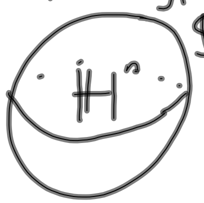
john.mackay@maths.ox.ac.uk

0. Outline

1. Gromov hyperbolic spaces,
2. Gromov hyperbolic groups
3. Boundaries of Gr. hyp. groups

Picture: $\mathbb{H}^n$ hyperbolic space



S^{n-1} sphere at infinity

quasi-isometries of groups

$\updownarrow$

quasi-Möbius homeomorphisms of the boundary

↑ nice analytic structure

4. Some analysis: conformal homeos, quasi-conformal homeos, ...
5. Mostow rigidity:

   Theorem M_1, M_2 closed hyperbolic n-manifolds, $n \geq 3$. Then $\pi_1(M_1) \cong \pi_1(M_2)$ $\Rightarrow M_1$ is isometric to M_2.

6. Sullivan/Tukia/Gromov/Cannon-Cooper

   Theorem if finitely gen. $G \overset{q.i.}{\simeq} \mathbb{H}^n$ for $n \geq 3$, then G acts on $\mathbb{H}^n$ isometrically, properly, cocompactly.

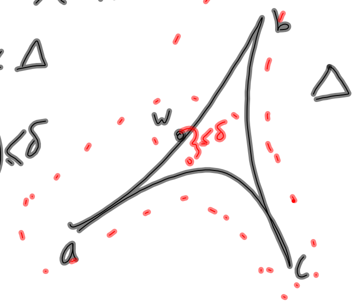
7. Other topics?
   - Dimensions of boundaries:
     Pansu: conformal dimension
     Bestvina-Mess: topological dimension
   - Cut points (Bowditch) $\leftrightarrow$ JSJ decomposition.

1. Gromov hyperbolic spaces

Def 1.1 A metric space is geodesic if $\forall x, y \in X$ $\exists$ isometry $\gamma : [0, d(x,y)] \to X$, $\gamma(0) = x$, $\gamma(d(x,y)) = y$

Write $[x,y]$ for the image of γ. (may not be unique.)

Def 1.2 A geodesic metric space is δ-hyperbolic, some $\delta \geq 0$, if every geodesic triangle in X is δ-slim: i.e. for all geodesic Δ, $w \in [a,b]$ then $d(w, [b,c] \cup [c,a]) \leq \delta$



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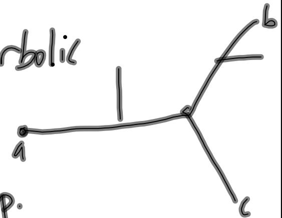
Examples

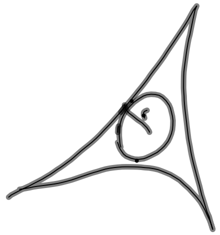
(a) Trees are 0-hyperbolic



(b) $\mathbb{R}^n$, $n \geq 2$ is not δ-hyp.

(c) $\mathbb{H}^2$ is δ-hyp. (with $\delta = 2$ say)

Since $\text{Area}(\Delta) \leq \pi$
$\text{Area}(B_1) > \pi$



(d) $\mathbb{H}^n_\kappa$ is δ-hyp

$\kappa < 0$ sectional curvature.

(can set $\delta = \frac{2}{\sqrt{-\kappa}}$.)

(e) X CAT(κ) space if geodesic triangles in X are thinner than in $\mathbb{H}^2_\kappa$

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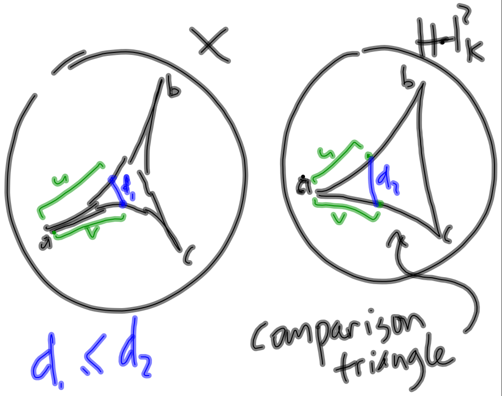

So CAT(K), K<0, spaces are δ-hyp.

