Proof (Malý '99)

WLOG $m=1$

Let $\{B_i\}$ = all balls in U with rational center, rational radius and $f|_{B_i}$ bounded.

This covers ~~U~~. an open set of full measure

Let $u_i(x) = \inf\{u(x) : u$ is i-Lip with $u \geq f$ on $B_i\}$.

$v_i(x) = \sup\{v(x) : v \dots$
$\dots v \leq f$ on $B_i\}$.

These are i-Lip.

Feb 23-13:01

Let $Z = \bigcup_{i=1}^{\infty} \{x \in B_i :$
u_i or v_i not diff at $x\}$.

This has measure zero by Rademacher's theorem.

- If $\operatorname{Lip} f(a) < \infty$, then $\exists\, r>0,\ M<\infty$ s.t.
  $|f(a) - f(x)| \leq M|x-a|$
  for all $x \in B(a,r)$
- If $a \notin Z$ also, then $\exists\, i > M$ and $a \in B_i \subset B(a,r)$.

Feb 23-13:09

$\Rightarrow$

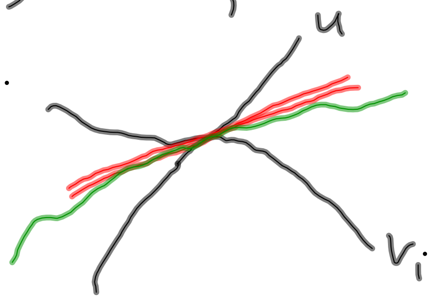
$f(a) - i|x-a| \leq v_i(x)$

$\leq u_i(x) \leq f(a) + i|x-a|$

Let $x \to a$; see that

$v_i(a) = u_i(a)$.

This implies that f is diff. at a.



$\square$

Feb 23-13:14

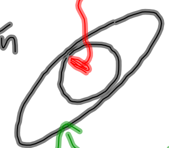
Proof of Theorem 7.3

$$\limsup_{r\to 0} \frac{L_f(x,r)}{r}$$

$$\leq \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)} \cdot \frac{l_f(x,r)}{r}$$

$$\leq K \limsup_{r\to 0} \frac{(l_f(x,r)^n)^{\frac{1}{n}}}{r}$$

$$\leq C(n) K \limsup_{r\to 0} \left(\frac{\mathcal{L}(f(B(x,r)))}{r^n}\right)^{\frac{1}{n}}$$



$f(B(x,r))$

Radon-Nikodym derivative of $f^*\mathcal{L}$ w.r.t. $\mathcal{L}$.

f homeomorphism, so

Feb 23-13:18

this is finite a.e.

Stepanov's theorem completes the proof.

<u>Fact</u> $\dot{D}f$ is measurable.

<u>Def 7.7</u> $U \subset \mathbb{R}^n$ meas. set is <u>dense</u> at $a \in \mathbb{R}^n$ if $\lim_{r \to 0} \frac{\mathcal{L}(U \cap B(a,r))}{\mathcal{L}(B(a,r))} = 1.$

Feb 23-13:23

<u>Theorem 7.8</u> (Lebesgue)

$U \subset \mathbb{R}^n$ meas

$\Rightarrow$ a.e. $a \in U$ is a density point of U.

<u>Def 7.9</u> $f : U \overset{\subseteq \mathbb{R}^n}{\longrightarrow} \mathbb{R}^m$

is <u>approximately continuous</u> <u>at a</u> if $\exists$ meas. V with $a \in V$ density pt. and $f|_V$ continuous at a.

Feb 23-13:26

Theorem 7.10 (Lusin)

$f: U \to \mathbb{R}^m$ meas

$\Rightarrow$ f approx. cts. at a.e. $a \in U$.

Def 7.11 A conformal structure on $\mathbb{R}^n$ is a choice of inner product, up to scale.

· The space of all such

Feb 23-13:31

is $SL(n,\mathbb{R})/SO(n)$

$= \{A : n\times n$ matrix $\det A = 1$, positive definite$\} =: S$

Since $SL(n,\mathbb{R}) \curvearrowright S$ by $X \cdot A = X^T A X$ with stabilizer of $I = SO(n)$.

· S is a symmetric space of non-compact type,

Feb 23-13:35

has $SL(n,\mathbb{R})$-invariant CAT(0) metric

$$d(I,A) = d(A)$$

$$= \frac{\sqrt{n}}{2}\left((\log \lambda_1)^2 + \cdots \cdots (\log \lambda_n)^2 \right)^{\frac{1}{2}}$$

(where $\lambda_1, .., \lambda_n$ eivals of A)

$$\asymp \max\left\{ \log \frac{1}{\lambda_1}, \log \lambda_n \right\}.$$

(See Tukia '85, Helgason for more discussion.)

