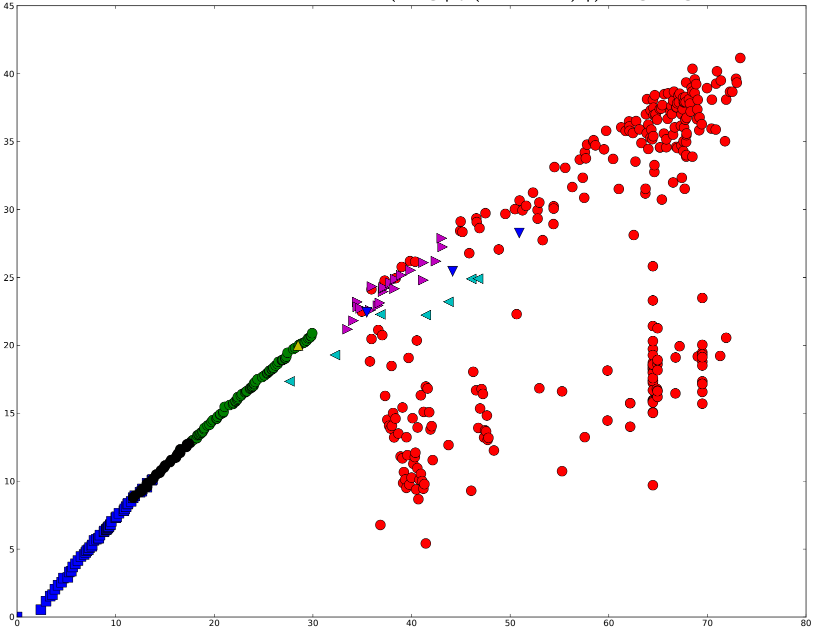

Some values of $\log|\zeta(.5+it)|$
10
8
6
4
2
0
−10
0
10
20
30
40
50
60
70
80
$\log\ t$

Possible nonsense analysis.

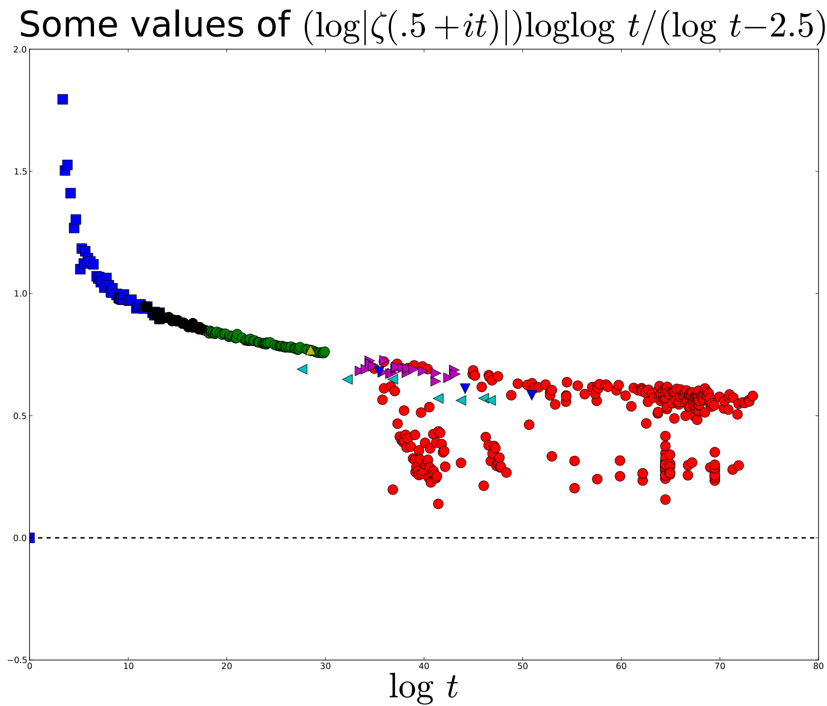

Some values of $(\log|\zeta(.5+it)|)^2$
100
80
60
40
20
0
0
10
20
30
40
50
60
70
80
$\log\, t$


Some values of $(\log|\zeta(.5+it)|)^2 / (\log t - 2.5)$
2.0
1.5
1.0
0.5
0.0
−0.5
−10
0
10
20
30
40
50
60
70
80
$\log t - 2.5$