(f) Cartan-Alexandrov-Toponogov:
M complete Riemannian manifold
all sectional curvatures $\le K < 0$
then M is locally CAT(K),
and $\tilde{M}$ is CAT(K);

Ex $\mathbb{CH}^n$ has sec. curv. $\in [-4,-1]$.
It's Gr. hyp.

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Def 1.3 X, Y metric spaces,
$f: X \to Y$ function is a
(λ, C)-quasi-isometric embedding
$(\lambda \ge 1,\ C \ge 0)$ if $\forall x, x' \in X$

$$\frac{1}{\lambda} d_X(x,x') - C \le d_Y(f(x), f(x')) \le \lambda d_X(x,x') + C$$

We write $f: X \overset{q.i.}{\hookrightarrow} Y$.

If in addition $\forall y \in Y$
$\exists x \in X$ s.t. $d_Y(f(x), y) \le C$,
say f is a (λ, C)-quasi-isometry.

If such an f exists, we write
$X \overset{q.i.}{\simeq} Y$.

Lemma 1.4 $\overset{q.i.}{\simeq}$ is an equivalence relation.

Ex $(1,0)$-q.i. are isometries
$(\lambda, 0)$-q.i. are λ-bi-Lipschitz homeomorphisms.

$f: \mathbb{R} \to \mathbb{Z}$, $f(x) = \lfloor x \rfloor$ is a q.i.

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Def 1.5. X metric space,

$I = \begin{cases} [a,b] \\ [0,\infty) \\ \mathbb{R} \end{cases}$, $\gamma : I \to X$ q.i. embed.

Say image $\gamma(I)$ is a quasi-geodesic $\begin{cases} \text{segment} \\ \text{ray} \\ \text{line} \end{cases}$.

Def 1.6 $A, B \subset X$ metric space

$N_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\}$

The Hausdorff distance between A, B is

$d_H(A,B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B), B \subset N_\varepsilon(A)\}$.

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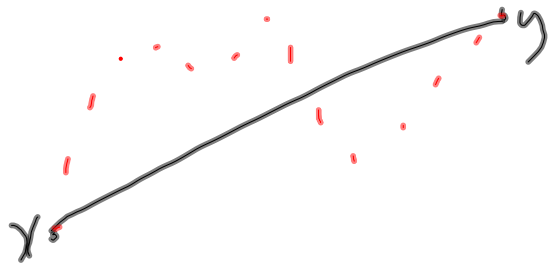
Ex $d_H(\mathbb{R}, \mathbb{Z}) = \frac{1}{2}$.

Theorem 1.7 (Morse Lemma)

X δ-hyp. geod. mtr. space, $S \subseteq X$ q.geod segment with end points x, y, Set $\gamma = [x,y]$. Then $\exists C = C(\delta, \lambda_S, K_S)$ s.t. $d_H(S, \gamma) \leq C$.



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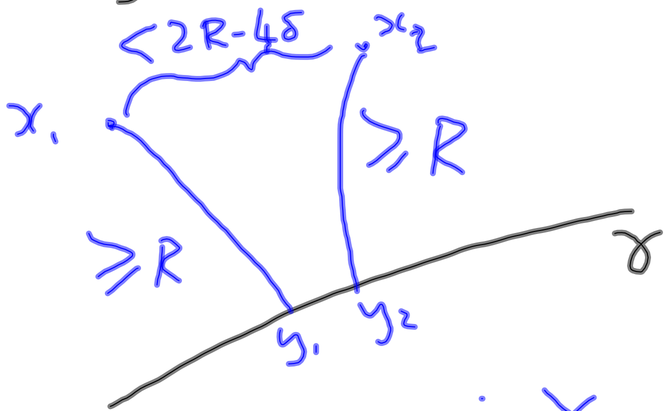
Lemma 1.8 (Projection)

γ geodesic segment.



y_i closest to x_i in γ

Then: $d(y_1, y_2) \leq 8\delta$.

