For $\overline{\mathbb{R}^n}$, define

$$[x_1, x_2, x_3, \infty] = \frac{d(x_1, x_3)}{d(x_1, x_2)}.$$

<u>Prop 4.3</u>

$f: \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ Möb

$\Longleftrightarrow$ f preserves cross-ratios.

<u>Proof</u> $\Rightarrow$ reflections ✓

inversions in $S(a,r)$

σ

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$$[\sigma(x_1), \ldots, \sigma(x_4)]$$

$$= \frac{|\sigma(x_1) - \sigma(x_3)|\,|\sigma(x_2) - \sigma(x_4)|}{|\sigma(x_1) - \sigma(x_4)|\,|\sigma(x_2) - \sigma(x_3)|}$$

$$= \frac{r^4 |x_1 - x_3|\,|x_2 - x_4| / (|x_1 - a| \cdots |x_4 - a|)}{r^4 |x_1 - x_4|\,|x_2 - x_3| / (|x_1 - a| \cdots |x_4 - a|)}$$

$$= [x_1, x_2, x_3, x_4].$$

$\Leftarrow$ WLOG $f(\infty) = \infty$

$$[x, y, z, \infty] = [f(x), f(y), f(z), \infty]$$

$$\Rightarrow \frac{|x - y|}{|x - z|} = \frac{|f(x) - f(y)|}{|f(x) - f(z)|}.$$

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Also $\dfrac{|x-z|}{|u-z|} = \dfrac{|f(x)-f(z)|}{|f(u)-f(z)|}$

$\Rightarrow \dfrac{|x-y|}{|u-z|} = \dfrac{|f(x)-f(y)|}{|f(u)-f(z)|}$

$\Rightarrow |x-y| = |f(x)-f(y)| \cdot \left(\dfrac{|u-z|}{|f(u)-f(z)|}\right)$

$\Rightarrow f$ similarity

$\Rightarrow f$ Möb. $\square$

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Prop 4.4

Möbius transformations send generalised spheres to generalised spheres.

Proof reflections ✓

dilatations, translations ✓

Check σ inversion in $S(0,1)$.

What does it do to $S(a,r)$?

$|x-a|^2 = r^2$

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$$r^2 = |\sigma^{-1}(x) - a|^2$$

$$\sigma(x) = \frac{x}{|x|^2}$$

$$r^2 = \frac{1}{|x|^2} - \frac{2a\cdot x}{|x|^2} + |a|^2$$

$$1 - 2a\cdot x + (a\cdot a - r^2)\,x\cdot x = 0$$

If $a\cdot a - r^2 = 0$, image is plane

If not, image sphere:

$$0 = (a\cdot a - r^2)\left|x - \frac{a}{a\cdot a - r^2}\right|^2 + 1 - \frac{a\cdot a}{a\cdot a - r^2} \quad \square$$

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Def 4.5 A $n\times n$ matrix is conformal if

- A preserves angles

$\Leftrightarrow$ • $A = \lambda Q$, $Q \in O(n)$, $\lambda \neq 0$

$\Leftrightarrow$ • $\|A\|^n = |\det A|$

where $\|A\| = \sup_{|x|=1} |Ax|$.

A C^1 map $f: U \to V$, $U, V \subset \mathbb{R}^n$ is conformal if Df is conformal ~~everywhere~~ except at finitely many points.

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Lemma 4.6 Let $\mathrm{Conf}(\overline{\mathbb{R}^n})$ be the group of conformal homeos of $\overline{\mathbb{R}^n}$. Then $\text{Möb}(\overline{\mathbb{R}^n}) \leq \mathrm{Conf}(\overline{\mathbb{R}^n})$

Proof Suffices to check $\sigma(x) = \frac{x}{|x|^2}$.

$$\frac{\partial}{\partial x_j}\left(\frac{1}{|x|^2} x_i\right) = \frac{-2x_j}{|x|^4} x_i + \frac{1}{|x|^2}\delta_{ij}$$

$$\Rightarrow D_x\sigma = \frac{1}{|x|^2}\underbrace{\left(I - \frac{2xx^t}{|x|^2}\right)}_{\text{orthogonal}}$$

$\square$

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Liouville's Theorem 4.7

$f: U \to V$ conformal, C^3

$U, V \subset \mathbb{R}^n$, $n \geq 3$

$\Rightarrow f$ is a restriction of a Möbius transformation.

