Remark

If $\gamma: \mathbb{R} \to X$ geodesic

Set $\gamma_+ : [0,\infty) \to X$

$\gamma_+(t) = \gamma(t)$

$\gamma(+\infty) := \gamma_+(\infty)$

and $\gamma_- : [0,\infty) \to X$

$\gamma_-(t) = \gamma(-t)$

$\gamma(-\infty) := \gamma_-(\infty)$.

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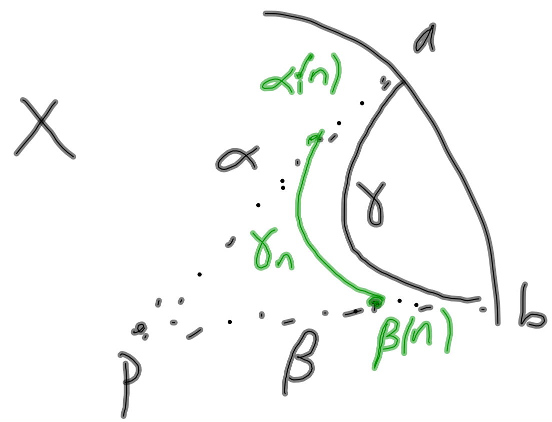
Lemma 3.5

X δ-hyp. proper.

$\forall a, b \in \partial_\infty X$, $a \neq b$.

$\exists \gamma : \mathbb{R} \to X$ geodesic

s.t. $\gamma(-\infty) = a$, $\gamma(+\infty) = b$



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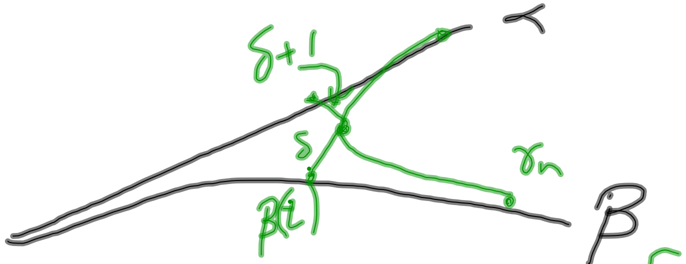
Proof Since $a \neq b$,

$$\sup_t d(\alpha(t), \beta(t)) = \infty$$

Choose T s.t. $\forall t \geq T$,

$$d(\alpha(t), \beta) \geq 2\delta + 2$$
$$d(\beta(t), \alpha) \geq 2\delta + 2$$



$n \geq T$ $\quad d(p, \gamma_n) \leq T + \delta$

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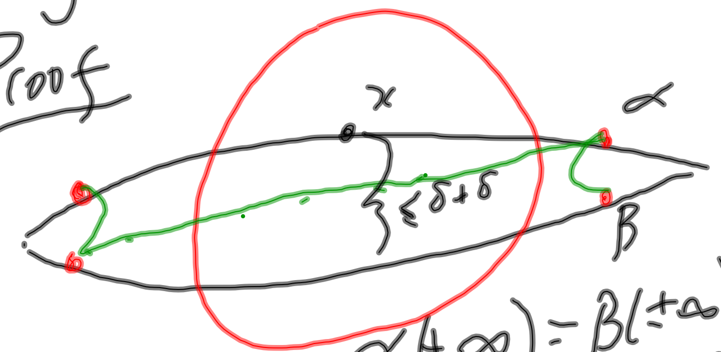
So reparametrize γ_n so that $\gamma_n : [s_n, t_n] \to X$ isometry, $\gamma_n(0) \in B(p, T+\delta)$.

Use Theorem 3.3. $\square$

Lemma 3.6 Bi-infinite geodesic bigons are 2δ-slim.

Proof



$\alpha(\pm\infty) = \beta(\pm\infty)$ $\square$

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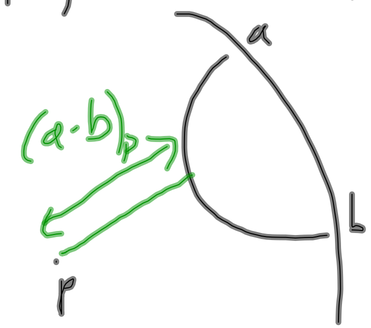
Def 3.7 $a, b \in \partial_\infty X$

