Non-associative algebras of minimal cones and axial algebras

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There will be no obvious connections of my talk to representations of simple finite groups.

On the other hand, there are only few established examples of commutative nonassociative algebras with nice fusion rules. There are some further common properties and features.

Is this mere coincidence?

My talk is dedicated to the memory of Sergei Natanovich Bernstein, 1880-1968.

S.N. Bernstein (1880 – 1968)

A Russian and Soviet mathematician (doctoral dissertation, submitted in 1904 to the Sorbonne under supervision of Emil Picard and David Hilbert), known for contributions to partial differential equations, differential geometry, probability theory, and approximation theory:

- 1904 solved Hilbert's 19th problem (a C^3-solution of a nonlinear elliptic analytic equation in 2 variables is analytic)
- 1910s introduced a priori estimates for Dirichlet's boundary problem for non-linear equations of elliptic type
- 1912 laid the foundations of constructive function theory (Bernstein's theorem in approximation theory, Bernstein's polynomial).
- 1915 the famous 'Bernstein's Theorem' on entire solutions of minimal surface equation.
- 1917 the first axiomatic foundation of probability theory, based on the underlying algebraic structure (later superseded by the measure-theoretic approach of Kolmogorov)
- 1924 introduced a method for proving limit theorems for sums of dependent random variables
- 1923 axiomatic foundation of a theory of heredity: genetic algebras (Bernstein algebras): $x^2x^2 = \omega(x)^2x^2$

Some important questions

How incident (important, relevant) that the certain *commutative non-associative* algebraic structures coming from a) finite simple groups, b) geometry of minimal cones, c) PDEs (truly viscosity solutions)

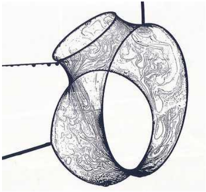
- have a distinguished Peirce spectrum
- have distinguished (in particular, graded) fusion rules
- are axial (generated by 'good' idempotents)
- are metrized (i.e. carrying an associating symmetric bilinear form)
- satisfy certain restrictions like the Norton inequality
- etc

One possible point of view is to put these algebras in a broader context of general non-associative metrized algebras with 'small' Peirce spectrum.

What is this all about?

A **minimal surface** (in a wider sense, a string) is a critical point of the area functional. Geometrically, this means that the **mean curvature** $= 0$. If $x_{n+1} = u(x)$ is a minimal graph over $\mathbb{R}^n$ then

$$\operatorname{div}(1 + |Du|^2)^{-\frac{1}{2}} Du = 0.$$



Bernstein's theorem (1915): u is an affine function $n = 2$. The result is still true for $n \leq 7$ (Almgren, De Giorgi, Simons) but it fails for $n = 8$ (Bombieri-de Giorgi-Giusti).

A **minimal cone** is a typical singularity of a minimal surface. All known minimal cones are algebraic, i.e. zero level sets of a homogeneous polynomial $u \in \mathbb{R}[x_1, \ldots, x_n]$:

- the Clifford-Simons cone, $u(x) := (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$ (**the norm for split octonions**).
- The **triality polynomials** $\mathrm{Re}((z_1 z_2) z_3), z_i \in \mathbb{K}_d, d = 1, 2, 4, 8$ are examples of **cubic minimal cones** in $\mathbb{R}^{3d}$.
- The **generic norm** on the trace free subspace of the **cubic Jordan algebra** $\mathscr{H}_3'(\mathbb{K}_d)$

Problem: How to characterize algebraic minimal cones?

Hsiang minimal cones

W.-Y. Hsiang (*J. Diff. Geometry*, **1**, 1967): Let u be a homogeneous polynomial in $\mathbb{R}^n$. Then $u^{-1}(0)$ is a minimal cone iff

$$\Delta_1 u := |Du|^2 \Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle \equiv 0 \mod u.$$

- In $\deg = 2$: $\{(x,y) \in \mathbb{R}^{k+m} : (m-1)|x|^2 = (k-1)|y|^2\}$
- The first non-trivial case: $\deg u = 3$ and then

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \qquad (1)$$

- In fact, all known irreducible cubic minimal cones satisfy very special equation:

$$\Delta_1 u = \lambda |x|^2 \cdot u(x) \qquad (2)$$

Hsiang problem: Classify all cubic polynomial solutions of (2).

A homogeneous cubic solution of (1) is called a **Hsiang cubic**.

Some explicit examples of Hsiang cubics

- $u = \mathrm{Re}(z_1 z_2) z_3$, $z_i \in \mathbb{A}_d$, $d = 1, 2, 4, 8$, the triality polynomials in $\mathbb{R}^{3d}$ where

$$\mathbb{A}_1 = \mathbb{R}, \ \ \mathbb{A}_2 = \mathbb{C}, \ \ \mathbb{A}_4 = \mathbb{H}, \ \ \mathbb{A}_8 = \mathbb{O}$$

  are the classical Hurwitz algebras. The example with $d = 1$ also appears as Example 2 below.

- $u(x) = \begin{vmatrix} \frac{1}{\sqrt{3}}x_1 + x_2 & x_3 & x_4 \\ x_2 & \frac{-2}{\sqrt{3}}x_1 & x_5 \\ x_4 & x_5 & \frac{1}{\sqrt{3}}x_1 - x_2 \end{vmatrix}$ = a Cartan isoparametric cubic in $\mathbb{R}^5$

  It is the generic norm in the Jordan algebra of 3×3 symmetric matrices over $\mathbb{R}$

- $u(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix}$

  (equivalently, the generic norm in the Jordan algebra of 4×4 symmetric *traceless* matrices over $\mathbb{R}$)

Thus, Hsiang cubics are nicely encoded by certain algebraic structures. Which ones?

Nonassociative algebras and singular solutions

Evans, Crandall, Lions: Let B be the unit ball, ϕ continuous on ∂B, F uniformly elliptic operator. Then the Dirichlet problem $F(D^2u) = 0$ in B, $u = \phi$ on ∂B has a unique **viscosity solution** u which is *continuous* in B.