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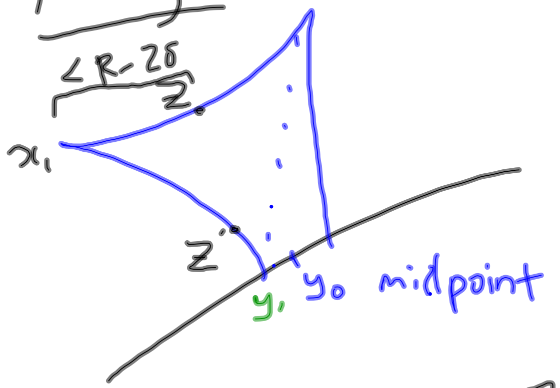
Proof



Case 1 $\exists z \in [x_1, x_2]$
$d(z, y_0) \leq 2\delta$

$\Rightarrow d(y_0, x_1) < (R-2\delta)+2\delta$
$= R$ ※.

Case 2 $\exists z' \in [x_1, y_1]$
$d(z', y_0) \leq 2\delta$

$d(y_1, y_0) \leq d(y_1, z) + d(z, y_0)$
$\leq 2 d(z, y_0) \leq 4\delta$

$\Rightarrow d(y_1, y_2) < 8\delta$ $\square$.

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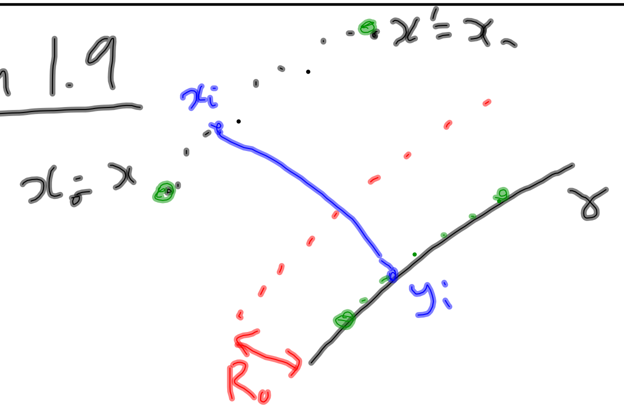
Lemma 1.9



$\phi : [0,L] \to X$ (λ, K)-q.geod.

$\phi([0,L])$ outside $N_{R_0}(\gamma)$

$\exists R_0 = R_0(\delta, \lambda, K), C_0 = C_0(\delta, \lambda, K)$

s.t.

$$L \leq C(d(x,\gamma) + d(x',\gamma)) + C$$

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Proof $T = 1 + 8\delta\lambda$

W.L.O.G. $L = Tn$

Set $x_i := \phi(Ti)$,

choose R_0 large enough

that $d(x_i, x_{i+1}) < 2R_0 - 4\delta$

Set $y_i =$ closest point to x_i in γ

$$L \leq \lambda\, d(\phi(0), \phi(L)) + K$$
$$\leq \lambda(d(x,\gamma) + d(x',\gamma)) + K + \lambda(d(y_0,y_1) + \cdots + d(y_{n-1},y_n))$$
$$< \lambda(d(x,\gamma) + d(x',\gamma)) + \underbrace{\lambda 8\delta n}_{= \frac{\lambda 8\delta}{T} L} + K$$

Take to Left hand side ↑ ∎

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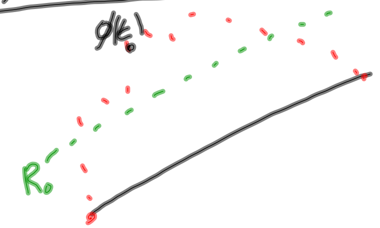
Proof of Thm. 1.7



Suppose $\exists t$: $d(\phi(t), \gamma) > R_0 +$ [illegible]

$t \in [a,b]$ maximal outside

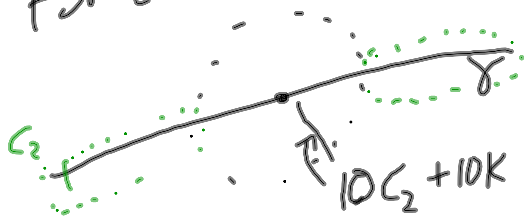
$N_{R_0}(\gamma)$, by L 1.9

$|b-a| \lesssim C_2$

$\Rightarrow d(\phi(t), \gamma) \lesssim C_2$.