Basepoint $p \in X$.
The Gromov product of a and b (w.r.t. p)

$$(a \cdot b)_p = \inf \left\{ d(p, \gamma) : \begin{array}{l} \gamma : \mathbb{R} \to X \text{ geod}, \\ \gamma(+\infty) = b, \gamma(-\infty) = a \end{array} \right\}$$



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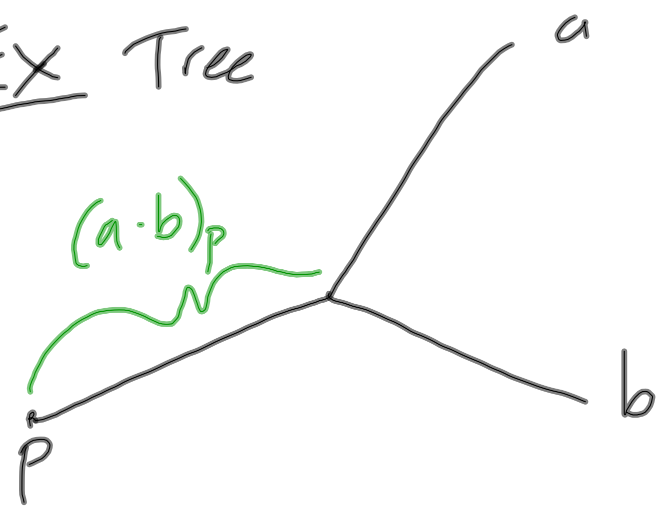
Remarks:

- $(a \cdot b)_p = +\infty \iff a = b$
- Can ignore inf up to 2δ-error.

Ex Tree



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Detour: Tree approximations

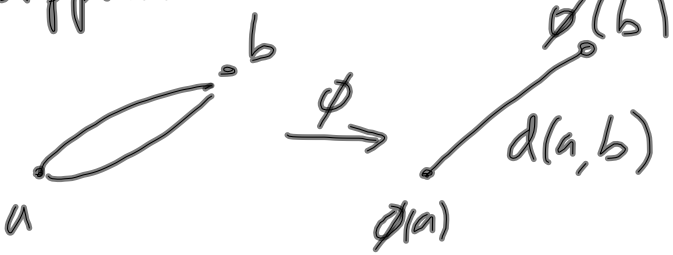
Def 3.8 $F \subset X \cup \partial_\infty X$ finite set. A <u>tree approx.</u> for F (with constant C) is a geodesic 0-hyp metric space T that is a finite union of intervals and rays, together with a $(1,C)$-q.i.

$$\phi_T : Geod(F) \to T$$

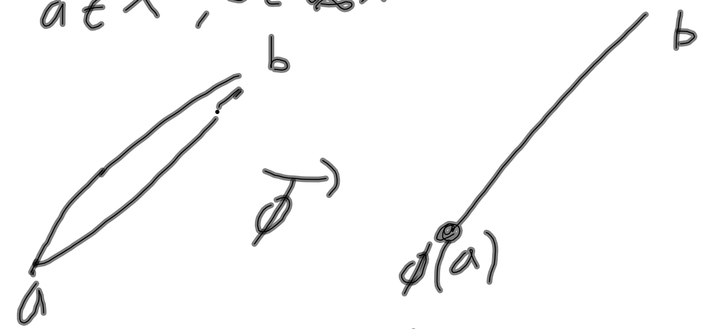
where $Geod(F)$ = Union of geodesics joining points of F.

Ex (a) $a, b \in X$

Then there is a δ-tree approx

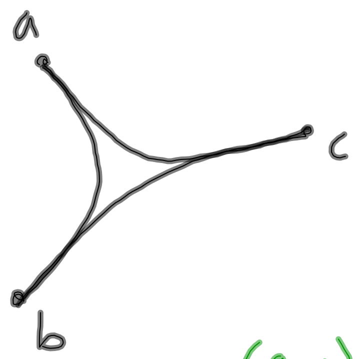


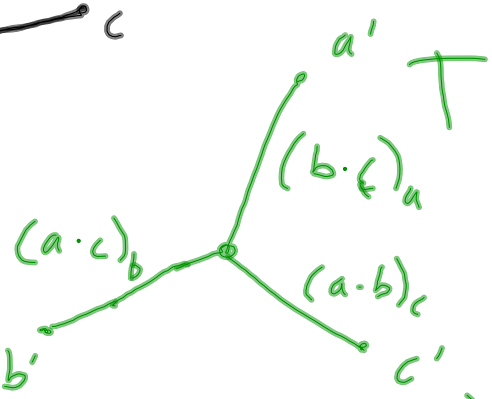
(b) $a \in X$, $b \in \partial_\infty X$



ϕ is a $(1, 2\delta)$-q.i.

