Theorem 5.8

(Liouville for C^3 homeos, 18??, Gehring $n=3$ '62, Reshetnyak $n\geq 4$ '67, Mostow '67; Tukia-Väisälä '85)

$f : \mathbb{S}^n \to \mathbb{S}^n$ 1-qc $n \geq 2$ then $f \in \text{Möb}(\mathbb{S}^n)$.

Remark Suffices $f : \mathbb{R}^n \to \mathbb{R}^n$ 1-qc $\Rightarrow f$ similarity.

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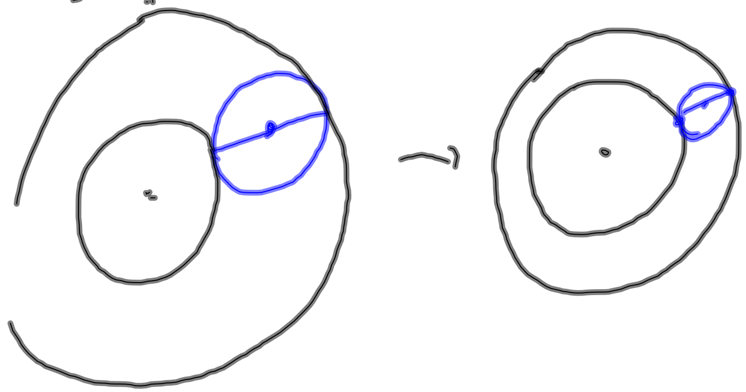
Lemma 5.9

$f : \mathbb{R}^n \to \mathbb{R}^n$ homeo that sends spheres to spheres preserving their centres. Then f is a similarity.

Proof

- f preserves midpoints



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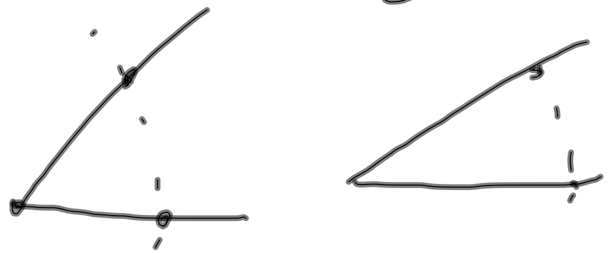
So f is a similarity on every line.

Must stretch distances on all lines by same amount



So $\exists \lambda > 0$ so that $\forall x, y \in \mathbb{R}^n$

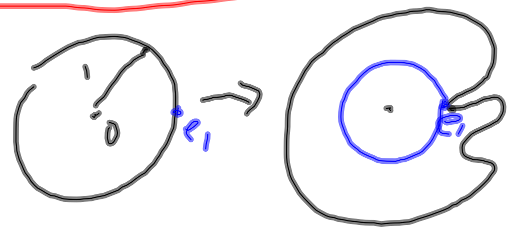
$$\lambda |x - y| = |f(x) - f(y)|$$

$\square$.

Proof of Theorem 5.8

Want to show spheres go to spheres.

WLOG $x = 0$, $f(x) = 0$, $r = 1$, $f(e_1) = e_1$, $\mathbb{B}^n \subset f(\mathbb{B}^n)$



We only need show $f(\mathbb{B}^n) = \mathbb{B}^n$.

Let $\mathcal{F} = \{h : \mathbb{R}^n \to \mathbb{R}^n$ 1-qc with $\star$ $\}$.

Then F forms a Semi-group under composition.
Also F is non-empty, equicontinuous (fixes $0, e_1, \infty$; apply Prop. 5.7), pointwise bounded (in spherical metric). So any sequence in F has a convergent subsequence.

$\exists h \in F$ s.t.
$$\mathrm{vol}(h(\mathbb{B}^n)) = \max\{\mathrm{vol}(g\mathbb{B}^n) : g \in F\}.$$
$$= M < \infty.$$

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If $h(\mathbb{B}^n) \supsetneq \mathbb{B}^n$,
then $h(h(\mathbb{B}^n)) \supsetneq h(\mathbb{B}^n) \supsetneq \mathbb{B}^n$

$h^2 \in F$, but
$\mathrm{vol}(h^2(\mathbb{B}^n)) > M$ ※ $\square$

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6. Gromov hyp. groups and QI $\longleftrightarrow$ QM

Theorem 6.1 (Efremovich-Tihomirova, Mostow, Margulis...)

