Converse to Theorem 6.1:

Theorem 6.14 (Paulin ~'95, see also Bonk-Schramm)

Suppose X, X' Gr. hyp. proper, geodesic metric spaces admitting cocompact isom. actions. Suppose $h: \partial_\infty X \to \partial_\infty X'$ is an η-qM homeo. Then $\exists\ \Phi(h): X \to X'$ (λ, C)-q.i. s.t. $\partial_\infty \Phi(h) = h$, where λ, C depend only on relevant data

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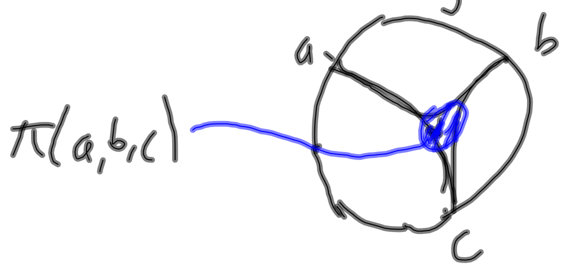
Cor. 6.15 G, H Gr. hyp groups, $\partial_\infty G \overset{qM}{\cong} \partial_\infty H$
$\Rightarrow G \overset{q.i.}{\cong} H$.

Proof of 6.14

① Let $\partial^3_\infty X = \{(a,b,c) \in (\partial_\infty X)^3 :$ a,b,c distinct$\}$.

Define $\pi: \partial^3_\infty X \to X$ by

$(a,b,c) \longmapsto$ center of ideal triangle



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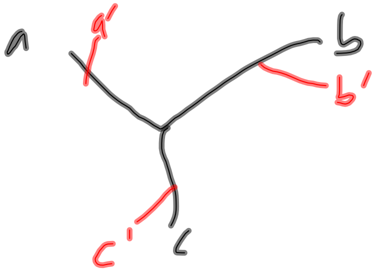
Note: π "quasi-continuous":

if (a',b',c') sufficiently close to (a,b,c), then

$$d(\pi(a',b',c'),\pi(a,b,c)) \leq C$$

Pic:



- π equivariant w.r.t. isometries of X:

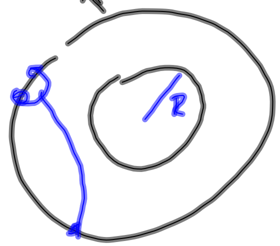
if $f \in \mathrm{Isom}(X)$,

$$d(f(\pi(a,b,c)), \pi(\partial_\infty f(a), \partial_\infty f(b), \ldots)) \leq C.$$

- π is proper

$\pi^{-1}(B(x,R))$ is compact.

if not, two points in preimage are getting close in boundary, so center of ideal triangle going to infinity too, $\times$



(2) $\exists D$ s.t. $\forall x \in X$,

$$\pi_X^{-1}(B(x,D)) \neq \emptyset.$$

because X has cocpt group of isometries:

③ Let $E(x) = \pi_{x'} \circ h \circ \pi_x^{-1}(B(x,D))$

By ②, $E(x) \neq \emptyset$. In fact, $\operatorname{diam} E(x) \lesssim C_1(\text{data})$.

<u>Proof</u> Fix base points $p_x, p_{x'}$, parameters $\varepsilon_x, \varepsilon_{x'}$, visual metrics.

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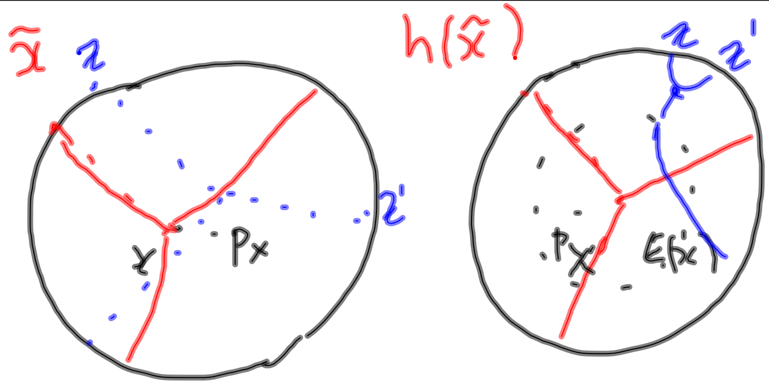