Remark Very false for $n=2$.

Def 4.8 $\text{Möb}(\mathbb{B}^n) = \{f \in \text{Möb}(\overline{\mathbb{R}^n})$ preserving $\mathbb{B}^n = \{x : |x| < 1\}\}$;

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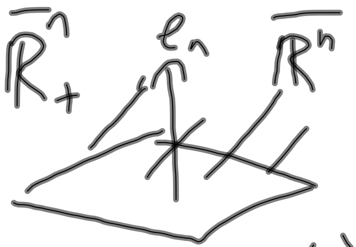
• $\text{Möb}(\overline{\mathbb{R}^n_+}) = \dots$

$\dots$



• $\text{Möb}(\mathbb{S}^{n-1}) = \{ f \in \text{Homeo}(\mathbb{S}^{n-1})$ preserving crossratios $\}$

Theorem 4.9

$\text{Isom}(\mathbb{H}^n) = \text{Möb}(\mathbb{B}^n)$
$= \text{Möb}(\mathbb{S}^{n-1})$.

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Proof ① $\text{Möb}(\mathbb{S}^{n-1})$
$= \text{Möb}(\overline{\mathbb{R}^{n-1}})$

Stereographic projection: invert in $S(e_n, \sqrt{2})$, then reflect in $x_n = 0$

Note: sends $\mathbb{B}^n \longrightarrow \overline{\mathbb{R}^n_+}$.

Apply Prop. 4.3.

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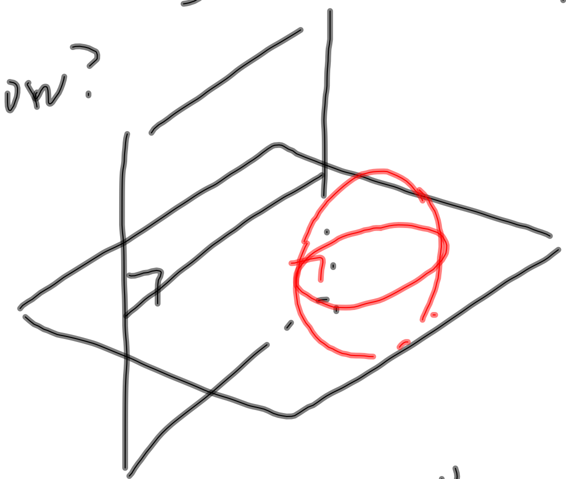
② $\text{Möb}(\mathbb{S}^{n-1}) = \text{Möb}(\mathbb{B}^n)$

Suffices to extend $f \in \text{Möb}(\overline{\mathbb{R}^{n-1}})$ to something in $\text{Möb}(\overline{\mathbb{R}^n_+})$.

How?



Extend orthogonally.

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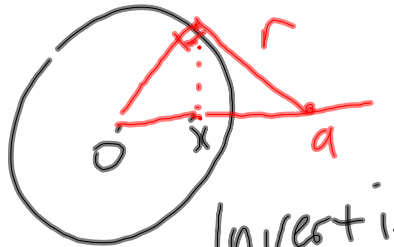
③ So $\text{Möb}(\mathbb{B}^n)$, $\text{Möb}(\mathbb{S}^{n-1})$ are generated by reflections in generalised spheres $\perp$ to $\mathbb{S}^{n-1}$.

④ $\text{Möb}(\mathbb{B}^n) \curvearrowright \mathbb{B}^n$ transitively with $O(n)$ stabilizer.