X, X' Gr. hyp, proper, geodesic metric spaces. $f: X \to X'$ (λ, c)-q.i. embedding. Then f extends to $\partial_\infty f: \partial_\infty X \hookrightarrow \partial_\infty X'$ which is η-qM embedding,

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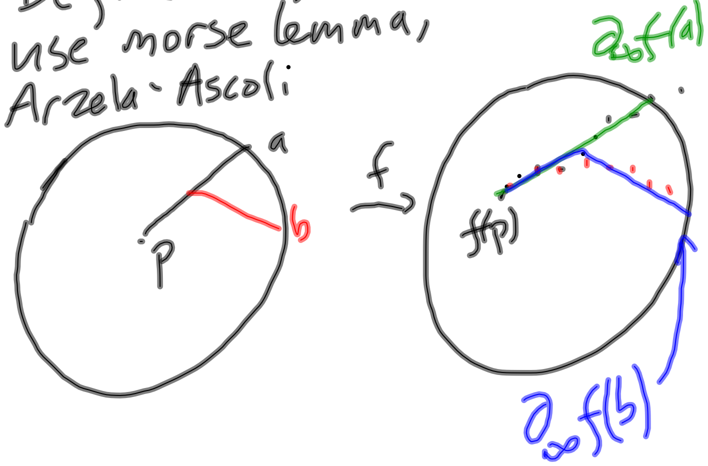
(η depends on λ, C, δ_X, $\delta_{X'}$, visual metric constants, ε/ε'. Does not depend on basepoints)

Proof

① Define $\partial_\infty f$
use morse lemma, Arzela-Ascoli



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② This is well defined

③ Use tree approximations to see that

$$(\partial_\infty f(a) \cdot \partial_\infty f(b))_{f(p)}$$

$$\leq \lambda (a \cdot b)_p + C$$

$$\geq \frac{1}{\lambda} (a \cdot b)_p - C$$

So $\partial_\infty f$ is top. embedding

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④ To show η-qM,

suffices to prove for "metrics"

$d(a,b) = e^{-(a \cdot b)_p}$ on $\partial_\infty X$

$d(a',b') = e^{-(a' \cdot b')_q}$ on $\partial_\infty X'$

So suppose $a,b,c,d \in \partial_\infty X$ are distinct.

Let

$\langle a,b,c,d \rangle = (a \cdot c)_p + (b \cdot d)_p - (a \cdot d)_p - (b \cdot c)_p$

"cross-distance"

Note: $[a,b,c,d] = e^{-\langle a,b,c,d \rangle}$.

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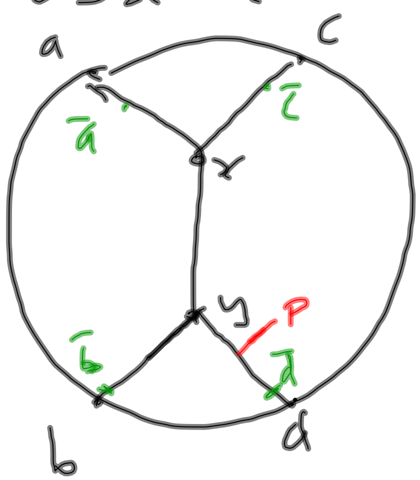
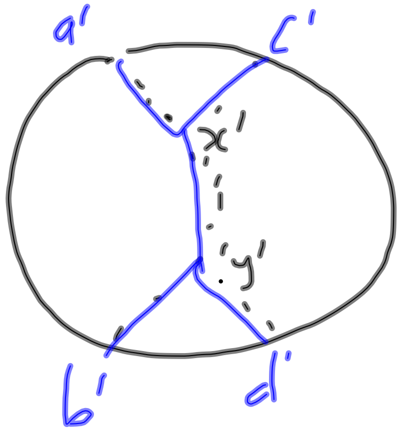
(Let $a' = \partial_\infty f(a)$, etc.)