Choose $\tilde{x} \in \pi_x^{-1}(B(x,D))$

WLOG $d(x, p_x)$,
$d(\pi_{x'}(h(\tilde{x})), p_{x'}) \lesssim C$.

Now need to show
$E(x) \subset B(p_{x'}, R)$.

depending only on data.

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If false, $\exists z, z' \in \partial_\infty X'$ so that $h^{-1}: \partial_\infty X' \to \partial_\infty X$ takes z, z', which are close, to points that are far. Impossible by Prop. 5.5.

So $\Phi h: X \to X'$
$x \mapsto E(x)$
is well-defined up to bounded distance.

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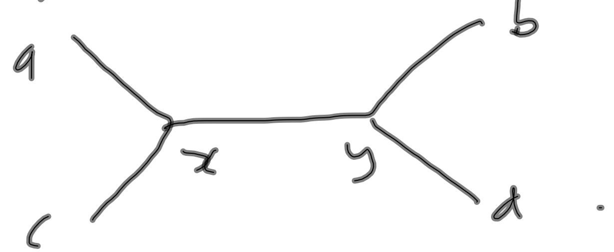
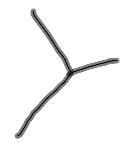
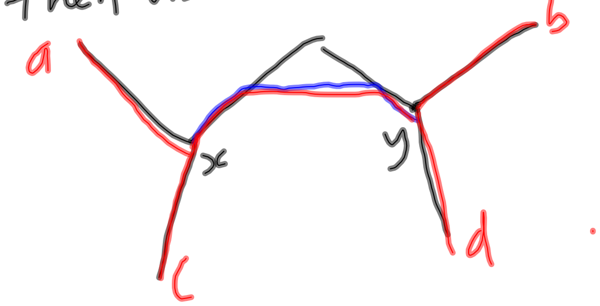
④ Given $x, y \in X$
$\exists a, b, c, d \in \partial_\infty X$
s.t. have



up to controlled finite Hausdorff distance.

Proof $|\partial_\infty X| \geq 3$ so have 
then use action:



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(5) Φh is a q.i.

For $x, y \in X$ find a, b, c, d as in (4).

Recall from Theorem 6.1 that

$[a,b,c,d] \asymp e^{-\varepsilon_x d(x,y)}$

Let $a' = h(a)$, etc.

$[a',b',c',d'] \lesssim \eta([a,b,c,d])$

So provided $d(x,y)$ is large enough, image will look like

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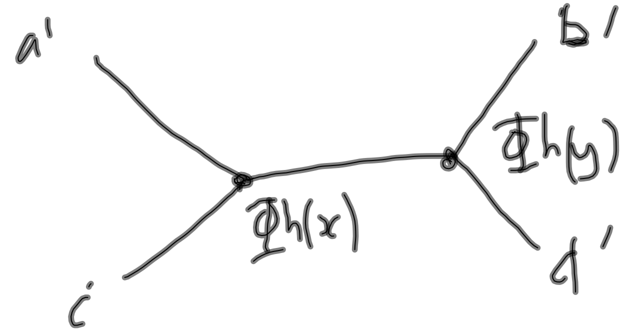

So

$$e^{\varepsilon_x \cdot d(\Phi h(x), \Phi h(y))} \asymp [a', b', d', c'] \lesssim \eta([a,b,d,c]) \lesssim \eta(Ce^{\varepsilon_x d(x,y)})$$

So for A large enough,

$$d(x,y) \leq A \Rightarrow d(\Phi h(x), \Phi h(y)) \lesssim \eta(Ce^{\varepsilon_x A}).$$

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As X is geodesic,
Φh is coarsely Lipschitz

(6) Likewise $\Phi(h^{-1})$ is coarsely Lipschitz,
also see that
$\Phi(h^{-1}) \circ \Phi h$ is a bounded distance to Id_X, etc.