<u>Proof</u> Send x to 0 by



Invert in $S(a,r)$

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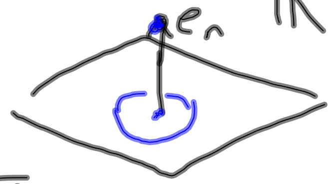
Stabilizer of 0 ?

$= \mathrm{Stab}(e_n)$ in $\mathrm{M\ddot{o}b}(\overline{\mathbb{R}}^n_+)$

WLOG $f(\infty)=\infty$



f Möb of $\overline{\mathbb{R}}^n_+$

Fixing ∞. and fixes e_n.

So similarity fixing e_n

So $f \in O(n-1)$.

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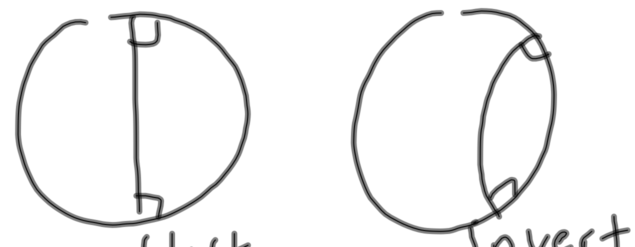
(5)

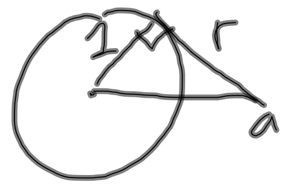


are in $\mathrm{Isom}(\mathbb{H}^n)$

Proof. reflection ✓

· For inversion in $S(a,r)$

$r^2 = |a|^2 - 1$.



Suffices

$$\frac{1-|x|^2}{2}\,\|D_x\sigma\|\,\frac{2}{1-|y|^2}=1$$

for $y=\sigma(x)$.

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In Lemma 4.6, essentially showed

$$D_x\sigma = \frac{r^2}{|x-a|^2}\left[I - \frac{2}{|x-a|^2}(x-a)(x-a)^t\right]$$

$$\|D_x\sigma\| = \frac{r^2}{|x-a|^2} = \frac{|a|^2-1}{|x-a|^2}.$$

Check:

$$(1-|y|^2)\left(\frac{|x-a|^2}{|a|^2-1}\right) = 1-|x|^2.$$

by substituting

$$y=\sigma(x) = a + \frac{|a|^2-1}{|x-a|^2}(x-a).$$

(6) $\text{Möb}(\mathbb{B}^n) = \text{Isom}(\mathbb{H}^n)$

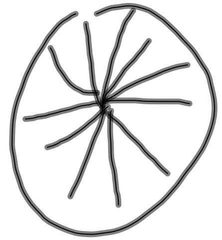
For $\supseteq$, for $f \in \text{Isom}(\mathbb{H}^n)$

$D_0 f \in O(n)$, so can find $\varphi \in \text{Möb}(\mathbb{B}^n)$ s.t.

$\varphi^{-1}\circ f$ fixes 0, has

$D_0(\varphi^{-1}\circ f) = Id$

$\Rightarrow$ fixes geodesics



$\Rightarrow \varphi^{-1}\circ f = Id.$

$\square$.

5. Quasi-conformal & Quasi-Möbius homeomorphisms

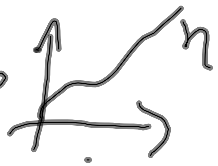
Recall Def 4.2:

$$[x_1, x_2, x_3, x_4] = \frac{d(x_1,x_3)\,d(x_2,x_4)}{d(x_1,x_4)\,d(x_2,x_3)}$$

Def 5.1 (Väisälä ~81)

$f: X \to X'$ homeo is η-quasi-Möbius (η-qM)

where $\eta: [0,\infty) \circlearrowleft$ homeo



called a distortion function.