Feb 23-13:38

Def 7.12

A smooth manifold M has a bundle $E \to M$ with fibers S of possible conformal structures on TM.

(lives in projectivised $S^2 T^* M$)

A <u>measurable</u> <u>conformal</u> <u>structure</u> on M is

Feb 23-13:41

a measurable section of this bundle.

Write as $\mathcal{E}$, with $\mathcal{E}_p$ the value at $p \in M$.

- $\mathcal{E}$ is <u>bounded</u> if $d(\mathcal{E}_p) < C$ a.e., for some $C < \infty$.

<u>Theorem 7.13</u> QC maps preserve sets of measure zero.

Feb 23-13:46

<u>Cor 7.14</u>

$f : \mathbb{S}^n \to \mathbb{S}^n$ qM or QC, $\mathcal{E}$ meas. conf. structure on $\mathbb{S}^n$,

Then $Df^* \mathcal{E}$, $Df_* \mathcal{E}$ give well defined meas. conf. structures.

Feb 23-13:48

Proof of Theorem 7.1

(1) Given $f : G \xrightarrow{\cong} G'$

Fix $O \in \mathbb{H}^n$ (s.t. stabilizer is trivial)

Define $F(g \cdot O) = f(g) \cdot O$ $(*)$

Extend to $F : \mathbb{H}^n \to \mathbb{H}^n$ quasi-isometry. (see Theorem 2.4).

(2) Let $h = \partial_\infty f : S^{n-1} \to S^{n-1}$.

h is qM.

$(*) \Rightarrow h \circ g = f(g) \circ h$ on S^{n-1}.

Feb 23-13:52

(3) If h was 1-qc, we'd be done:

By thm 4.9, 5.8

$h = \partial_\infty \alpha$ some $\alpha \in \mathrm{Isom}(\mathbb{H}^n) = \text{Möb}(S^{n-1})$.

$h = \alpha$

$\Rightarrow \forall g \in G$

$h \circ g \circ h^{-1} = f(g)$ on $\partial_\infty \mathbb{H}^n$

$\alpha \circ g \circ \alpha^{-1} = f(g)$ on $\partial_\infty \mathbb{H}^n$

$\Rightarrow f$ extends to $i_\alpha \in \mathrm{Inn}(\mathrm{Isom}(\mathbb{H}^n))$.

Feb 23-13:55

(4) h is qc (prop 5.4)

so diff. a.e. (Thm 7.3)

and Df meas.

(5) Let $\bar{\mathcal{E}}$ be round conformal structure on S^{n-1}.

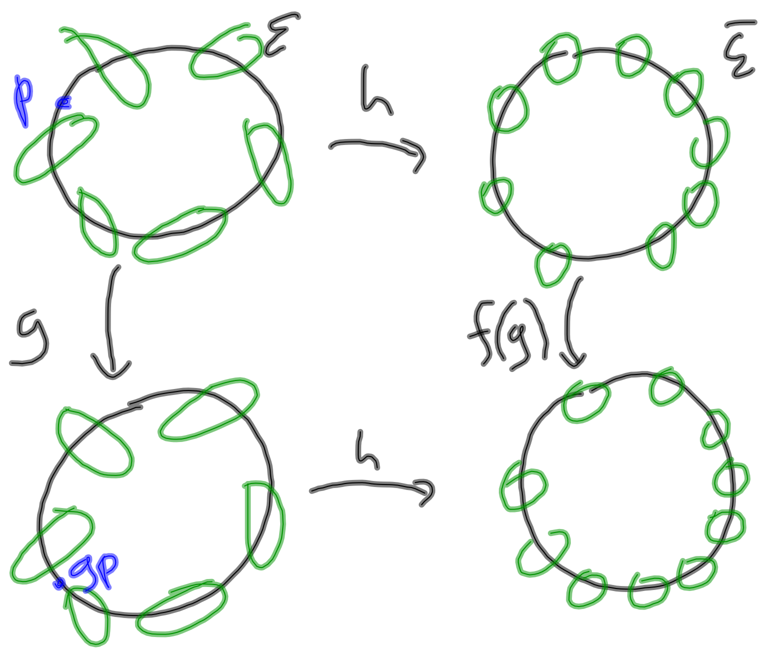
Let $\mathcal{E} = (Dh)^* \bar{\mathcal{E}}$.

Note: (†) $\Rightarrow$

$$Dh\, Dg = Df(g)\, Dh \quad \text{(a.e.)}$$

i.e. $\mathcal{E}$ is G-equivariant.