- Nirenberg, 50's: if $n = 2$ then u is classical (C^2) solution (Abel Prize, 2015)
- Krylov-Safonov, Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$

A problem of crucial importance is **when a viscousity solution is a classical solution**. Nadirashvili, Vlăduț, 2007-2011: if $n \geq 12$ then there are solutions which are not C^2.

Theorem (N. Nadirashvili, V.T., S. Vlăduț, Adv. Math. 2012)

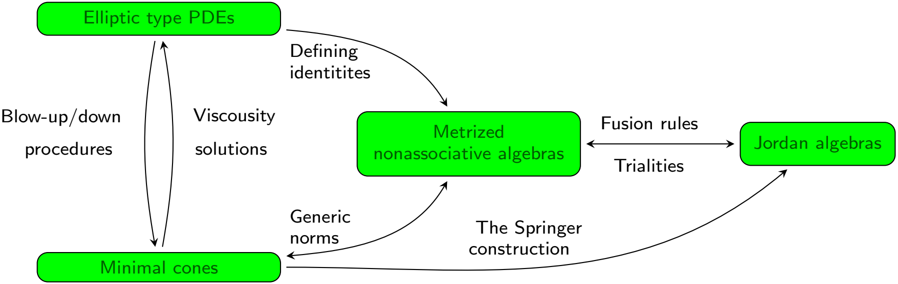
The function $w(x) := \frac{u_1(x)}{|x|}$ where u_1 is the Cartan isoparametric cubic

$$u_1(x) = x_5^3 + \tfrac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \tfrac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

is a singular viscosity solution of the uniformly elliptic Hessian equation

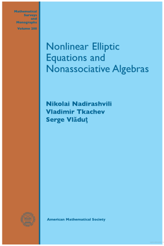
$$(\Delta w)^5 + 2^8 3^2 (\Delta w)^3 + 2^{12} 3^5 \Delta w + 2^{15} \det D^2(w) = 0,$$

How it works (on algebraic level)



Further reading on both analytic and algebraic account is here:

N. Nadirashvili, V.T., S. Vlăduţ, *Nonlinear elliptic equations and nonassociative algebras*, Math. Surveys and Monographs, v. 200, AMS, 2015.



Notations

- V denotes a commutative nonassociative algebra with multiplication denoted by juxtaposition;
- $L_v : x \to vx$ is the multiplication operator (also denoted as ad_v)
- $c \in V$ is called an idempotent if $c^2 = c$;
- $\sigma(c) =$ the spectrum of L_c;
- c is called semi-simple if V is the direct sum of (simple) L_c-invariant subspaces
- a symmetric bilinear form $\langle , \rangle$ on V is associating if $\langle xy, z\rangle = \langle x, yz\rangle$
- V is metrized if it carries a symmetric associating bilinear form ($\approx$ Frobenius)
- If V is metrized then L_c is self-adjoint. In particular, all idempotents in V are semi-simple.

Why $\frac{1}{2}$?

The eigenvalues $1, 0$ and $\frac{1}{2}$ are very distinguished:

- Power-associative algebras, $x^2x^2 = xx^3$ and Jordan algebras, $x^2(xy) = x(x^2y)$:

$$\sigma(c) \subset \{0, 1, \tfrac{1}{2}\}.$$

- certain axial algebras
- pseudocomposition algebras, i.e. $x^3 = b(x)x$: $\sigma(c) = \{1, -1, \frac{1}{2}\}$, always primitive
- nonassociative rank 3 algebras, i.e. $x^3 = a(x)x^2 + b(x)x$:

$$\sigma(c) = \{1, -b(c), \tfrac{1}{2}\}, \quad \text{always primitive}$$

- Algebras of cubic minimal cones,

$$4xx^3 + x^2x^2 - 3\langle x, x\rangle x^2 - 2\langle x^2, x\rangle x = 0 \quad \text{and} \quad \operatorname{tr} L_x = 0,$$

  then $\sigma(c) \subset \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$, always primitive.
- Bernstein algebras $x^2x^2 = \omega(x)^2x^2$, $\sigma(c) = \{1, 0, \frac{1}{2}\}$, always primitive

All the above algebras have nice **fusion rules** (some are $\mathbb{Z}/2$-graded)

Fusion rules for algebras of cubic minimal cones

$\star$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1		1	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
$-\frac{1}{2}$			$1, -\frac{1}{2}$	$-1, \frac{1}{2}$
$\frac{1}{2}$				$1, -1, -\frac{1}{2}$

- Not $\mathbb{Z}/2$-graded in general
- Two distinguished subalgebras: $V_c(1) \oplus V_c(-1)$ (carries a hidden Clifford algebra structure) and $V_c(1) \oplus V_c(-\frac{1}{2})$ (carries a hidden rank 3 Jordan algebra structure)
- Primitive idempotents w in the hidden Jordan algebra ($w \bullet w = w$) are exactly 2-nilpotents in V ($w^2 = w$) with the fusion rules

$\star$	$0'$	-1	1
$0'$	$-1, 1$	$0'', 1$	$0'', -1$
-1		$0'$	$0''$
1			$0'$

$(0 = 0' \oplus 0'',\ 0' = \mathrm{Span}(w))$

Fusion rules for algebras of cubic minimal cones

However, it turns out that the fusion rules *are* $\mathbb{Z}/2$-graded *a posteriori*. Let $V = V^0 \oplus V^1$ be the standard $\mathbb{Z}/2$-grading. Let

$$n_1 = \dim V_c(-1), \qquad n_2 = \dim V_c(-\tfrac{1}{2}),$$

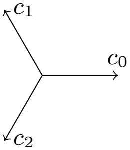
Then

- if V is polar ('Clifford type'), see Definition 5 below, then $V^0 V^0 = 0$; in this case V^0 is an isomorphic image of $V_c(1) \oplus V_c(-1)$.
- if $n_2 = 0$ then $V^0 = V_c(1) \oplus V_c(-1)$ and $V^1 = V_c(\frac{1}{2})$
- if $n_1 = 0$ then $V^0 = V_c(1) \oplus V_c(-\frac{1}{2})$ and $V^1 = V_c(\frac{1}{2})$
- if $n_1 = 1$ then $V^0 = V_c(1) \oplus V_c(-\frac{1}{2})$ and $V^1 = V_c(-1) \oplus V_c(\frac{1}{2})$
- if $n_1 = 4$ then $n_2 = 5$ (the algebra V has dimension 21 and comes from the Albert exceptional Jordan algebra) then grading is explicit but more subtle