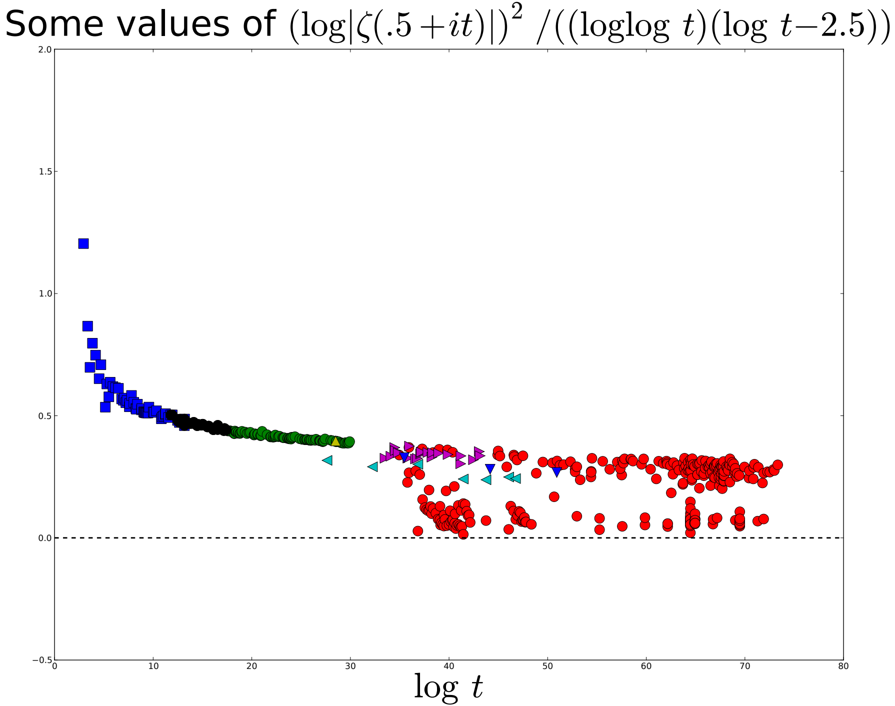
Some values of $(\log|\zeta(.5+it)|)\log\log\, t$

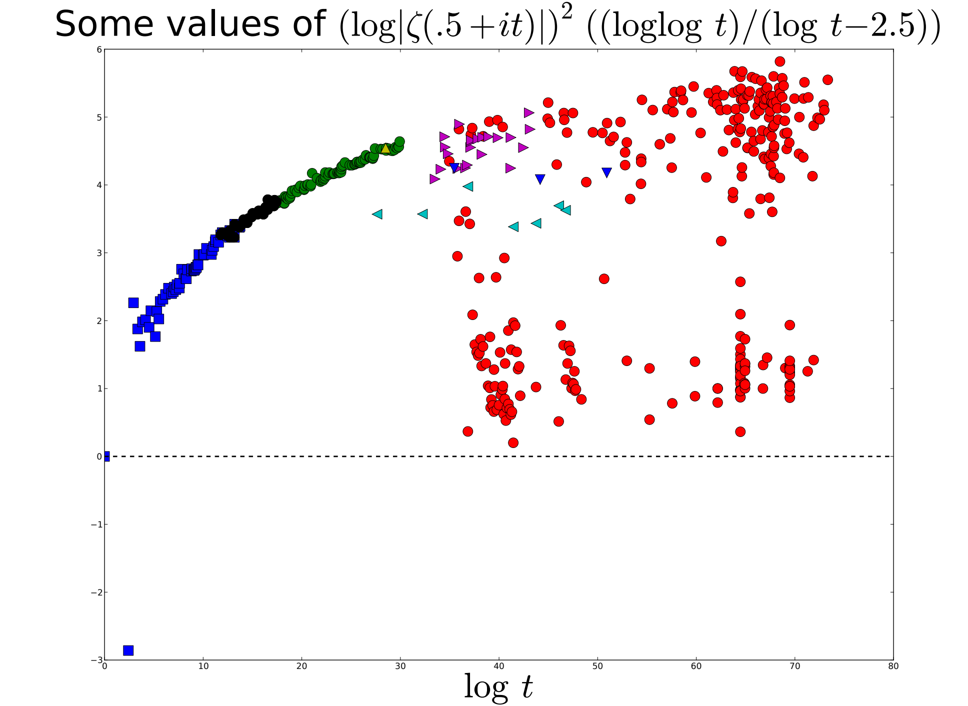


$\log\, t$



Some values of $(\log|\zeta(.5+it)|)\log\log t/(\log t-2.5)$
2.0
1.5
1.0
0.5
0.0
−0.5
0
10
20
30
40
50
60
70
80
$\log t$


Some values of $(\log|\zeta(.5+it)|)^2 / ((\log\log t)(\log t - 2.5))$
2.0
1.5
1.0
0.5
0.0
−0.5
0
10
20
30
40
50
60
70
80
$\log t$



Some values of $(\log|\zeta(.5+it)|)^2\ ((\log\log\ t)/(\log\ t-2.5))$
6
5
4
3
2
1
0
−1
−2
−3
0
10
20
30
40
50
60
70
80
log t

Brief discussion of accuracy

In all computations of the zeta function, perhaps one of the best signs of correctness is that no violations of the Riemann Hypothesis have been found.
Odlyzko reports having deliberately introduced errors into his programs to see what would happen, and violations of the Riemann Hypothesis were always found.

Brief discussion of accuracy

I have found no need to do this deliberately.