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If for any 4 distinct points $x_1, .., x_4$ (with images $x_1', .. x_4'$) we have

$$[x_1', x_2', x_3', x_4'] \leq \eta([x_1, .., x_4])$$

Ex • $f: S^n \to S^n$ Möbius is η-qM with $\eta(t) = t$.

• $f: X \to X'$ is L-bi Lipschitz, it is η-qM with $\eta(t) = L^4 t$.

• $Id: (X,d) \to (X, d^\varepsilon)$ for $\varepsilon \in (0,1]$ is η-qM, $\eta(t) = t^\varepsilon$.

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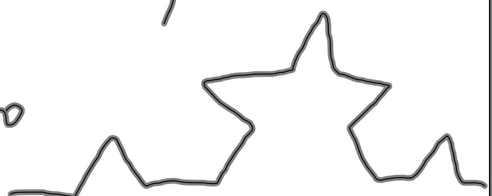
Note: (X, d^{ε}), $\varepsilon \in (0,1]$
is a metric space by
Minkowski's inequality.

This is called a <u>snowflake transformation</u> because
$\left([0,1], d^{\log 3/\log 4}\right)$
is bi-Lip to



• Remember $(\partial_\infty X, \rho_\varepsilon) \to (\partial_\infty X, \rho_{\varepsilon'})$
can check this is a η-qM
for $\eta(t) = C t^{\varepsilon'/\varepsilon}$.

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So a geodesic proper
δ-hyp. space X has
all visual metrics on $\partial_\infty X$
qM-equivalent.

<u>Prop 5.2</u> f η-qM
$\Rightarrow f^{-1}$ is $\tilde{\eta}$-qM
where $\tilde{\eta}(t) = \dfrac{1}{\eta^{-1}\left(\frac{1}{t}\right)}$.

<u>Proof</u> $[x_1, x_2, x_3, x_4]^{-1}$
$= [x_2, x_1, x_3, x_4]$

$[x_1', x_2', x_3', x_4'] \le \eta([x_1, x_2, x_3, x_4])$

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$\Longleftrightarrow$

$$[x_2, x_1, x_3, x_4] \leq \tilde{\eta}([x_2', x_1', x_3', x_4'])$$

$\square$

Def 5.3 $f: X \to X'$ homeo is K-<u>quasiconformal</u> $(K \geq 1)$ if $\forall x \in X$, $\limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \leq K$.

$$L_f(x,r) = \sup\{d(y', x') : d(y,x) \leq r\}$$
$$l_f(x,r) = \inf\{d(y', x') : d(y,x) \geq r\}.$$

In otherwords, infinitesimal balls get mapped with controlled distortion.

ratio $\leq K$.

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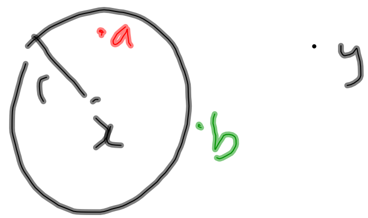
Ex $f: \mathbb{R}^n \to \mathbb{R}^n$ conformal is 1-quasi-conformal.

Prop 5.4 $f: X \to X'$ η-qM $\Rightarrow f$ $\eta(1)$-qc.

Proof Take $x \in X$ non-isolated. Fix $y \neq x$, $0 < r < \min\{1, d(x,y)\}$.



Choose $a, b \in X$ so that $a \in B(x,r)$, $b \notin B(x,r)$

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$$d(a',x') \geq (1-r)L_f(x,r)$$
$$d(b',x') \leq (1+r)\,l_f(x,r)$$