(5) Observe



$$\langle a,b,c,d\rangle \approx \frac{1}{2}\left(d(\bar{a},\bar{d})+d(\bar{b},\bar{c}) - d(\bar{a},\bar{c})-d(\bar{b},\bar{d})\right)$$

$$\approx d(x,y)$$



Similarly,

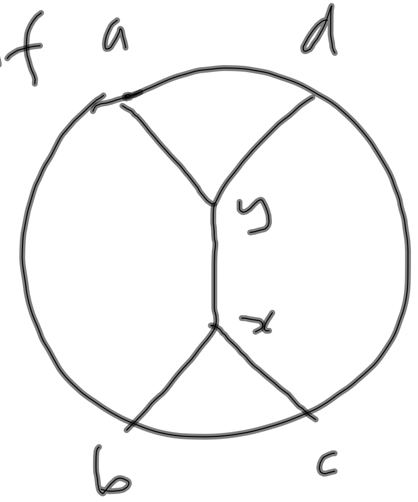
$$\langle a',b',c',d'\rangle \approx d(x',y')$$

$$\gtrsim \frac{1}{\lambda} d(x,y)$$

$$\Rightarrow [a',b',c',d'] \lesssim e^{-\frac{1}{\lambda}\langle a,b,c,d\rangle}$$

$$= [a,b,c,d]^{\frac{1}{\lambda}}.$$

(6) If



$$\langle a,b,c,d\rangle \approx -d(x,y)$$

and

$$\langle a',b',c',d'\rangle \approx -d(x',y') \gtrsim -\lambda d(x,y)$$

$$\Rightarrow [a',b',c',d'] \lesssim e^{-\langle a',b',c',d'\rangle} \lesssim [a,b,c,d]^{\lambda}$$

$\square$

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Cor 6.2

(a) G f.g. δ-hyp.
$\Rightarrow \partial_\infty G$ has canonical metric up to qM.

(b) $G \curvearrowright X$; X proper, geod, δ-hyp. Action proper, cocompact, by isometries. Then G f.g. δ-hyp. and $\partial_\infty G \overset{qm}{\cong} \partial_\infty X$.

Proof (a) Lem 2.2, Thm 6.1
(b) Thm 2.4, Thm 6.1 $\square$.

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Cor 6.3

Isometric action $G \curvearrowright \Gamma(G,S)$ induces uniformly qM action on $\partial_\infty G$.

Proof In Thm 6.1, η is independent of base point.

Some examples

ⓐ G finite. $\partial_\infty G = \emptyset$

ⓑ $G = \mathbb{Z} \Rightarrow \partial_\infty G = \{\pm\infty\}$.

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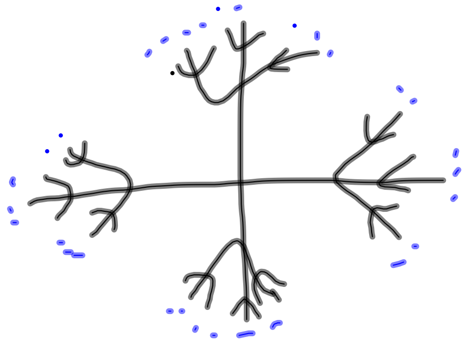
ⓒ $G = \pi_1\left(\mathbb{H}^n / \Gamma \right)$ closed hyp. manifold

$\partial_\infty G \overset{qM}{\cong} S^{n-1}$

ⓓ $G = F_2$

$\partial_\infty G$



Cantor set

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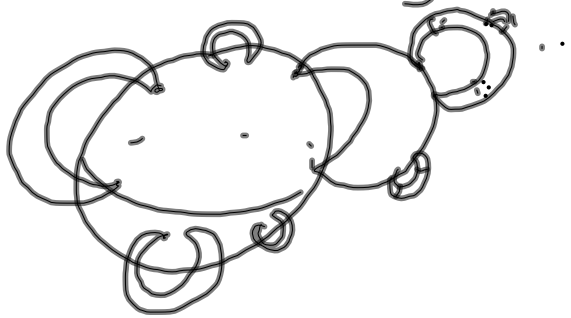
(e) $G = \pi_1\Big($ M^3 hyp  $\sqcup$ M^2 $\Big)$

$X =$ scc. geod glue

Rough idea:

$\tilde{X}$ is a tree of $\mathbb{H}^3$'s glued to $\mathbb{H}^2$'s along $\mathbb{R}$'s.