$\Rightarrow \phi \subset N_{C_2}(\gamma)$

For converse, consider



ϕ can't jump from left to right

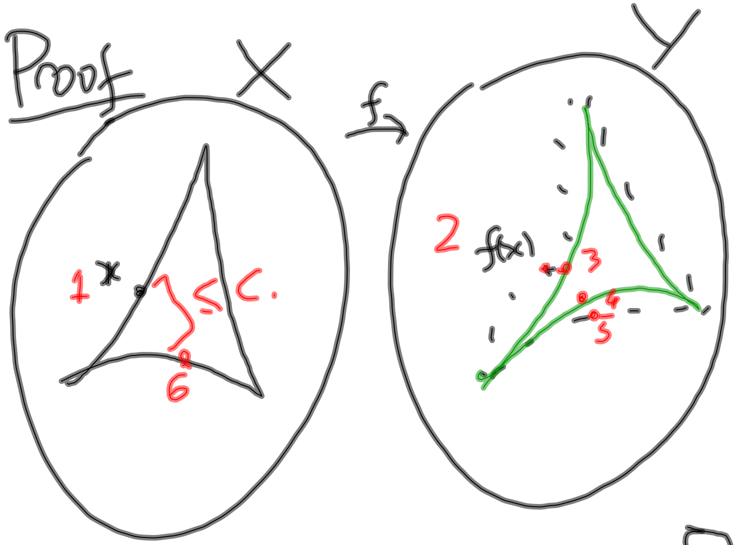
$\boxtimes$

Cor 1.10 X, Y geod.

Y Gr. hyp. $X \overset{q.i.}{\longleftrightarrow} Y$

$\Rightarrow$ X Gr. hyp.

Proof



$\square$

2. Gromov hyp. groups

Def 2.1 G discrete group

S finite gen. set for G

- The Cayley graph $\Gamma(G,S)$ is the graph with vertex set G and edges $\{g, gs\}$ $\forall g \in G, s \in S$.
- Set each edge isometric to $[0,1]$; put path metric on $\Gamma(G,S)$, denote by d_S, a word metric on G.

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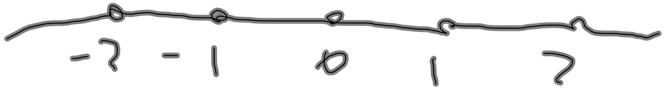
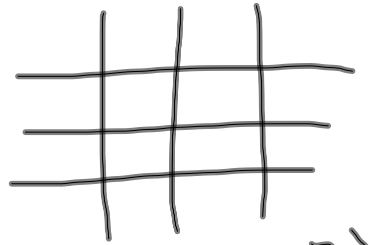
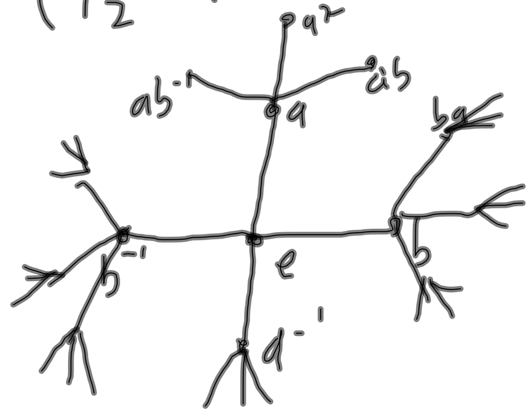
Ex (a) $\Gamma(\mathbb{Z}, \{1\})$