Then

$$[a,b,x,y] = \frac{d(a,x)\,d(b,y)}{d(a,y)\,d(b,x)}$$

$$\leq \frac{d(b,y)}{d(a,y)} \xrightarrow{r\to 0} 1$$

$$\frac{L_f(x,r)}{l_f(x,r)} \leq \underbrace{\boxed{\frac{(1+r)}{(1-r)}}}_{\to 1}\;\boxed{\frac{d(a',x')}{d(b',x')}\frac{d(b',y')}{d(a',y')}}\cdot\underbrace{\boxed{\frac{d(a',y')}{d(b',y')}}}_{\to 1}$$

$$\leq \eta(\underbrace{[a,b,x,y]}_{\to 1}) \to \eta(1)$$

$\square$

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Remark

$$\begin{array}{ccc} \text{Möb} & \Longrightarrow & qM \\ \Downarrow\Uparrow & & \Downarrow\Uparrow \\ \text{Conf} & \Longrightarrow & qC \end{array}$$

This is true for $S^n, n \geq 2$, see below.

This is true for nice spaces (have "calculus") e.g. $\mathbb{R}^n$, Heisenberg groups with Carnot-Carathéodory metrics, certain boundaries of certain buildings.

(Heinonen-Koskela, Pansu...)

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Normalisation

Prop 5.5 X, X' bounded

$f: X \to X'$ η-qM.

$\forall \lambda > 0$ $\exists \mu = \mu(\eta, \lambda)$

distortion function so that

if $\otimes$ $\exists z_1, z_2, z_3 \in X$

s.t. for $i \neq j$ $d(z_i, z_j) \geq \frac{\operatorname{diam} X}{\lambda}$

and $d(z_i', z_j') \geq \frac{\operatorname{diam} X'}{\lambda}$,

then $\forall x, y \in X$

$$d(x', y') \leq \operatorname{diam}(X') \mu\left(\frac{d(x,y)}{\operatorname{diam} X}\right).$$

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(i.e. gives equicontinuity control.)

Proof WLOG $\operatorname{diam}(X) = \operatorname{diam}(X') = \lambda$.

Take $x' = f(x)$, $y' = f(y)$.

Case 1 $d(x, z_1) \leq \frac{1}{2}$, $d(y, x) \leq \frac{1}{4}$

$$\tau_1 := [y, z_2, x, z_3] = \frac{d(y,x)\, d(z_2, z_3)}{d(y, z_3)\, d(z_2, x)} < \frac{d(x,y)\lambda}{\frac{1}{4} \cdot \frac{1}{2}}$$

$$\tau_1' \geq \frac{d(y', x') \cdot 1}{\lambda \cdot \lambda}$$

so $d(x', y') \leq \lambda^2 \eta\left(8 \lambda\, d(x,y)\right)$.

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Case 2 $d(x,y) \geq \frac{1}{4} \Rightarrow$

$d(x',y') < \lambda$

$\leq 4\lambda\, d(x,y)$.

Case 3 $d(x,y) < \frac{1}{4}, d(x,z_1) \geq \frac{1}{2}$.

Take $z \in \{z_2, z_3\}$ so that $d(y,z) \geq \frac{1}{2}$.

$\tau_2 = [x, z, y, z_1] \leq \dfrac{d(x,y)\lambda}{\frac{1}{2}\cdot\frac{1}{2}}$

$\tau_2' \geq \dfrac{d(x',y')\cdot 1}{\lambda \quad \lambda}$

So $d(x',y') \leq \lambda^2 \eta(4\lambda d(x,y))$

$\square$.

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Cor 5.6 $\mathcal{F} = \{f_i : X \to X'\}$

family of η-qM homeo, satisfying $\circledast$ for some $\lambda > 0$ fixed (usually with same three points)

then any seq in $\mathcal{F}$ has a uniformly convergent subsequence, limit also a η-qM homeo.

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Prop 5.7 If

$\mathcal{F} = \{ f_i : \mathbb{S}^n \to \mathbb{S}^n \}$

K-qc homeos with

$\circledast$ holding for fixed $\lambda > 0$,

fixed z_1, z_2, z_3.

Same conclusion holds.

~~Proof~~: see Väisälä

"Lectures on n-dimensional

Quasiconformal mappings" '71.

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