ⓒ $a, b, c \in X$





Where $(a\cdot b)_c = \frac{1}{2}\left(d(a,c) + d(b,c) - d(a,b)\right)$

Note: $(a\cdot c)_b + (b\cdot c)_a = d(a,b)$

Define $\phi : \mathrm{Geod}(\{a,b,c\}) \to T$ to be an isometry on $[a,b]$.

Check ϕ is a $(1, C)$-q.i.

$(C = 20\delta)$.

Theorem 3.9

X δ-hyp, proper, geod.

$\forall n \in \mathbb{N}\ \exists\, C(n)$ s.t.

$\forall F \subset X \cup \partial_\infty X$, $|F| \le n$,

$\exists$ $C(n)$-tree approx

$\phi_T : \mathrm{Geod}(F) \to T$.

Proof Induction

- $n=2$ ✓
- $F = \{x_1, .., x_{n-1}\} \subset X \cup \partial_\infty X$

Have $(n-1)$-tree approx

$\phi_T : \mathrm{Geod}(F) \to T$.

- Given $x_n \in X \cup \partial_\infty X$.

If $x_n \in X$, set $x_n' = x_n$

If $x_n \in \partial_\infty X \setminus F$, set x_n' in $[x_1, x_n)$ at distance $\geq 10\delta$ from $\mathrm{Geod}(F)$.

- Take $q \in \mathrm{Geod}(F)$ minimizing $d(x_n', \mathrm{Geod}(F))$.

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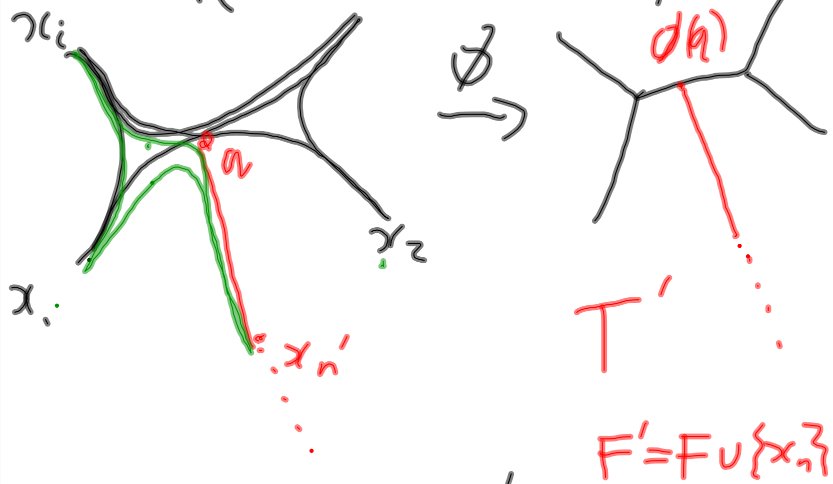
Geod(F) T



- This defines T'.

To define $\phi_{T'} : \mathrm{Geod}(F') \to T'$

assume $q \in [x_1, x_2]$

Map $[q, x_n)$ by isometry, $\mathrm{Geod}(F')$ in 2δ-nbhd of $\mathrm{Geod}(F) \cup [q, x_n)$. □

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<u>Remark</u> This proof gives $C(n) \sim C\delta n$.

Can get $C(n) \sim C\delta \log n$

(See Ghys-de la Harpe).

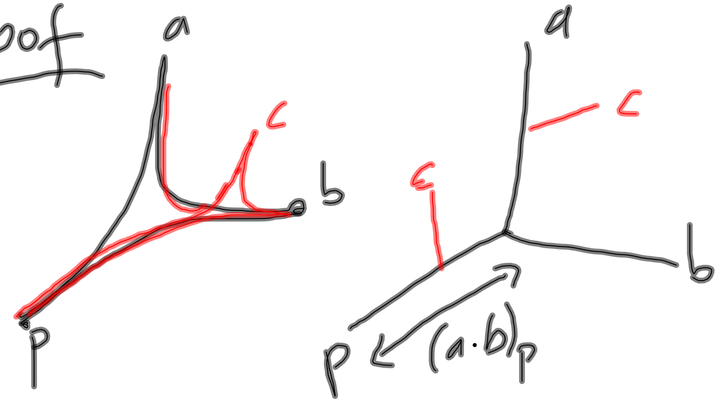
<u>Cor 3.10</u> (Gromov product definition of δ-hyp.)