(b) $\Gamma(\mathbb{Z}^2, \{(0,1),(1,0)\})$



(c) $\Gamma(F_2 = \langle a,b \rangle, \{a,b\})$



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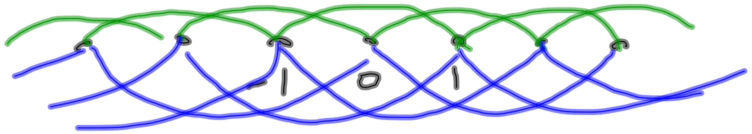
(d) $\Gamma(\mathbb{Z}, \{2, 3\})$



Lemma 2.2 S, S' fin. gen sets for G, then $\Gamma(G,S) \overset{q.i.}{\simeq} \Gamma(G,S')$

Proof Ex.

Def 2.3 G f.g. group is Gromov hyp. if $\Gamma(G,S)$ is Gr. hyp for ~~some~~ any fin. gen set S.

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Ex • $\mathbb{Z}$, F_2 are Gr. hyp.

• $\mathbb{Z}^2$ is not Gr. hyp.

Theorem 2.4 (Fund. Lemma of Geometric Group theory)

X proper, geod. metric sp.

$G \curvearrowright X$ isometrically properly cocompactly.

$\left(\begin{array}{l} \alpha : G \to \mathrm{Isom}(X) \\ G \times X \to X \\ g \cdot x \mapsto \alpha(g)(x). \end{array} \right)$

Then G is fin. gen and $G \overset{q.i.}{\simeq} X$, via $\begin{array}{l} G \to X \\ g \mapsto g \cdot x_0 \end{array}$.

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Proof

$G \curvearrowright X$ cocompact so

$\exists B = \bar{B}(x_0, R) \subset X$

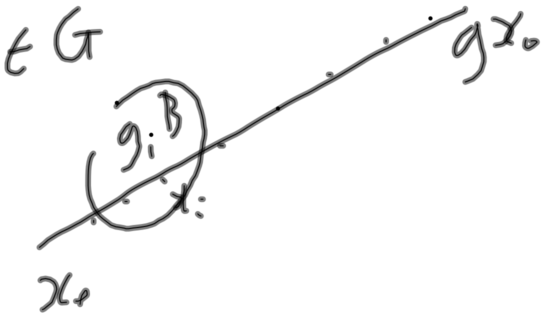
s.t. $\bigcup_{g \in G} gB = X$

Set $S = \{ g \in G : \underset{g \neq 1}{gB \cap B \neq \emptyset} \}$

Let $a = \max_{s \in S} d(x_0, s x_0) < \infty$

$b = \inf_{g \notin S \cup \{1\}} d(B, gB) > 0$

Take $g \in G$



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Choose x_i s.t. $d(x_i, x_{i+1}) < b$,

$x_n = g x_0$, $n \leq \frac{d(x_0, g x_0)}{b} + 1$

$\exists g_i \in G$ s.t. $x_i \in g_i B$

$d(g_i B, g_{i+1} B) < b$

$\Rightarrow g_i B \cap g_{i+1} B \neq \emptyset$

$\Rightarrow g_{i+1} = g_i s_i$, some $s_i \in S$

$\Rightarrow g = g_n = g_{n-1} s_n = \cdots s_1 \cdots s_n$

$\Rightarrow S$ fin. gen. set for G.

$d_S(e, g) \leq n \leq \frac{d(x_0, g x_0)}{b} + 1$

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Finally, $d_S(e,g)=n$, $g=s_1\cdots s_n$

$$d(x_0, gx_0) \leq d(x_0, s_1x_0) + \cdots + d((s_1\cdots s_{n-1})x_0, (s_1\cdots s_{n-1})s_nx_0)$$

$$\leq an = a\, d_S(e,g)$$

$\square$.

Cor 2.5 (i) G f.g. Gr. hyp.

$\Leftrightarrow$ (ii) $G \curvearrowright X$, X proper geodesic Gr. hyp.

isom., proper. cocpt.