$\square$.

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Remark (See Bonk-Schramm)
If h is bi-Lip, get
Φh is $(1, C)$-q.i. ;
lots more results in this direction.

Remark M compact metric space, $G \curvearrowright M$ by homeos, so that induced action distinct triples
$G \curvearrowright \mathrm{Triples}(M)$ is prop. discontinuous, say
G is a convergence group.

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If, in addition, this action is cocompact, say G is a uniform convergence group. (Gehring-Martin)

Theorem 6.16 (Bowditch)

$G \curvearrowright M$ perfect, cpt metric space; unif. cvgc. action

$\iff$ G hyp. $\partial_\infty G \overset{\text{homeo}}{\cong} M$.

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Theorem 6.17 $|\partial_\infty G| \geq 3$.

(1) $G \curvearrowright \partial_\infty G$ is minimal: every orbit is dense.

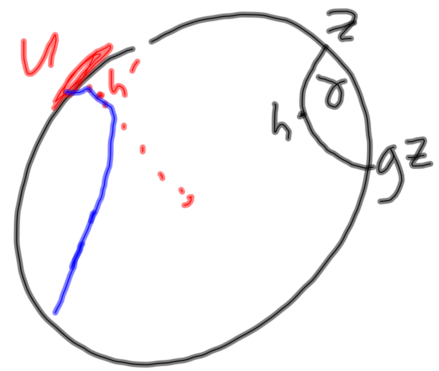
(2) $G \curvearrowright \partial^2_\infty G$ is top. transitive: $\exists$ dense orbit.

Proof of (1)

Take $z \in \partial_\infty G$. $\exists g \in G$ s.t. $gz \neq z$.

Then $\forall$ open $U \subset \partial_\infty G$,

look at (one end of) $h'h^{-1}\gamma$



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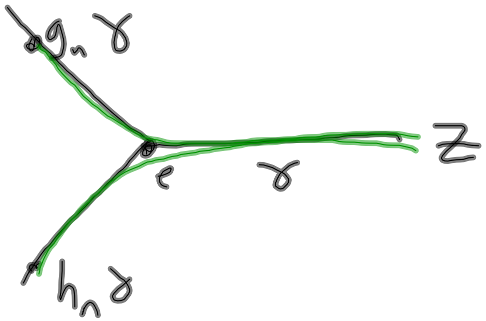
Such g exists, if not



i.e. $g_n h_n^{-1}$ essentially fixes e for all n, ※ to properness.

either $h'h^{-1}z$ or $h'h^{-1}gz$ lies in U for h' chosen large enough □.

Proof of (2) omitted.

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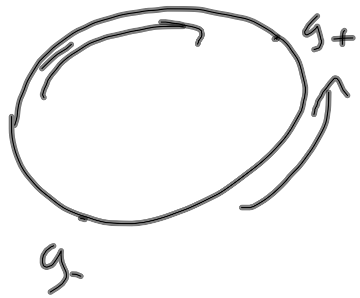
Theorem 6.18 $g\in G$ infinite order, then $\langle g\rangle \curvearrowright \partial_\infty G$ with North-South dynamics: i.e. $\exists\, g_-, g_+ \in \partial_\infty G$ fixed by g s.t. $\forall$ compact $K \subset \partial_\infty G \setminus \{g_-\}$, $g^n|_K \longrightarrow g_+$ uniformly as $n\to+\infty$.



Proof Note: $G \curvearrowright \partial_\infty G$ prop. disc. so $|\mathrm{Fix}(g)| \le 2$.