Two basic examples in dimensions 2 and 3

Example 1. Let V be the 2 dimensional algebra generated by three **idempotents** c_i, $i = 0, 1, 2$ which can be realized as unit vectors in $\mathbb{R}^2$ subject to the conditions:

- $\langle c_i, c_j \rangle = -\frac{1}{2}$, $i \neq j$,
- $c_0 + c_1 + c_2 = 0$



Then for any triple $\{i, j, k\} = \{1, 2, 3\}$ we have

$$c_k = c_k^2 = (-c_i - c_j)^2 = c_i + c_j + 2c_i c_j = -c_k + 2c_i c_j$$

hence $c_i c_j = c_k$ and $c_k(c_i - c_j) = -(c_i - c_j)$. This implies $V = V_{c_i}(1) \oplus V_{c_i}(-1)$, the both Peirce subspaces being 1-dimensional. The corresponding fusion rules are

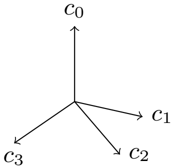
$\star$	1	-1
1	1	-1
-1		1

The Peirce dimensions are $n_1 = 1, n_2 = n_3 = 0$, the ambient dimension $n = 2$. The minimal cone is given by $x_1^2 x_2 = 0$, i.e. pair of two orthogonal planes in $\mathbb{R}^2$

Two basic examples in dimensions 2 and 3

Example 2. Similarly, let V be the 3 dimensional algebra generated by four **idempotents** c_i, $i = 0, 1, 2, 3$ realized as unit vectors in $\mathbb{R}^3$ subject to the conditions:

- $c_i + c_j$ is a 2-nilpotent, i.e. $(c_i + c_j)^2 = 0$ $(i \neq j)$



Then similarly to the above, one easily verifies that

$$V = V_{c_i}(1) \oplus V_{c_i}(-\tfrac{1}{2}),$$

where $\dim V_{c_i}(1) = 1$ and $\dim V_{c_i}(-\frac{1}{2}) = n_2 = 2$.

The corresponding fusion rules are

$\star$	1	$-\frac{1}{2}$
1	1	$-\frac{1}{2}$
$-\frac{1}{2}$		$1, -\frac{1}{2}$

The underlying algebra structure after a 1-rank perturbation becomes a Jordan algebra of Clifford type. The minimal cone is given by $x_1x_2x_3 = 0$, i.e. the triple of coordinate planes in $\mathbb{R}^3$.

$\frac{1}{2}$ in the spectrum of metrized algebras

Let V be a commutative metrized algebra over $\mathbb{R}$ with positive definite associating form $\langle x, y\rangle$. Then

(a) the set of idempotents of V is nonempty

Proof. Let x be a stationary point of $f(x) = \langle x, x^2\rangle$ on the unit sphere $\langle x, x\rangle = 1$ (obviously a nonempty set). By Lagrange's principle, $\nabla\langle x, x^2\rangle = 3x^2$ and $\nabla\langle x, x\rangle = 2x$ must be proportional $\Rightarrow$:

$$x^2 = kx \quad \Rightarrow \quad c^2 = c, \quad \text{where } c := \frac{x}{\langle x, x^2\rangle} \text{ is an idempotent!}$$

(b) if c is an extremal idempotent then it is *primitive*. In fact, the spectrum of $L_c : x \to cx$ on the orthogonal complement $c^\perp$ is a subset of $(-\infty, \frac{1}{2}]$

Proof. Consider variation of $\langle x, x^2\rangle$, with $x = x_0 + y$ and $y \perp x_0$.

(c) if c is an idempotent with the smallest length then $V_c(\frac{1}{2})V_c(\frac{1}{2}) \subset V_c(\frac{1}{2})^\perp$.

(d) if all idempotents c have the same length and $\frac{1}{2} \in \sigma(c)$ then the $\frac{1}{2}$-fusion rule holds!

$\frac{1}{2}$ in algebras with identities

Let a commutative algebra V satisfy an identity of the kind

$$\sum_{\alpha} \phi_\alpha(x) x^\alpha = 0, \qquad x^\alpha \in N(x), \tag{3}$$

where

$$N(x) = \{x^\alpha : x,\ x^2,\ x^3,\ x^2x^2,\ xx^3,\ x^2x^3,\ x(x^2x^2), \ldots\}$$

is the commutative groupoid generated by x.

Theorem A.

Let V be a commutative algebra satisfying identity (3) and let c be a nonzero idempotent in V. Then $\frac{1}{2} \in \sigma(c)$ in the sense that $\frac{1}{2}$ is a root of the characteristic polynomial of the linearization of (3).

$\frac{1}{2}$ in algebras with identities

Proof by linearization: given a NA polynomial $f(x)$ in x, there exists a unique endomorphism $Df(x) : V \to V$ such that

$$f(x + \epsilon y) \equiv f(x) + Df(x)(y)\epsilon \mod \epsilon^2.$$

Similarly, given a homogeneous function $\phi : V \to K$, there exists a unique linear form $D\phi(x) \in V^*$ such that

$$\phi(x + \epsilon y) \equiv \phi(x) + D\phi(x)(y)\epsilon \mod \epsilon^2.$$

We also have

$$D(\phi f) = \phi Df + f \otimes D\phi$$

where

$$(a \otimes b)(y) = a \cdot b(y), \qquad a \in V,\ b \in V^*.$$

Example 1. We have

$$(x + \epsilon y)^2 \equiv x^2 + 2xy\epsilon \mod \epsilon^2 \qquad \Rightarrow \quad D(x^2) = 2L_x$$
$$(x + \epsilon y)^3 \equiv x^3 + (x^2y + 2x(xy))\epsilon \mod \epsilon^2 \qquad \Rightarrow \quad D(x^3) = L_{x^2} + 2L_x^2,$$