Proof $\Leftarrow$ Thm 2.4

$\Rightarrow$ Cayley graph. $\square$

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Cor 2.6 M compact Riem. manifold, sec. curv < 0,

$\Rightarrow \pi_1(M)$ Gr. hyp.

Proof sec. curv $< \chi < 0$

$\tilde{M}$ CAT(χ), so Gr. hyp

$\pi_1(M) \curvearrowright \tilde{M}$ satisfies assumptions of Thm 2.4.

$\blacksquare$

Remark Lots more to say, we won't for now:

- Linear isoperimetric ineq.
- solvable word problem.
- "Generic"

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3. Boundaries at infinity

Def 3.1 X δ-hyp. geod. metric space (proper!). The boundary at infinity of X is

$$\partial_\infty X = \{\gamma:[0,\infty)\to X \text{ geod.}\}/\sim$$

$$\gamma_1\sim\gamma_2 \iff \sup d(\gamma_1(t),\gamma_2(t))<\infty.$$

If $\gamma_1\sim\gamma_2$ we write $\gamma_1(\infty)=\gamma_2(\infty)$.

Ex X bounded $\overset{q.i.}{\simeq}\{*\}$

$$\partial_\infty X=\emptyset.$$

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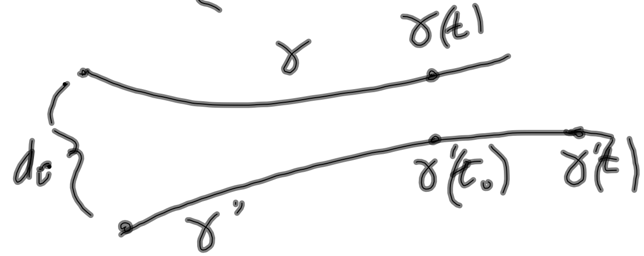
Ex $\partial_\infty\mathbb{R}=\{-\infty,\infty\}$

Lemma 3.2 $\gamma(\infty)=\gamma'(\infty)$

$$\iff d_H(\gamma,\gamma')<\infty$$

Proof $\Rightarrow$ ✓

$\Leftarrow$



$r=d_H(\gamma,\gamma')<\infty$

$t\le d_0+t_0+r \Rightarrow |t-t_0|\le d_0+r.$

Other case similar. ∎

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Theorem 3.3 (Arzelà-Ascoli)

X, Y proper metric,

$\mathcal{F} \subset C(X,Y) = \{f : X \to Y \text{ cts.}\}$

- **pointwise bounded**:

$\forall x \in X \quad \operatorname{diam}\left(\{f(x)\}_{f \in \mathcal{F}}\right) < \infty$

- **equicontinuous**

$\forall K \subset X$ compact, $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall f \in \mathcal{F}, x, x' \in K$

$d(x, x') < \delta \Rightarrow d(f(x), f(x'))$.

Then:

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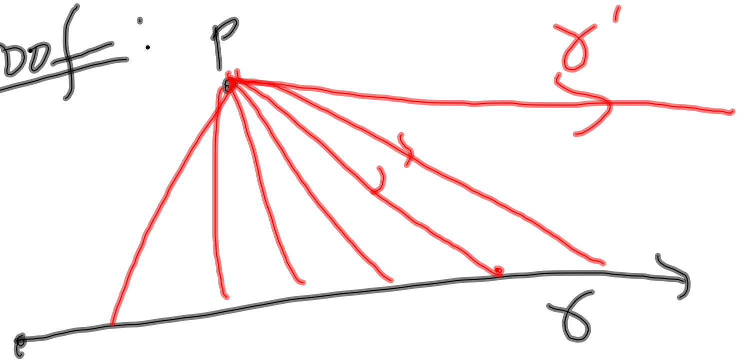
Any sequence in $\mathcal{F}$ has a subsequence that converges to $f \in C(X,Y)$ unif. on compact sets.

Cor 3.4 In definition of $\partial_\infty X$, can assume all $\gamma(0) = p$ base point.

Proof:



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