If X δ-hyp. $\exists C(\delta)$ s.t. $\forall a,b,c,p \in X$

$$(a\cdot b)_p \geq \min\{(a\cdot c)_p, (c\cdot b)_p\} - C(\delta)$$

(*)

<u>Proof</u>



(Can take $C(\delta) = 6\delta$.) $\square$

<u>Remark</u> (*) holds for $p \in X$ $\forall a,b,c \in \partial_\infty X$.

Def 3.11 basepoint p.

A visual metric on $\partial_\infty X$ with parameter $\varepsilon > 0$, constant C_p is a metric ρ on $\partial_\infty X$ so that $\forall a,b \in \partial_\infty X$

$$\frac{1}{C} e^{-\varepsilon (a\cdot b)_p} \le \rho(a,b) \le C e^{-\varepsilon (a\cdot b)_p}$$

<u>Note</u>: $\rho(a,b) = 0 \Leftrightarrow (a\cdot b)_p = \infty$
$\Leftrightarrow a = b$.

Convention 3.12

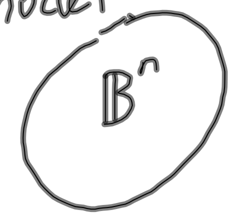
- $A \lesssim B \Leftrightarrow A \le B + C$
  where C depends on data only

- $A \simeq B \Leftrightarrow A \lesssim B,\ B \lesssim A$.
- $A \precsim B \Leftrightarrow A \le C B$
- $A \asymp B \Leftrightarrow A \precsim B,\ B \precsim A$.

Another detour:

Example 3.13 $\mathbb{H}^n$.

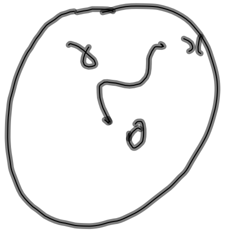
Poincaré ball model $\mathbb{R}^n$



metric
$x, y \in \mathbb{H}^n \overset{\text{set}}{=} \mathbb{B}^n$

$$d_{\mathbb{H}^n}(x,y) = \inf \left\{ \int_\gamma \frac{2}{1-|x|^2}\, ds \ : \ \gamma \text{ joins } x \text{ to } y \text{ piecewise } C^1 \right\}$$

Note



$\gamma : [0,1] \to \mathbb{H}^n$

$$\text{length}(\gamma) := \int_0^1 \frac{2}{1-|\gamma(t)|^2} |\gamma'(t)|\, dt$$

$\gamma(t) = (r(t), \theta(t))$ in polar

$$= \int_0^1 \frac{2}{1-r(t)^2} \sqrt{r'(t)^2 + r(t)^2 \theta'(t)^2}\, dt$$

$$\geq \int_0^1 \frac{2}{1-r(t)^2} |r'(t)|\, dt$$

$$\geq \int_0^{|x|} \frac{1}{1-s^2}\, ds = \log \frac{1+|x|}{1-|x|}.$$

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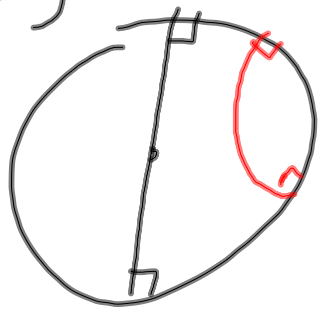
So $d_{\mathbb{H}^n}(0,x) = \log \frac{1+|x|}{1-|x|}$,

unique geodesic is Euclidean line segment.

Later: all geodesics are segments of



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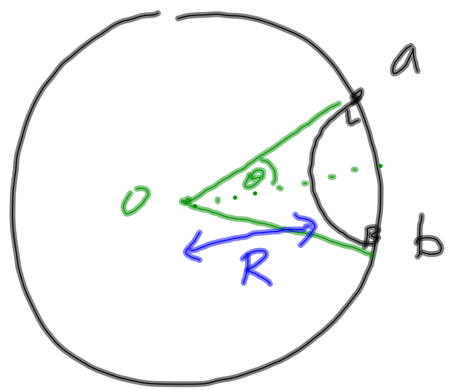
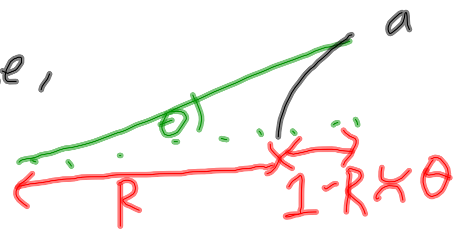
So $a,b \in \partial_\infty \mathbb{H}^n$



