Permutations groups – Exercise sheet 2

Unless otherwise stated, in these exercises, G is a group which acts on a set Ω.

1. Show that a group G is k-transitive if and only if G is transitive and G_α is $(k-1)$-transitive on $\Omega - \{\alpha\}$, for any $\alpha \in \Omega$.

2. (a) Show that S_n is n-transitive.
   (b) Show that A_n is $(n-2)$-transitive, provided $n \geq 3$.

3. Show that if G is 2-transitive, then it is primitive.

4. Let $G = PGL_2(q)$. We may view $\mathbb{P}_1$ as being $\mathbb{F}_q \cup \{\infty\}$, where

$$\langle (x, y) \rangle \mapsto \begin{cases} x/y & \text{if } y \neq 0 \\ \infty & \text{if } y = 0 \end{cases}$$

Then, $PGL_2(q)$ acts by so called fractional linear transformations:

$$z \mapsto \frac{az + c}{bz + d} = \frac{a + c/z}{b + d/z}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(q)$ and the above should be interpreted where ∞ obeys the usual rules: $1/0 = \infty$, $1/\infty = 0$, $\infty + a = \infty$ and, if $a \neq 0$, $\infty a = \infty$. We use either the first or the second form to avoid meaningless expressions such as ∞/∞.

   (a) Show that the kernel of the action is indeed the scalar matrices.
   (b) Show that $PGL_2(q)$ acts sharply 3-transitively on $\mathbb{F}_q \cup \{\infty\}$.
   (c) Conclude that $PSL_2(q)$ acts 3-transitively if $q = 2^a$ is a power of 2 and 2-transitively otherwise.

You may use that $|GL_2(q)| = q(q^2-1)(q-1)$ and $|PGL_2(q)| = q(q^2-1)$.

5. Let $n \geq 2$. Show that

   (a) $AGL(V)$ is 2-transitive,

   (b) $AGL_n(q)$ is 3-transitive if and only if $q = 2$,

   (c) $AGL_n(q)$ is 4-transitive only if $q = n = 2$.

   Can you say what is $AGL_n(q)$ isomorphic to when $q = n = 2$?

6. Let G be a sharply 4-transitive group of degree $p + 2$, where p is an odd prime. Suppose that S is a subgroup of G which is generated by a p-cycle.

   (a) By considering how such a subgroup acts on Ω, deduce that $N_G(S)$ has order $2p(p-1)$. (You may use without proof that G acts transitively by conjugation on the set of all such subgroups S in G and that groups H of order $p(p-1)$ have exactly one subgroup of order p and this is normal in H.)

   (b) Name a subgroup of G which $N_G(S)$ must contain.

   (c) Consider the conjugation action of $N_G(S)$ on non-trivial elements of S and deduce that the centraliser, $C_G(S) \leq N_G(S)$, contains a transposition. (Hint: If $s = (1a_2 \ldots a_p)$ is a p-cycle, then where 1 is mapped determines the power of s and vice versa)

   (d) Conclude that G is the full symmetric group and hence that there are no sharply 4-transitive groups of degree 7, or 9.

7. Prove Lemma 7.2: Let G act on Ω and $\Sigma \subseteq \Omega$. Show that the following are all equivalent:

   (1) Σ is a base for G.

   (2) Σg is a base for G, for all $g \in G$.

   (3) For all $g, h \in G$, if $\alpha g = \alpha h$ for all $\alpha \in \Sigma$, then $g = h$.

   (4) $\Sigma \cap \operatorname{supp}(g) \neq \emptyset$, for all $1 \neq g \in G$.

8. Show that the smallest base for $AGL_n(F)$ has size $n + 1$.

9. Let G be a primitive group acting on Ω. Suppose $g \in G$ such that $|\operatorname{supp}(g)| = m$ and g has exactly s non-trivial disjoint cycles. Show that, for any $\alpha \in G$, the maximal number of orbits of G_α on Ω is $m-s+1$. [Hint: Show one may pick $\alpha \in \operatorname{supp}(g)$ and then $G = \langle g, G_\alpha \rangle$.]

   Show that this is not necessarily true for transitive groups.