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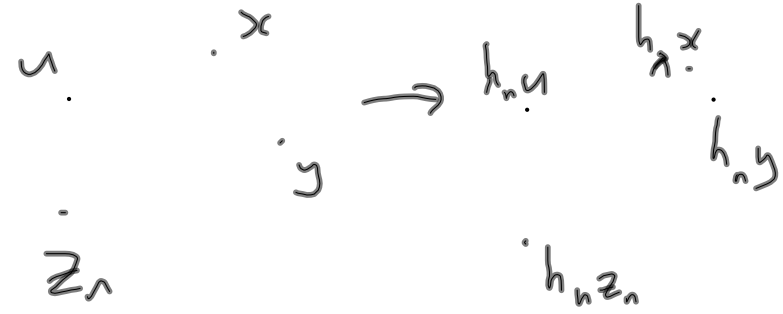
Lemma 6.19

$\{h_n\}_{n\in\mathbb{N}} \subset G$ infinite, distinct elements.

Then, up to subsequence, $\exists\, a, b \in \partial_\infty G$, $a \neq b$, s.t. $\forall$ compact $K \subset \partial_\infty G \setminus \{b\}$, $h_n|_K \to a$ unif. as $n \to \infty$.

Proof Pick x, y, z distinct in $\partial_\infty G$, $z_n \to z$, so that (after subseq.)
$h_n x \to a$
$h_n y \to a$
$h_n z_n \to b$, $a \neq b$.

Then $h_n|_{\partial_\infty G\setminus\{z\}} \to a$ loc. uniformly: if $u \neq z$



$$\frac{d(h_n y, h_n u)\, d(h_n x, h_n z_n)}{d(h_n y, h_n x)\, d(h_n u, h_n z_n)} \asymp \frac{d(y,u)\, d(x,z_n)}{d(y,x)\, d(u,z_n)} \lesssim \frac{1}{d(u,z)}$$

$$\frac{d(h_n y, h_n u)}{d(h_n y, h_n x)} \lesssim$$

As $n \to \infty$,
$d(h_n y, h_n x) \to 0$
so $d(h_n y, h_n u) \to 0$ too;
i.e. $h_n u \to a$.

If $a = b$, choose
$c \in \partial_\infty G \setminus \{a\}$,
let $w_n = h_n^{-1} c$,
WLOG $w_n \to w$.
Can use w instead of z.

etc. $\square$

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<u>Remark</u> actually
$\mathbb{Z} \cong \langle g \rangle \overset{q.i.}{\hookrightarrow} G$.
(See Bridson-Haefliger)

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7. Mostow Rigidity

Theorem 7.1 Suppose have uniform lattices $G, G' < \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$.

(i.e. G, G' discrete groups, acting cocompactly on $\mathbb{H}^n$)

Then if $\exists f : G \xrightarrow{\cong} G'$

$\exists \alpha \in \mathrm{Isom}(\mathbb{H}^n)$ so that f extends to $i_\alpha \in \mathrm{Inn}(\mathrm{Isom}\,\mathbb{H}^n)$

i.e. $\forall g \in G$, $f(g) = \alpha g \alpha^{-1}$.

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Cor 7.2 $M_1^{n_1}, M_2^{n_2}$ closed hyp. manifolds, $n_1 \geq 3$, and $\pi_1(M_1) \overset{\text{isom}}{\cong} \pi_1(M_2)$

then $M_1 \overset{\text{isom.}}{\cong} M_2$.

Proof

$\partial_\infty(\pi_1(M_i)) = S^{n_i - 1}$,

so $n_1 = n_2$.

(See below to extend $\overset{\text{isomorphism}}{\cong}$ to q.i.)

Let $G_i = \pi_1(M_i) < \mathrm{Isom}(\mathbb{H}^n)$

$f : G_1 \to G_2$ isomorphism.