$\partial_\infty \tilde{X}$:



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Lemma 6.4 $H \underset{f.i.}{<} G$ f.g.

Then $H \overset{q.i.}{\simeq} G$, H f.g.

Proof $H \curvearrowright \Gamma(G,S)$. $\square$

Prop 6.5 G δ-hyp, $|\partial_\infty G| = 2$

$\Longleftrightarrow$ G virtually $\mathbb{Z}$.

Proof $\Leftarrow$ ✓

$\Rightarrow$ $\exists$ bi-infinite geodesic, γ and $d_H(\Gamma(G,S), \gamma) < 2\delta$

Consider $\langle g \rangle$, where g fixes $\partial_\infty G$, translates e by 10δ.

$\square$

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Theorem 6.6

$$|\partial_\infty G| = \begin{cases} 0 & \leftrightarrow G \text{ finite} \\ 2 & \leftrightarrow G \text{ v.} \mathbb{Z} \\ 2^{\aleph_0} & \end{cases}$$

If G finite or v.$\mathbb{Z}$ say G is elementary.

If G is non-elementary, $\partial_\infty G$ is uniformly perfect.

Def 6.7 A metric space is uniformly perfect if $\exists C < \infty$ s.t. $\forall x \in X$, $0 < R \leq \operatorname{diam} X$,

$$B(x, R) \setminus B(x, \tfrac{R}{C}) \neq \emptyset.$$

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Proof ① G infinite $\Rightarrow$ $\exists$ bi-infinite geodesic (L3.5) $\Rightarrow |\partial_\infty G| \geq 2$.

② $|\partial_\infty G| = 2 \Rightarrow G$ v. $\mathbb{Z}$.

③ $|\partial_\infty G| \geq 3 \Rightarrow \partial_\infty G$ is uniformly perfect (for fixed visual metric).

④ Given ③, $|\partial_\infty G| \geq 2^{\aleph_0}$

(build Cantor set)

$\partial_\infty G$ compact metric space $\Rightarrow |\partial_\infty G| \leq 2^{\aleph_0}$,

so $|\partial_\infty G| = 2^{\aleph_0}$.

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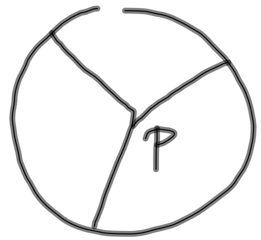
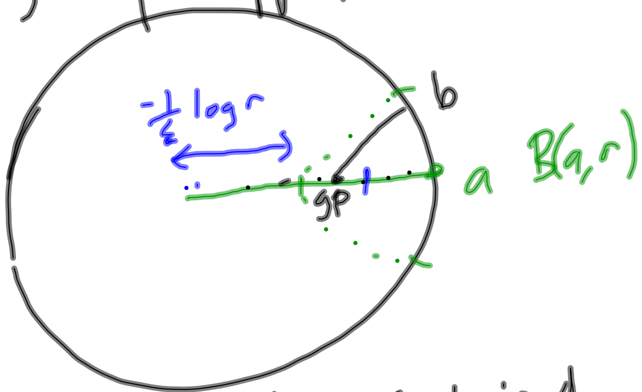
(5) Show (3):



$\exists$ ideal tripod

Given $a \in \partial_\infty G$, $r < \operatorname{diam} \partial_\infty G$:

shift p appropriately



One of three legs of tripod centered at gp goes to be $B(a,r) \setminus B(a, \frac{r}{C})$

some $C < \infty$ $\square$

Connected components of $\partial_\infty G$.

<u>Def 6.8</u> · X proper geodesic

$$\mathrm{Ends}(X) = \{\gamma : [0,\infty) \to X \text{ geodesic}, \gamma(0) = p\} / \sim'$$

p basepoint.