$\frac{1}{2}$ in algebras with identities

Now, let $c \neq 0$ be an idempotent. Then $c^\alpha = c$ yields

$$\sum_\alpha \phi_\alpha(c) = 0.$$

Also, the linearization followed by substitution $x = c$ yields

$$\sum_\alpha \phi_\alpha(x) D(x^\alpha) + x^\alpha \otimes D(\phi_\alpha) = 0$$
$$\sum_\alpha \phi_\alpha(c) P_{x^\alpha}(L_c) = c \otimes \Phi, \quad \Phi \in V^*$$

where $P_{x^\alpha}(L_c) = D(x^\alpha)|_{x=c}$ is a polynomial in L_c. For example,

$$D(x^3) = L_{x^2} + 2L_x^3 \quad \Rightarrow$$
$$D(x^3)|_{x=c} = L_c + 2L_c^2 \quad \Rightarrow$$
$$P_{x^3}(t) = 2t^2 + t.$$

$\frac{1}{2}$ in algebras with identities

Some characteristic polynomials (note that $D(x^\alpha)$ are very complicated but $P_{x^\alpha}(t)$ not):

x^α	$D(x^\alpha)$	$P_{x^\alpha}(t)$	$P_{x^\alpha}(1)$	$P_{x^\alpha}(\frac{1}{2})$
x	1	1	1	1
x^2	$2L_x$	$2t$	2	1
x^3	$L_{x^2} + 2L_x^2$	$2t^2 + t$	3	1
x^4	$L_{x^3} + L_x L_{x^2} + 2L_x^3$	$2t^3 + t^2 + t$	4	1
x^2x^2	$4L_{x^2}L_x$	$4t^2$	4	1
x^5	$L_{x^4} + L_x L_{x^3} + L_x^2 L_{x^2} + 2L_x^4$	$2t^4 + t^3 + t^2 + t$	4	1

- In particular, $P_z(1) = \deg z$.
- A NA groupoid of characteristic polynomials generated by $P_x = 1$ by virtue of

$$D(x^\alpha x^\beta) = L_{x^\alpha} D(x^\beta) + L_{x^\beta} D(x^\alpha), \quad \text{i.e.}$$
$$P_{x^\alpha x^\beta} = t(P_{x^\alpha} + P_{x^\beta})$$

- This in particular yields (by induction on $\deg x^\alpha$) that

$$P_{x^\alpha x^\beta}(\tfrac{1}{2}) = \tfrac{1}{2}(P_{x^\alpha}(\tfrac{1}{2}) + P_{x^\beta}(\tfrac{1}{2})) = \tfrac{1}{2}(1+1) = 1$$

$\frac{1}{2}$ in algebras with identities

Return to the identity:

$$\sum_{\alpha} \phi_\alpha(c) P_{x^\alpha}(L_c) = c \otimes \Phi$$

If $L_c y = \lambda y$ and $y \notin Kc$ this yields

$$\chi_c(\lambda) := \sum_{\alpha} \phi_\alpha(c) P_{x^\alpha}(\lambda) = 0$$

Since $\sum_\alpha \phi_\alpha(c) = 0$ and $P_{x^\alpha}(\frac{1}{2}) = 1$ we conclude that

$$\chi_c(\tfrac{1}{2}) = 0 \qquad \Rightarrow \quad \tfrac{1}{2} \in \sigma(c)$$

How to connect nonassociative algebras to PDEs?

Commutative metrized algebras

Let A be a commutative K-algebra on V. A K-bilinear symmetric form Q on a vector space V is called **associating** if

$$Q(x,y) = 0 \quad \forall y \in V \quad \Rightarrow \quad x = 0,$$
$$Q(xy,z) = Q(x,yz) \qquad \forall x,y,z \in V.$$

An algebra (A,Q) is called **metrized** if Q is associating. In that case we have

$$L_y^* = L_y \quad \text{for all } y \in V.$$

In particular, there holds the Peirce decomposition

$$V = \bigoplus_{\lambda \in \sigma(L_y)} V_y(\lambda)$$

Examples:

- a full matrix algebra with its trace $Q(x,y) = \operatorname{tr} xy$
- a real semisimple Lie algebra with its Killing form $Q(a,b) = \operatorname{tr} \operatorname{ad}_a \operatorname{ad}_b$
- a real semisimple Jordan algebra with its trace form $Q(a,b) = \operatorname{tr} ab$

In what follows $Q(x,y) = \langle x,y \rangle$.

Commutative metrized algebras

In this setting, the study of V is essentially equivalent to study of the cubic form

$$N(x) := \frac{1}{6}\langle xx, x\rangle = \frac{1}{6}\langle x^2, x\rangle$$

Then the (*commutative*) multiplication structure is recovered by linearization:

$$\langle xy, z\rangle = N(x, y, z) = N(x+y+z) - N(x+y) - N(x+z) - N(y+z) + N(x) + N(y) + N(z).$$

Conversely, if $N(x)$ is a cubic form on an inner product space $(V, \langle,\rangle)$ then the multiplication is uniquely determined and turns V into a commutative metrized algebra.