R = Euclidean distance

So $(a\cdot b)_0 = \log \frac{1+R}{1-R}$

If a,b close,



So $e^{-(a\cdot b)_0} = \frac{1-R}{1+R} \asymp \frac{\theta}{2} \asymp d_{Euc}(a,b)$

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So d_{Euc} on $\partial_\infty \mathbb{H}^n$ is a visual metric with $\varepsilon = 1$, base point O.

<u>Remark</u> For a slight variation of $(a\cdot b)_0$, and X any CAT(-1) space, $e^{-(a\cdot b)_p}$ is a visual metric. (Bourdon).

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Theorem 3.14

X δ-hyp, $\exists\, \varepsilon_0 = \varepsilon_0(\delta)$ s.t. $\forall\, \varepsilon \in (0, \varepsilon_0)$, $\exists$ visual metric on $\partial_\infty X$ with parameter ε.

Lemma 3.15

Z set, $\rho : Z \times Z \to [0, \infty)$

s.t.
- $\rho(a,b) = \rho(b,a)$
- $\rho(a,b) = 0 \iff a = b$
- $\rho(a,b) \leq A \max\{\rho(a,c), \rho(c,b)\}$

for some $A \in [1, \sqrt{2}]$.

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Then $\exists$ metric d on Z, $d \simeq \rho$.

Proof of Thm 3.14

Recall $\otimes$

$(a \cdot b)_p \geq \min\{(a \cdot c)_p, (c \cdot b)_p\} - ($

Set $\rho_\varepsilon(a,b) = e^{-\varepsilon (a \cdot b)_p}$

Then $\rho_\varepsilon(a,b) \leq e^{\varepsilon c} \max\{\rho_\varepsilon(a,c), \rho_\varepsilon(c,b)\}$

Choose ε small enough $\left(\frac{1}{4\delta}\right)$.

Apply L.3.15. $\square$

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Proof of Lemma 3.15

Set $d(a,b) = \inf \{ \rho(a_0,a_1)+\rho(a_1,a_2)+\cdots + \rho(a_{n-1},a_n) : a_0,\ldots,a_n \in Z,\ a_0=a,\ a_n=b \}$

- $d \leq \rho$, $d(a,b) = d(b,a)$, triangle inequality, $d(a,a)=0$
- Show $d \geq \frac{1}{C}\rho$.

Proof by induction.

Assume

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$\frac{1}{C}\rho(a_0,a_m) \leq \rho(a_0,a_1)+\cdots + \rho(a_{m-1},a_m)$

$\forall m<n$.

- Consider $S = \rho(a_0,a_1)+\cdots+\rho(a_{n-1},a_n)$

Take m max s.t.

$\rho(a_0,a_1)+\cdots+\rho(a_{m-1},a_m) \leq \frac{S}{2}$

- $\rho(a_0,a_n) \leq A^2 \max\{\rho(a_0,a_m),\ \rho(a_m,a_{m+1}),\ \rho(a_{m+1},a_n)\}$

$\leq A^2 \max\{C(\rho(a_0,a_1)+\cdots+\rho(a_{m-1},a_m)),\ \rho(a_m,a_{m+1}),\ C(\rho(a_{m+1},a_{m+2})+\cdots)\}$.

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$$\lesssim A^2 \max\left\{ C\frac{S}{2}, S, C\frac{S}{2} \right\}$$

$$\lesssim CS \max\left\{ \frac{A^2}{2}, \frac{A^2}{C} \right\}$$

$A \leq \sqrt{2}$; $C := A^2$.

$$\lesssim CS.$$

$\blacksquare$

Remark Choices of basepoint, ε, visual metric don't change topology of $\partial_\infty X$.

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Lemma 3.16

$\forall p, q \in X$, $a, b \in \partial_\infty X$

$$(a \cdot b)_p \underset{C + d(p,q)}{\approx} (a \cdot b)_q$$

Cor 3.17

$\rho_{\varepsilon,p}$, $\rho_{\varepsilon';q}$ visual metrics

then $\rho_{\varepsilon,p} \asymp \left(\rho_{\varepsilon';q}\right)^{\varepsilon/\varepsilon'}$.

i.e. well-defined Hölder structure.