Feb 23-13:59



$$\mathcal{E}_{gp} = (Dh_{gp})^* \bar{\mathcal{E}}$$

$$= (Dg^{-1})^* (Dh_p)^* \cancel{(Df(g))^*} \bar{\mathcal{E}}$$

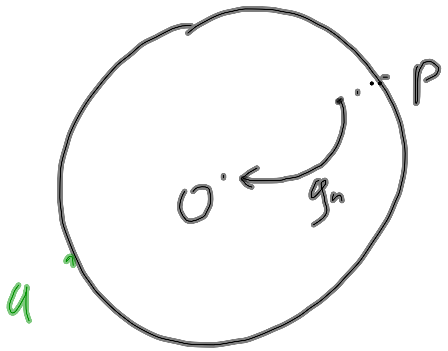
$$= (Dg^{-1})^* \mathcal{E}_p = (Dg)_* \mathcal{E}_p.$$

Feb 23-14:03

⑥ $\mathcal{E}$ meas $\Rightarrow$ approx. continuous (Thm 7.10)

$\Rightarrow \exists\, p$ s.t. $\mathcal{E}_p$ approx. cts. at p.

Take sequence $g_n \in G$ s.t. $g_n^{-1} \cdot O \to p$

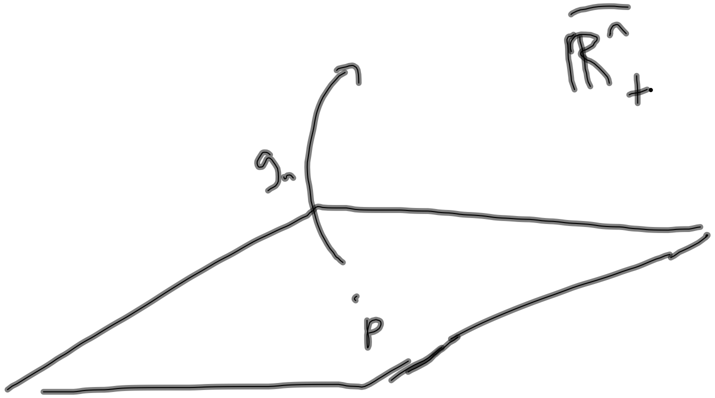


Feb 23-14:09

Apply (a subsequence of) g_n to $\mathbb{S}^{n-1}$ (via Lemma 6.19). Get $a, b\,(=p)$ in $\mathbb{S}^{n-1}$ s.t. $g_n|_{\partial_\infty G \setminus \{b\}} \to a$

Stereographically project a to infinity.

Feb 23-14:11



$(Dg_n)_* \mathcal{E} = \mathcal{E}$, and p point of approx continuity, so in limit get $\mathcal{E} = \hat{\mathcal{E}}_c$ a.e., where $\hat{\mathcal{E}}_c$ is a constant field of ellipses in these coordinates.

(7) If $\hat{\mathcal{E}}_c$ not spherical, $\times$.

$\square$

Feb 23-14:13

Remark

Any Gromov-Hausdorff tangent to $\partial_\infty G$ is quasi-Möbius to $\partial_\infty G \setminus \{*\}$ (Bonk-Kleiner).

Feb 23-14:30

8. Sullivan, Tukia Theorems

Theorem 8.1 (Gromov, Cannon-Cooper)

$n \geq 3$, $G \overset{q.i.}{\simeq} \mathbb{H}^n$

$\Rightarrow G \curvearrowright \mathbb{H}^n$ disc. cocpt, properly, isometrically.

Cor. $\partial_\infty G \overset{qM}{\simeq} S^{n-1}$, same conclusion.

Proof: Paulin, $\square$.

Feb 23-14:32

Prop 8.2 (Circumcentres)

X CAT(0) geod.

$A \subset X$ bounded.

Then A has a unique circumcenter $p(A) \in X$, and the map

$p : \{A \subset X \text{ bounded}, d_{\mathcal{H}} \text{ topology}\} \longrightarrow X$ is continuous.

Def 8.3 A bounded, $x \in X$

$$r(x, A) = \sup_{y \in A} d(x, y)$$

$$r(A) = \inf_{x \in X} r(x, A),$$

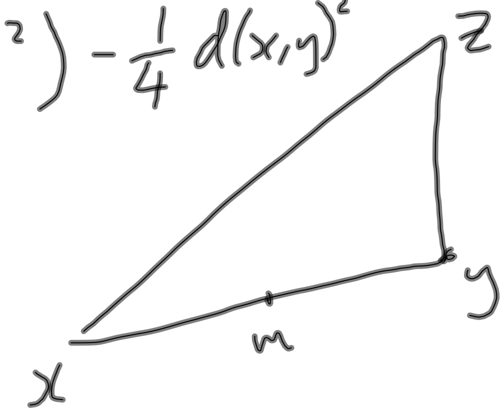
Feb 23-14:35

and x is a circumcenter for A if $r(A) = r(x,A)$.