While a CMA is not power associative in general ($x^2x^2 \neq x^3x$), the moments of x of order ≤ 5 are well defined:

$$\langle x^2, x^2\rangle = \langle x, x^3\rangle,$$
$$\langle x, x^2x^2\rangle = \langle x^2, x^3\rangle.$$

Commutative metrized algebras

Let $K = \mathbb{R}$, $(V, \langle\cdot,\cdot\rangle)$ be an inner product vector space and $u(x)$ be a cubic form V. Denote by $V = \mathrm{CMA}(u)$ the corresponding metrized algebra, i.e.

$$u(x, y, z) = \langle xy, z\rangle, \qquad \forall z \in V.$$

In this setting,

- $u(x) = \frac{1}{6}\langle x, x^2\rangle$
- $Du(x) = \frac{1}{2}x^2$
- $xy = (D^2u(x))y$, or $L_x = D^2u(x)$
  i.e. the (left) multiplication operator by x is the *Hessian* of u at x
- L_x is **self-adjoint**: $\langle L_x y, z\rangle = \langle y, L_x z\rangle$
- If $\langle,\rangle$ is positive definite then the set of idempotents of V is nonempty.

a cubic form u + a PDE $\quad = \quad$ a metrized algebra $V(u)$ with a defining identity

Examples

- **A trivial example.** $V = \mathbb{R}^1$ with $\langle x, y\rangle = xy$ and $u(x) = \frac{1}{6}x^3$. One has

$$u(x; y, z) = \partial_z \partial_y u(x) = xyz, \qquad \Rightarrow \qquad x \bullet y = xy$$

therefore $\bullet$ is the usual multiplication.

- **A less trivial example (a Jordan spin-factor)** Let $V = \mathbb{R}^2$ with the standard Euclidean inner product $\langle x, y\rangle$. Let $u(x) = \frac{1}{2}x_1^2 x_2$. Then

$$u(x; y; z) = x_2 y_1 z_1 + x_1 y_2 z_1 + x_1 y_1 z_2 \quad \Rightarrow \quad x \bullet y = (x_1 y_2 + x_2 y_1,\, x_1 y_1)$$

- **A non-trivial example (H. Freudenthal 1954, T. Springer 1961)** Let $V = \mathscr{H}_3(\mathbb{F}_d)$ be the vector space of self-adjoint 3×3-matrices with coefficients in a normed division algebra $\mathbb{F}_d$ and

$$u(x) = \mathrm{Det}(x) := \tfrac{1}{6}((\operatorname{tr} x)^3 - 3 \operatorname{tr} x \operatorname{tr} x^2 + 2 \operatorname{tr} x^3).$$

Then $V(u)$ is a Jordan algebra w.r.t. the multiplication

$$x \bullet y = \tfrac{1}{2}(xy + yx).$$

Two basic examples

(A) The Cartan-Münzner equations (describe isoparametric hypersurfaces with $g = 3$ distinct principal curvatures):

$$\begin{cases} |Du(x)|^2 &= 9|x|^4 \\ \Delta u(x) &= 0 \end{cases} \implies \begin{cases} \langle x^2, x^2 \rangle &= 36|x|^4 \\ \operatorname{tr} L_x &= 0, \qquad \forall x \in V \end{cases}$$

(B) Hsiang (1967) asked to classify all cubic homogeneous solutions of

$$|Du|^2 \Delta u - \tfrac{1}{2} \nabla u \cdot \nabla |\nabla u|^2 = \lambda |x|^2 u \tag{4}$$

This equation asserts that the cone $u^{-1}(0)$ has zero mean curvature in $\mathbb{R}^n$.

$$\implies \begin{cases} \langle x^2, x^3 \rangle &= \langle x, x^2 \rangle |x|^2 \\ \operatorname{tr} L_x &= 0 \qquad \text{(a nontrivial implication)} \end{cases}$$

How to connect Cartan-Münzner eqs with Jordan algebras

Given a cubic form $u : V \to \mathbb{K}$, consider its linearizations

- $u(x, y, z) = u(x + y + z) - u(x + y) - u(x + z) - u(y + z) + u(x) + u(y) + u(z)$
- $\partial_y u(x) = u(x; y) = \frac{1}{2} u(x, x, y)$

The Springer Construction (McCrimmon, 1969)

A cubic form $N : V \to \mathbb{K}$, $N(e) = 1$, is called a admissible if the bilinear form

$$T(x; y) = N(e; x)N(e; y) - N(e; x; y)$$

is a *nondegenerate* and the map $\# : V \to V$ uniquely determined by $T(x^{\#}; y) = N(x; y)$ satisfies the adjoint identity

$$(x^{\#})^{\#} = N(x)x.$$

If N is Jordan and $x \# y = (x + y)^{\#} - x^{\#} - y^{\#}$ then

$$x \bullet y = \frac{1}{2}(x \# y + N(e; x)y + N(e; y)x - N(e; x; y)e)$$

defines a Jordan algebra structure on V and

$$x^{\bullet 3} - N(e; x)x^{\bullet 2} + N(x; e)x - N(x)e = 0, \quad \forall x \in V.$$

How to connect Cartan-Münzner eqs with Jordan algebras

Let us drop the second (harmonictiy) equation. Then

Theorem (V.T., *J. of Algebra*, 2014). There is a natural correspondence between

- cubic solutions of $|\nabla u(x)|^2 = 9|x|^4$, and
- rank 3 formally real semisimple Jordan algebras

such that congruent solutions corresponds to isomorphic Jordan algebras.

Proof. Let $V = \mathrm{CMA}(u)$, then $u(x) = \frac{1}{6}\langle x^2, x\rangle$ and $\langle x^2, x^2\rangle = 36|x|^4$. Let $W = \mathbb{R} \oplus V$ and define

$$N(\boldsymbol{x}) = x_0^3 - \frac{3}{2}x_0|x|^2 + \tfrac{1}{6\sqrt{2}}\langle x^2, x\rangle, \quad \boldsymbol{x} = (x_0, x).$$