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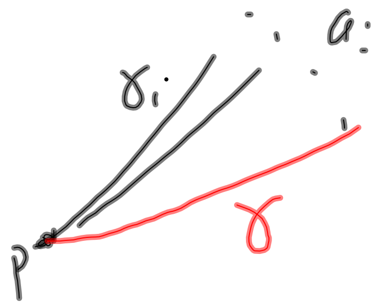
Prop 3.18

X proper δ-hyp, then $\partial_\infty X$ is compact.

Proof. Suppose visual metric $\rho = \rho_{\varepsilon, p}$ on $\partial_\infty X$.

- Suppose (a_i) seq. in $\partial_\infty X$ represented by $\gamma_i : [0,\infty) \to X$ $\gamma(0) = p$.
- By Thm 3.3, $\gamma_i \to \gamma$



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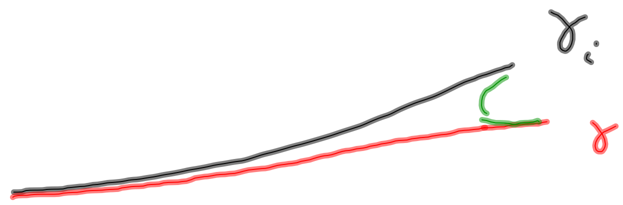
Check: $(\gamma_i \cdot \gamma)_p \xrightarrow{i \to \infty} \infty$



$\square$.

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4. $\mathrm{Isom}(\mathbb{H}^n) = \text{Möb}(\mathbb{B}^n)$
$= \text{Möb}(S^{n-1})$.

Let $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ be one-point compactification.

Def 4.1 $S(a,r) = \{x : |x-a| = r\}$.

Inversion in $S(a,r)$ is

$$\sigma(x) = \begin{cases} \infty & x = a \\ a & x = \infty \\ a + \dfrac{r^2}{|x-a|^2}(x-a) & \text{otherwise.} \end{cases}$$

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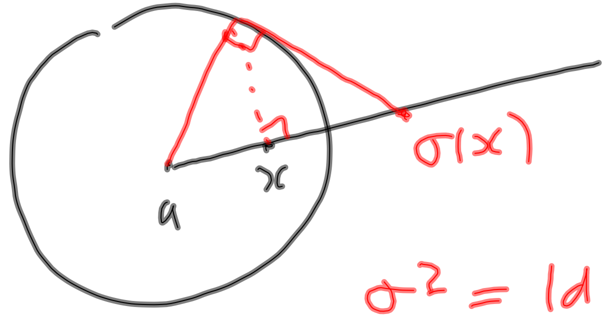

$\sigma^2 = \mathrm{Id}$.

- Generalised sphere = sphere or hyperplane.
- Reflection in generalised sphere is reflection in hyperplane or inversion in sphere. These generate the group of Möbius transformations $\text{Möb}(\overline{\mathbb{R}}^n)$.

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Remarks

(a) $\text{Möb}(\overline{\mathbb{R}^2}) = \text{Möb}(\hat{\mathbb{C}})$

$$= \left\{ \frac{az+b}{cz+d},\ \frac{a\bar{z}+b}{c\bar{z}+d} : \begin{array}{l} a,b,c,d \in \mathbb{C}, \\ ad-bc \neq 0. \end{array} \right\}$$

(b) $|\sigma(x)-\sigma(y)| = \dfrac{r^2|x-y|}{|x-a||y-a|}$

Ex.

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(c) $\text{Möb}(\overline{\mathbb{R}^n})$ includes all similarities of $\mathbb{R}^n$:

reflections, rotations, translations ✓

Dilations:

σ_1 invert $S(0,1)$

σ_2 invert $S(0,\sqrt{\lambda})$,

$$\sigma_2 \circ \sigma_1(x) = \sigma_2\left(\frac{x}{|x|^2}\right)$$

$$= \frac{\lambda\ x/|x|^2}{|x/|x|^2|^2} = \lambda x.$$

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Def 4.2 The **cross-ratio** of distinct x_1, x_2, x_3, x_4 in a metric space X is

$$[x_1, x_2, x_3, x_4] = \frac{d(x_1, x_3)\, d(x_2, x_4)}{d(x_1, x_4)\, d(x_2, x_3)}.$$

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