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$M_1 = G_1 \backslash \mathbb{H}^n$, $M_2 = G_2 \backslash \mathbb{H}^n$

f extends to $\dot{\iota}_\alpha$ s.t. $g \in G_1$

$$\alpha: \begin{tikzcd} \mathbb{H}^n \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{H}^n \arrow[d, "f(g)"] \\ \mathbb{H}^n \arrow[r, "\alpha"'] & \mathbb{H}^n \end{tikzcd}.$$

So $\alpha : G_1 \backslash \mathbb{H}^n \longrightarrow G_2 \backslash \mathbb{H}^n$

$\quad\;\; \shortparallel \qquad\qquad\quad \shortparallel$

$\quad\; M_1 \qquad\qquad\; M_2$.

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First, preliminaries.

<u>Recall</u>: $f: X \to Y$ homeo K-qc

if $\displaystyle \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \le K$

$\forall x \in X$.

<u>Theorem 7.3</u> (Mori $n=2$, Gehring $n=3$, Väisälä, Mostow $n \geq 4$)

$U, V \subseteq \mathbb{R}^n$, $n \geq 2$; then

$f: U \to V$ qc $\Rightarrow$ f diff. a.e.

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Recall:

Theorem 7.4 (Lebesgue)

$f:[a,b]\to\mathbb{R}$ Lipschitz

$\Rightarrow f$ diff. a.e., and

$f(b)-f(a)=\int_a^b f'(t)\,dt$.

Theorem 7.5 (Rademacher)

$U\subseteq\mathbb{R}^n$, $f:U\to\mathbb{R}^m$

Lipschitz $\Rightarrow f$ diff. a.e.

Proof WLOG $m=1$, $U=\mathbb{R}^n$.

By Theorem 7.4 and Fubini,

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$\nabla f(x):=\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)$

exists a.e.

· In fact, for any fixed $v\in\mathbb{R}^n$,

$D_v f(x):=\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t}$

exists a.e., and $D_v f(x)=v\cdot\nabla f(x)$.

Proof for all test functions $\eta\in C_0^\infty(\mathbb{R}^n)$

$\int_{\mathbb{R}^n} D_v f(x)\,\eta(x)\,dx=$

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$$= \int_{\mathbb{R}^n} \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}\,\eta(x)\,dx$$

by DCT!

$$= -\int_{\mathbb{R}^n} f(x)\, D_v \eta(x)\,dx$$

$$= -\sum v_i \int_{\mathbb{R}^n} f(x) \frac{\partial \eta}{\partial x_i}(x)\,dx$$

$$= \int (v\cdot \nabla f(x))\,\eta(x)\,dx.$$

- Choose countable dense set of directions $\{v_i\}$, let

$A = \{x : D_{v_i} f(x) \text{ exists } \forall i\}$.

This has full measure.

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Take $a \in A$.

Set $D(v,t) = \frac{f(a+tv)-f(a)}{t} - v\cdot\nabla f(a)$.

Want to show this is small uniformly in v.

$\forall \varepsilon > 0$, choose $v_1, \dots, v_N \in S^{n-1}$ ε-dense, $0<\delta$ s.t.

$|D(v_i,t)| < \varepsilon$ for $|t|<\delta$.

Then
$$|D(v,t)| \le |D(v,t) - D(v_i,t)| + |D(v_i,t)|$$

where v_i ε-close.

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$$\leq \varepsilon + \left| \frac{f(a+tv)-f(a+tv_i)}{t} \right|$$

$$+ |v-v_i| \, |\nabla f(a)|$$

$$\leq \varepsilon + \mathrm{Lip}(f)\,\varepsilon + |\nabla f(a)|\,\varepsilon$$

indep of v. $\boxed{\cdot}$

Theorem 7.6 (Stepanov)

Let $\mathrm{Lip}\, f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r}$.

If $f: U \overset{\subseteq \mathbb{R}^n}{\longrightarrow} \mathbb{R}^m$ is a function s.t. $\mathrm{Lip}\, f < \infty$ a.e.

Then f is diff. a.e.

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