Then $\boldsymbol{e} = (1, 0)$ is a base point: $N(\boldsymbol{e}) = 1$, and the polarization yields:

$$\begin{aligned}
& N(\boldsymbol{x};\boldsymbol{y}) && = 3x_0^2 y_0 - 3x_0\langle x, y\rangle - \tfrac{3}{2}|x|^2 y_0 + \tfrac{1}{2\sqrt{2}}\langle x^2, y\rangle \\
\Rightarrow\quad & N(\boldsymbol{x};\boldsymbol{e}) && = 3x_0^2 - \tfrac{3}{2}|x|^2 \qquad \text{and} \qquad N(\boldsymbol{e};\boldsymbol{x}) = 3x_0 \\
\Rightarrow\quad & T(\boldsymbol{x},\boldsymbol{y}) && = N(\boldsymbol{e};\boldsymbol{x})N(\boldsymbol{e};\boldsymbol{y}) - N(\boldsymbol{x};\boldsymbol{y};\boldsymbol{e}) = 3(x_0y_0 + \langle x, y\rangle) = 3\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\
\Rightarrow\quad & \boldsymbol{x}^{\#} && = (x_0^2 - \tfrac{1}{2}|x|^2, \tfrac{1}{6\sqrt{2}}x^2 - x_0 x) \\
\Rightarrow\quad & (\boldsymbol{x}^{\#})^{\#} && = N(\boldsymbol{x})\boldsymbol{x} \quad \Rightarrow \quad N(\boldsymbol{x}) \text{ is admissible} \quad \square
\end{aligned}$$

An alternative approach

Let $V = \mathrm{CMA}(u)$. Then the defining relation and the subsequent polarizations are:

$$\langle x^2, x^2\rangle = 36|x|^4 \quad \Rightarrow \quad x^3 = |x|^2 x \quad \Rightarrow \quad 2L_x^2 + L_{x^2} = 2x \otimes x + |x|^2$$

If $c \neq 0$ is an idempotent of V then $|c|^2 = 1$ and $2L_c^2 + L_c - 1 = 2c \otimes c$ implying

$$\sigma(L_c) \subset \{-1, \tfrac{1}{2}, 1\} \quad \Rightarrow \quad \text{the Peirce decomposition:} \quad V = \mathbb{R}c \oplus V_c(-1) \oplus V_c(\tfrac{1}{2}),$$

A further polarization gives $x(cy) + c(xy) + (cx)y = c\langle x, y\rangle$, thus

$$V_c(t_1)V_c(t_2) \perp V_c(t_3) \qquad \text{unless} \quad t_1 + t_2 + t_3 = 0.$$

The fusion rules:

	1	-1	$\frac{1}{2}$
1	1	-1	$\frac{1}{2}$
-1		1	$\frac{1}{2}$
$\frac{1}{2}$			$1, -1$

- $L_x : V_c(\frac{1}{2}) \to V_c(\frac{1}{2})$ and $L_x^2 = \frac{3}{4}|x|^2$ for any $x \in V_c(-1)$, hence $(L_x, V_c(-1), V_c(\frac{1}{2}))$ is a symmetric Clifford system, implying that

$$d \leq \rho(d) \quad \Rightarrow \quad d \in \{1, 2, 4, 8\}!$$

Coming back to minimal cones and Hsiang algebras

Hsiang algebras

In the metrized algebra setup, the Hsiang problem (4) becomes equivalent to the classification of all **commutative Euclidean metrized algebras** V satisfying

$$\langle x^2, x^3 \rangle = \langle x, x \rangle \langle x^2, x \rangle,$$
$$\operatorname{tr} L_x = 0.$$

We call a commutative algebra with positive definite associating symmetric bilinear product a **Hsiang algebras** if the two above equations hold.

The correspondence:

V is a Hsiang algebra $\Leftrightarrow$ $u(x) = \frac{1}{6}\langle x, x^2 \rangle$ generates a Hsiang cubic minimal cone.

Examples of Hsiang algebras, I

Any commutative **pseudocomposition algebra**, i.e. an algebra with

$$x^3 = |x|^2 x, \qquad \operatorname{tr} L_x = 0$$

is Hsiang.

Remark. Appear in diverse contexts, for instance, genetic algebras or **isoparametric hypersurfaces** (the hypersurfaces M of the Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$ whose principal curvatures are constant along M). In the CMA[1] setup, the Cartan-Münzner equations for $g = 3$ distinct curvatures become the pseudocomposition algebra definition:

$$\begin{cases} |\nabla u(x)|^2 &= 9|x|^4 \\ \Delta u(x) &= 0 \end{cases} \Longrightarrow \begin{cases} \langle x^2, x^2 \rangle &= 36|x|^4 \\ \operatorname{tr} L_x &= 0, \qquad \forall x \in V \end{cases}$$

The first equation is essentially equivalent to $x^3 = 36|x|^2 x$.

[1] commutative metrized algebra

Examples of Hsiang algebras, II

Definition. A commutative metrized $\mathbb{Z}_2$-graded algebra $V = V_0 \oplus V_1$ is called polar if

$$V_0V_0 = \{0\} \qquad \text{and} \qquad L_x^2 = |x|^2 \text{ on } V_1, \quad \forall x \in V_0. \tag{5}$$

An equivalent description: start with a **symmetric Clifford system** $\mathscr{A} \in \mathrm{Cliff}(X, Y)$, i.e. symmetric matrices $\{A_1, \ldots, A_q\}$ with $A_i^2 = I$ and

$$A_iA_j + A_jA_i = 0, \quad i \neq j$$

The well-known obstruction:

$$q \leq 1 + \rho(p),$$

where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \mathrm{odd}$, $0 \leq b \leq 3$ is the **Hurwitz-Radon function**.

Proposition (the correspondence)

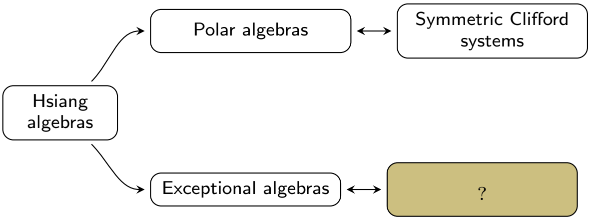
An algebra $V = V_0 \oplus V_1$ is polar iff it is isomorphic to CMA of the cubic form

$$u_{\mathscr{A}}(z) = \sum_{i=1}^{q} x_i \cdot y^t A_i y, \qquad z = (x, y) \in \mathbb{R}^q \times \mathbb{R}^{2p}$$

The correspondence is a bijection between isomoprhy classes and the classes of geometrically equivalent Clifford systems.

How to classify?