- $\gamma \sim' \gamma' \Leftrightarrow \gamma \sim'_r \gamma' \ \forall r > 0$
- $\gamma \sim'_r \gamma' \Leftrightarrow \gamma((r,\infty))$ and $\gamma'((r,\infty))$ are in same path component of $X \setminus B(p,r)$.
- $\gamma_n \to \gamma$ in $\mathrm{Ends}(X) \Leftrightarrow \forall r\ \exists N$ s.t. $\forall n \geq N$, $\gamma_n \sim'_r \gamma$.

- $\mathrm{Ends}(G) := \mathrm{Ends}(\Gamma(G,S))$ for G f.g.

Lemma 6.9 $G \overset{q.i.}{\simeq} H$ f.g.

$\Rightarrow \mathrm{Ends}(G) \overset{\text{homeo}}{\cong} \mathrm{Ends}(H)$

Proof Exercise, or see Bridson Haefliger.

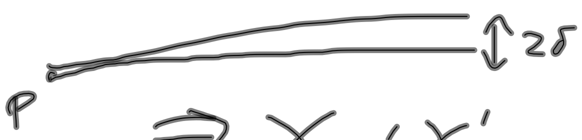
Prop 6.10 G δ-hyp.

$\pi : \partial_\infty G \to \mathrm{Ends}(G)$

is a continuous surjection and the preimages of points in $\mathrm{Ends}(G)$ are exactly the conn. components of $\partial_\infty G$.

Proof

- Well defined: $\gamma \sim \gamma'$

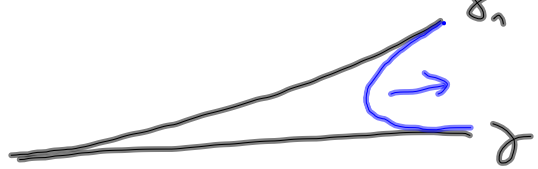


$\Rightarrow \gamma \sim' \gamma'$.

- continuous:

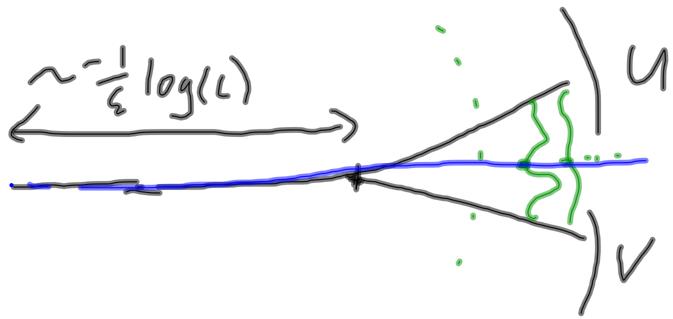
$\gamma_n \to \gamma$ in $\partial_\infty G$

$\Rightarrow \gamma_n \to \gamma$ in $\mathrm{Ends}(G)$



- $\mathrm{Ends}(G)$ is totally disconnected so just need to show $\pi^{-1}(a)$ is connected $\forall a \in \mathrm{Ends}(G)$.

If not, find disjoint closed $U, V \subset \partial_\infty G$ that cover $\pi^{-1}(\alpha)$ and are separated by some distance $c > 0$.



These geodesic rays are in the same end α; must split as in picture. Use Arzela-Ascoli to get contradiction: blue geodesic ray.

$\square$.

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Theorem 6.11 (Hopf, Stallings)

G f.g

$$|\mathrm{Ends}(G)| = \begin{cases} 0 & G \text{ finite} \\ 1 & ? \\ 2 & G \text{ v. } \mathbb{Z} \\ 2^{\aleph_0} & G \text{ splits over finite subgroups} \end{cases}$$

Theorem 6.12 (Dunwoody)

Every f. presented G has a graph of groups decomposition with finite edge groups, finite or one-ended vertex groups. (e.g. G hyp.)

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<u>Remark</u> So often assume
$\partial_\infty G$ is connected.

<u>Cor 6.13</u>
G f.g. group
G v. free
$\iff$ G hyp $\partial_\infty G$ totally disconn.
$\iff$ G hyp $\dim_{top} \partial_\infty G = 0$.

<u>Proof</u> We'll skip writing down details.

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