Permutations groups – Exercise sheet 1

Unless otherwise stated, G is a group which acts on a set Ω.

1. Let $\alpha, \beta \in \Omega$. Suppose $g \in G$, such that $\alpha g = \beta$. Show that $\beta g^{-1} = \alpha$.

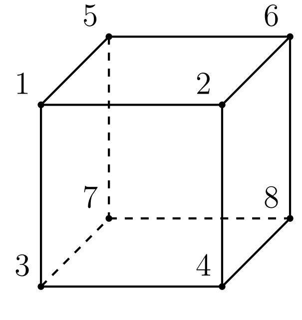
2. The right regular action of G on itself is defined by $\mu(x, g) = xg$, $g \in G$, $x \in \Omega = G$. How would you define the left regular action? Check carefully that it satisfies the definition.

3. Show that G acts on a finite set Ω of size $|\Omega| = n$ if and only if there exists a group homomorphism from G to S_n.

4. Let $H \leq G$ be a subgroup of G and consider the coset action of G on H. What is the kernel K of this action? Show that if $N \trianglelefteq G$ such that $N \leq H$, then N is contained in K. The kernel K is called the *core* of H in G, often denoted $\mathrm{core}_G(H)$.

5. Let G be a group with a subgroup H which has finite index n. Show that G has a normal subgroup of index at most $n!$. Use this to conclude that any subgroup of index 2 in a group is normal.

6. Let G be the group of symmetries of the cube.



   (a) By picking two rotations of the cube, show that G is transitive on Ω, the set of vertices of the cube.

   (b) What is the index of G_1 in G?

(c) Consider the group G_1 which is the stabiliser in G of 1. What are the orbits of G_1?

(d) What is the index of G_{12}, the stabiliser of 2 in G_1, in G_1?

(e) What is the order of G_{12}.

(f) Hence, show that $|G| = 48$.

(g) Show that G acts imprimitively by describing at least two non-trivial systems of imprimitivity.

7. By considering the action of $GL_2(3)$ on the points of $\mathbb{P}_1(3)$, show that $PGL_2(3) \cong S_4$.

8. Let G be a group of order $p^a \neq 1$ for p prime and for some $a \in \mathbb{N}$.

   (a) Show that
   $$Z(G) := \{g \in G : hg = gh \; \forall h \in G\} \neq 1$$
   (Hint: use conjugation action and argue by counting.)

   (b) Let H be a non-trivial subgroup of G. Show that $H \lneqq N_G(H)$. (Hint: use coset action.)

9. (Frattini argument) Let G be a group and $N \trianglelefteq G$. Suppose that $P \in Syl_p(N)$. Then, show that $G = N.N_G(P)$

   Fact: G acts transitively by conjugation on the set $Syl_p(G) := \{P \leq G : p \nmid |G|/|P|\}$. (Hint: Show that G also acts transitively on $Syl_p(N)$ then combine the two actions to get an element of $N_G(P)$.)

10. Let Ω be the set of all $n \times n$ matrices over a field F and $G = GL_n(F) \times GL_n(F)$.

    (a) Show that $\mu(\alpha, (x, y)) = x^t \alpha y$ defines an action of G on Ω, where x^t denotes the transpose of x .

    (b) Show that G has $n + 1$ orbits and describe these.

    (c) Choose a suitable $\alpha \in \Omega$ and describe G_α.

    (Hint: this is a well-known fact in linear algebra)

11. Show that G has a system of imprimitivity iff there is a G-congruence.

12. Show that a transitive group action of prime degree is primitive.

13. Check that the two definitions of the semidirect product are equivalent. For the first definition, check that it actually defines a group. What is the identity and what are inverses?