Definition. A Hsiang algebra V similar to a polar algebra is said to be of **Clifford type**; otherwise it is called **exceptional**.



The harmonicity

Theorem 1

Any non-trivial Hsiang algebra V is harmonic, i.e. $\operatorname{tr} L_x = 0$ for all $x \in V$. In particular,

- *In any Hsiang algebra*

  $$\langle x^2, x^3 \rangle = -\tfrac{2}{3}\lambda \langle x, x^2 \rangle |x|^2$$

  for some $\lambda < 0$.

- *All idempotents c have* **the same length:** $|c| = \sqrt{-\frac{3}{2\lambda}}$.

Definition

A Hsiang algebra is called **normalized** if $\lambda = -2$ (i.e. $|c|^2 = \frac{3}{4}$). Then

$$\langle x^2, x^3 \rangle = \tfrac{4}{3}\langle x, x \rangle \langle x, x^2 \rangle,$$
$$x x^3 + \tfrac{1}{4} x^2 x^2 - |x|^2 x^2 - \tfrac{2}{3}\langle x^2, x \rangle x = 0.$$

The Peirce decomposition

- Let $c \in \mathscr{I}(V)$ and $V_c(t) = \ker(L_c - tI)$, then $V_c(1) = \mathbb{R}c$ and

$$V = \mathbb{R}c \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2})$$

- The **Peirce dimensions**

$$\boxed{n_1(c) = \dim V_c(-1), \quad n_2(c) = \dim V_c(-\tfrac{1}{2}), \quad n_3(c) = \dim V_c(\tfrac{1}{2})}$$

satisfy

$$\begin{aligned} n_3(c) &= 2n_1(c) + n_2(c) - 2 \\ 3n_1(c) + 2n_2(c) - 1 &= \dim V = n. \end{aligned}$$

In particular, any of $n_i(c)$ completely determines two others.

Examples.

- If V is a *polar algebra* then $(n_1(c), n_2(c)) = (\dim V_0 - 1,\ \frac{1}{2}\dim V_1 - \dim V_0 + 2)$.
- If V is a *pseudocomposition algebra* then $(n_1(c), n_2(c)) = (\frac{1+\dim V}{3}, 0)$.

The Peirce decomposition

Proposition 1

Setting $V_0 = V_c(1), \quad V_1 = V_c(-1), \quad V_2 = V_c(-\frac{1}{2}), \quad V_3 = V_c(\frac{1}{2})$ *we have*

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ *and* $V_0 \oplus V_2$ *are subalgebras of* V. *Notice however that these subalgebras may be Hsiang subalgebras or not.*

The cubic trace identity

Traces of (powers of) multiplication operators in an algebra is an important tool to study invariant properties. We already have $\operatorname{tr} L_x = 0$ for any $x \in V$. The following property provides an effective tool to determine the Peirce dimensions.

Theorem 2

Any normalized Hsiang algebra satisfies the cubic trace identity

$$\operatorname{tr} L_x^3 = (1 - n_1(c))\langle x, x^2\rangle, \qquad \forall c \in \mathscr{I}(V), x \in V. \tag{6}$$

In particular, the Peirce dimensions $(n_1(c), n_2(c))$ are similarity invariants of a general Hsiang algebra and do not depend on a particular choice of an idempotent c.

In what follows, we write $(n_1(V), n_2(V))$, or just (n_1, n_2).

A ‘rough’ classification of Hsiang algebras

Theorem 3 (A hidden Clifford algebra structure)

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1),$$

where ρ is the Hurwitz-Radon function.

Proof. One can prove that $A(x) = \sqrt{3}L_x - (1 + \sqrt{3})(L_x L_c + L_c L_x)$, $x \in V_1$ satisfies

$$A(x)^2 = |x|^2 \quad \text{on } V_2 \oplus V_3$$

which implies $A \in \mathrm{Cliff}(V_1, V_2 \oplus V_3)$ and the desired obstruction. □

Corollary

Given $n_2 \geq 0$, there are finitely many admissible Peirce dimensions (n_1, n_2).

A ‘rough’ classification of Hsiang algebras

Theorem 4 (A hidden Jordan algebra structure)

Given $c \in \mathscr{I}(V)$, let us define the new algebra structure on $\Lambda_c = (V_0 \oplus V_2, \bullet)$ with the multiplication

$$x \bullet y = \tfrac{1}{2}xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c. \tag{7}$$

Then Λ_c is a Euclidean Jordan algebra with unit $c^* = 2c$, the associative trace form $T(x; y) = \langle x, y\rangle$ and

$$\operatorname{rk} \Lambda_c = \min\{3, n_2(V) + 1\} \leq 3.$$

Idea of the **Proof**: to verify that the cubic form $N(x) = \frac{1}{6}\langle x, x^2\rangle$ on $V_0 \oplus V_2$ with a basepoint $c^* = 2c$ is Jordan for any $c \in \mathscr{I}(V)$ and apply the Springer-McCrimmon construction.

□

A ‘rough’ classification of Hsiang algebras

Theorem 5 (The dichotomy of Hsiang algebras)

The following conditions are equivalent:

1. *A Hsiang algebra V is exceptional*
2. *The Jordan algebra $V_c(1) \oplus V_c(-\frac{1}{2})$ is simple for some c*
3. *The Jordan algebra $V_c(1) \oplus V_c(-\frac{1}{2})$ is simple for all c*
4. *The quadratic form $x \to \operatorname{tr} L_x^2$ has a single eigenvalue and $n_2(V) \neq 2$*

A ‘rough’ classification of Hsiang algebras

Combining Theorem 3 and Theorem 5, one obtains

Corollary

There are at most 24 classes of exceptional Hsiang algebras. For any such an algebras $n_2 \in \{0, 5, 8, 14, 26\}$ and the possible corresponding Peirce dimensions are

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

The cells in blue color represent non-realizable Peirce dimensions and the cells in gold color represent unsettled cases

The above dimensions come from the possible solutions of the Hurwitz-Radon obstruction in Theorem 3 if $n_2 = 0, 5, 8, 14, 26$. The pink-color dimensions are not realizable (it follows form a finer, tetrad representation, see Example 2 above for en example of a tetrad, and Theorem 7 below).