Proof of 8.2

CAT(0) $\Rightarrow$ $\forall x,y \in X$ $\exists$ "midpoint" $m \in X$ s.t. $\forall z \in X$

$$d(z,m)^2 \leq \frac{1}{2}\left(d(z,x)^2 + d(z,y)^2\right) - \frac{1}{4}d(x,y)^2$$



So

$$r(m,A)^2 \leq \frac{1}{2}\left(r(x,A)^2 + r(y,A)^2\right) - \frac{1}{4}d(x,y)^2.$$

$\Rightarrow$

$$d(x,y)^2 \leq 2\left(r(x,A)^2 + r(y,A)^2\right) - 4r(A)^2 \quad (\star)$$

$\Rightarrow$ $P(A)$, if it exists, is unique.

Choose $(x_n) \subset X$ s.t. $r(x_n, A) \to r(A)$.

$(\star) \Rightarrow (x_n)$ Cauchy, so $x_n \to x$.

$$r(x,A) \le d(x,x_n) + r(x_n,A)$$
$$\to r(A)$$

So $P(A) = x$ is unique circumcenter.

Suppose A, A' bounded, $d_H(A,A') < \delta$.

Then $A' \subset B(P(A), r(A)+\delta)$

and $A \subset B(P(A'), r(A')+\delta)$.

$\Rightarrow |r(A) - r(A')| \le \delta$.

Feb 23-14:43

$\circledast \Rightarrow$

$$d(P(A), P(A'))^2$$
$$\le 2\left(r(A)^2 + \boxed{r(P(A'),A)^2}\right) - 4r(A)^2$$

$$\le \big(\underbrace{r(P(A'),A')}_{\le r(A)+\delta} + \delta\big)^2$$

$$\le \delta(8r(A) + 8\delta)$$
$$\to 0 \text{ as } \delta \to 0$$

$\square$.

Feb 23-14:45

<u>Theorem 8.4</u> (Measurable Riemann Mapping Theorem; Morrey '38, Ahlfors-Bers '60)

$\mathcal{E}$ bounded meas. conf. structure on S^2

$\Longleftrightarrow$ $\exists f : S^2 \to S^2$ qc st. $\mathcal{E} = Df^* \bar{\mathcal{E}}$.

<u>Remark</u> $\Leftarrow$ is easy.

For a proof, see e.g. Ahlfors lectures on QC

Feb 23-14:49

mappings.

Proof works by solving the Beltrami equation:

$$\partial_{\bar{z}} f = \mu \, \partial_z f \text{ a.e.},$$

where $\mu : \hat{\mathbb{C}} \to \mathbb{C}$ is a measurable function and $\|\mu\|_\infty < 1$.

Feb 23-14:53

Theorem 8.5 (Sullivan '81, see also Tukia ~81)

Suppose G is a group of K-qc homeos of S^2. Then $\exists f : S^2 \to S^2$ qc s.t. $fGf^{-1} \subset \text{Möb}(S^2)$.

Theorem 8.6 (Gromov, Tukia 85)

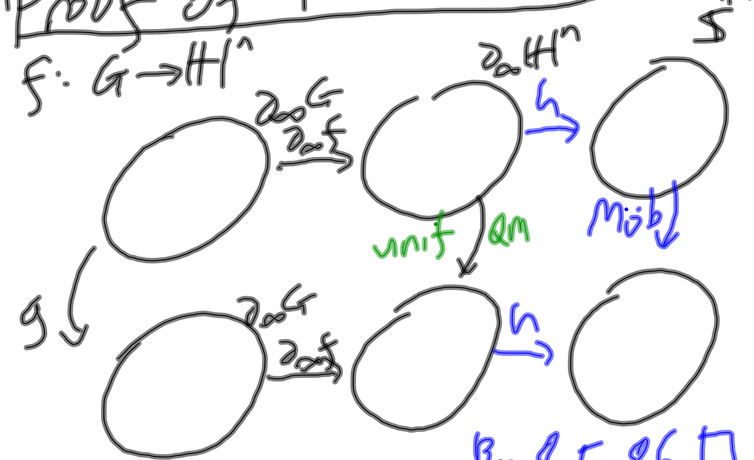
$n \geq 3$. Suppose G group of unif. qc homeos of S^n, acting cocpt on triples. Then $\exists f : S^n \to S^n$ qc s.t. $fGf^{-1} \subset \text{Möb}(S^n)$.

Feb 23-14:57

Remark False without the "cocompact on triples" assumption. (Tukia ~84)

Proof of Theorem 8.1

$f : G \to \mathbb{H}^n$



By 8.5, 8.6. $\square$

Feb 23-15:00

Feb 23-15:02