A key question: *Which Peirce dimensions in the above table are indeed realizable*?

A ‘rough’ classification of Hsiang algebras: the existence

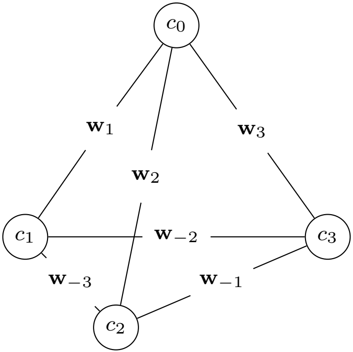
n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

White dimensions are realizable

- If $n_2 = 0$ then $n_2 \in \{2,5,8,14,26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$, $V = \mathscr{H}_3(\mathbb{K}_d) \ominus \mathbb{R}e$, $d = 0,1,2,4,8$.
- If $n_1 = 0$ then $n_2 \in \{5,8,14\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $\frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, where $z \to \bar{z}$ is the natural involution on $V = \mathscr{H}_3(\mathbb{K}_d)$, $d = 2,4,8$.
- If $n_1 = 1$ then $n_2 \in \{5,8,14,26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u(z) = \mathrm{Re}\langle z, z^2\rangle$, where $z \in V = \mathscr{H}_3(\mathbb{K}_d) \otimes \mathbb{C}$, $d = 1,2,4,8$.
- If $(n_1, n_2) = (4,5)$ then $V = V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$ on $\mathscr{H}_3(\mathbb{K}_8) \ominus \mathscr{H}_3(\mathbb{K}_1)$

Towards a finer classification: a tetrad decomposition

A quadruple of idempotents as in Example 2 on page 17 is called a **tetrad**, see picture below



Here w_i are 2-nilpotents. Remarkably, for each vertex c_i, the adjacent w_α are the *primitive idempotents* in the corresponding Jordan algebra $V_{c_i}(1) \oplus V_{c_i}(-\frac{1}{2})$ such that $2c_i$ is the Jordan algebra unit and $2c_i = \sum_{\text{adjacent}} w_\alpha$ is the Jordan frame.

Towards a finer classification: a tetrad decomposition

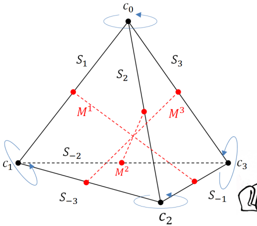
Theorem 6

$$V = S^1 \oplus S^2 \oplus S^3 \oplus M^1 \oplus M^2 \oplus M^3,$$

where $M^i := V_{w_i}(0) \cap V_{w_j}(0)^\perp \cap V_{w_k}(0)^\perp$ *and* $S^i := V_{w_i}(0)^\perp \cap V_{w_j}(0) \cap V_{w_k}(0)$. *If* V *is an* **exceptional** *Hsiang algebra with* $n_2 = 3d + 2$, $d \in \{1, 2, 4, 8\}$ *then*

- M^α *is a null-subalgebra,* $\dim M_\alpha = n_1 + 1$,
- $S^\alpha = S_\alpha \oplus S_{-\alpha}$, $\dim S_{\pm\alpha} = d$.
- *any 'vertex-adjacent' triple* $S_\alpha, S_\beta, S_\gamma$ *forms a triality:*

$$S_\alpha S_\beta = S_\gamma, \qquad |x_\alpha x_\beta|^2 = \tfrac{1}{2}|x_\alpha|^2 |x_\beta|^2,$$



Fusion rules of a tetrad

	E	S_1	S_2	S_3	S_{-3}	S_{-2}	S_{-1}	M_1	M_2	M_3
E	E	S_1	S_2	S_3	S_{-3}	S_{-2}	S_{-1}	$M_2 \oplus M_3$	$M_1 \oplus M_3$	$M_1 \oplus M_2$
S_1		$\mathbb{R}w_1$	S_3	S_2	S_{-2}	S_{-3}	M_1	$S_{-1} \oplus D_1$	M_1	M_1
S_2			$\mathbb{R}w_2$	S_1	S_{-1}	M_2	S_{-3}	M_2	$S_{-2} \oplus D_2$	M_2
S_3				$\mathbb{R}w_3$	M_3	S_{-1}	S_{-2}	M_3	M_3	$S_{-3} \oplus D_3$
S_{-3}					$\mathbb{R}w_3$	S_1	S_2	M_3	M_3	$S_3 \oplus D_{-3}$
S_{-2}						$\mathbb{R}w_2$	S_3	M_2	$S_2 \oplus D_{-2}$	M_2
S_{-1}							$\mathbb{R}w_1$	$S_1 \oplus D_{-1}$	M_1	M_1
M_1								0	$S^1 \oplus S^2 \oplus M^3$	$S^1 \oplus S^3 \oplus M^2$
M_2									0	$S^2 \oplus S^3 \oplus M^1$
M_3										0

Table:

Towards a finer classification: a tetrad decomposition

Define $T^\alpha := \mathrm{Span}[S^\alpha S^\alpha]$. Then

Theorem 7

- $T^\alpha \subset M^\alpha$
- $T^\alpha \cong T^\beta$
- T^α *admits a structure of a* commutative *real division algebra, in particular,* $\tau(V) := \dim T^\alpha \in \{1, 2\}$
- *If* $d > n_1$ *then* $\tau(V) = n_1$.
- *If* $n_1 \geq 1$ *and* $d \geq \rho(n_1) - 1$ *then* $\tau(V) = 1$.
- *If* $\tau(V) = 1$ *then* $n_1 \equiv 1 \mod 2$.
- *There is no exceptional Hsiang algebras with the blue Peirce dimensions.*

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
d	0	0	0	0	0	1	1	1	1	2	2	2	2	2	2	4	4	4	4	8	8	8	8	8

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THANK YOU FOR YOUR ATTENTION!