Proof of Theorem 8.5

- Let's assume G countable.
- Let $\bar{\xi}$ be round conformal structure on $\mathbb{S}^2$.
- For $z \in \mathbb{S}^2$, let

$$E(z) = \{ (Dg_z)^* \bar{\xi}_{gz} : g \in G \}$$

- defined a.e.
- bounded in space of conf. structures, as each g is K-q

- Let $\xi_z = P(E(z))$ circum-centre.

(Prop. 8.2)

Mar 1-13:02

- To see ξ is measurable, let $G = \{g_0, g_1, \ldots\}$

Let $\xi_z^{(n)} = P(\{ (Dg_i)_z^* \bar{\xi}_{g_i z} : i = 0, \ldots, n \})$

defined for a.e. $z \in \mathbb{S}^2$, measurable as Dg_i meas.

P continuous, so $\xi = \lim_{n \to \infty} \xi^{(n)}$ is measurable.

- Apply Thm 8.4 to find $f : \mathbb{S}^2 \to \mathbb{S}^2$ q.c. s.t.

$$\xi = (Df)^* \bar{\xi} \quad \text{a.e.}$$

Mar 1-13:12

$\cdot\ \forall g \in G$, a.e. $z \in \mathbb{S}^2$

$(Dg)_z^* E(gz)$

$= \{ (Dg_z)^* (Dh)_{gz}^* \mathcal{E}_{hgz} : h \in G \}$

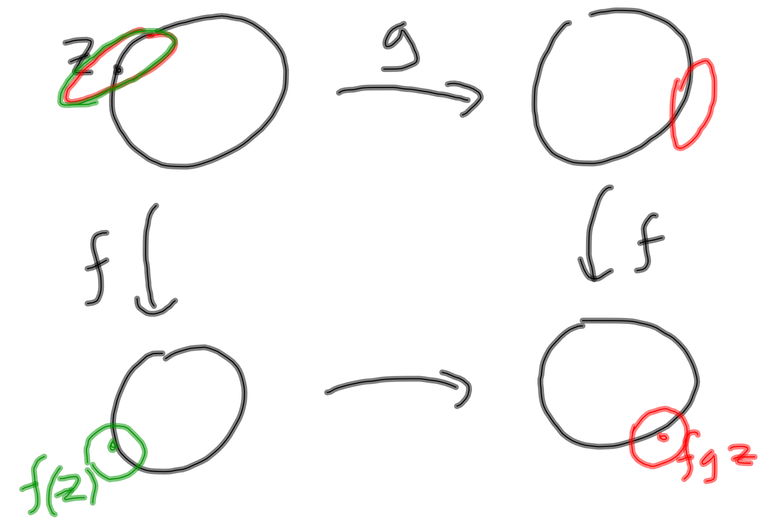
$= \{ (D(hg)_z)^* \mathcal{E}_{hgz} : hg \in G \}$

$= E(z)$.

So $(Dg_z)^* \mathcal{E}_{gz} = \mathcal{E}_z$ a.e.

$\Rightarrow (Dg)_z^* (Df_{gz})^* \mathcal{E}_{fgz} = (Df_z)^* \mathcal{E}_{fz}$ a.e.

Mar 1-13:17



So fgf^{-1} is 1-qc $\Rightarrow$ Möb.

(True without assuming G countable: take a countable dense subgroup; limit...)

$\square$

Mar 1-13:21

Outline of proof of theorem 8.6:

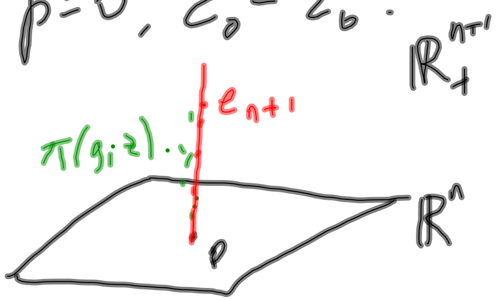
- Define $E(z)$ as above.
- Let $\mathcal{E}_z = P(E(z))$.

as before it's a G-inv. meas conformal structure.

- $\mathcal{E}$ is approx. cts a.e.

Let p be such a point.

WLOG $p=0$, $\mathcal{E}_0 = \overline{\mathcal{E}}_0$.



Mar 1-13:25

Pick a triple z.

Choose g_i s.t. projection $\pi(g_i z) \to p$ along line as in pic.

Choose $\lambda_i \to \infty$ s.t.

$$d_{\mathbb{H}^{n+1}}(e_{n+1}, \lambda_i \pi(g_i z))$$

uniformly bounded

- Let $f_i : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be $f_i(u) = \lambda_i g_i(u)$.
- This keeps triple spread out so by Prop 5.7, after

Mar 1-13:32

taking a subseq,
$f_i \longrightarrow f$ qc, uniformly
on $S^n = \overline{\mathbb{R}^n}$

- Show fgf^{-1} is 1-qc.

Basically, $\forall \varepsilon > 0$
$f_i g f_i^{-1}$ is $(1+\varepsilon)$-qc
off set of measure m_i in S^n,
and $m_i \downarrow 0$ as $\varepsilon \to 0$.
Tukia: in limit, fgf^{-1} is 1-qc.
Or: $Df_* \mathcal{E} = \lim_{i \to \infty} (Df_i)_* \mathcal{E}$
$= \lim_{i \to \infty} (\lambda_i Dg_i)_* \mathcal{E} \longrightarrow \mathcal{E}$.
//.

Mar 1-13:36

What about S^2?

Analogue of Theorem 8.5
for unif qm groups
(Hinkkanen '85). In fact:

Theorem 8.7 (Tukia, Gabai,
Casson-Jungreis ~'89)
$G \curvearrowright S^2$. Then
G is a convergence group
$\Longleftrightarrow$ action is topologically
conjugate to Fuchsian
action.

Cor 8.8 (Tukia, Freden) G hyp.
$\partial_\infty G \overset{\text{homeo}}{\cong} S^1 \Longleftrightarrow G \curvearrowright \mathbb{H}^2$ disc. cocpt.

Mar 1-13:42

<u>Remark</u> Solved Seifert fibred space conjecture: M compact orientable irred. 3 manifold, $|\pi_1 M| = \infty$. Then M is Seifert fibred space $\Leftrightarrow \exists\, \mathbb{Z} \cong H \triangleleft \pi_1 M$.

Connection: $G \curvearrowright \mathrm{Triples}(S^1) \overset{\text{homeo}}{\cong} S^1 \times \mathbb{R}^2$

Case S^2:

<u>Cannon's conjecture</u>: G hyp.

$\partial_\infty G \overset{\text{homeo}}{\cong} S^2 \Leftrightarrow G \curvearrowright \mathbb{H}^3$ geometric.

Mar 1-13:46

False for S^n, $n \geq 3$.

e.g. Mostow-Siu

· Gromov-Thurston: $\exists\, M^4$ cpt. that admits Riem. metrics of curvature $K \in [-1-\varepsilon, -1]$, but no hyperbolic metric.

· $G = \mathbb{C}\mathbb{H}^2/\Gamma$ cocpt lattice

then $\partial_\infty G \overset{\text{homeo}}{\cong} S^3$ but not $\overset{qm}{\not\cong} S^3$.

See conformal dimension later.

Mar 1-13:50

9. Cohomology & topological dimension

Def 9.1 X metric space, $R>0$. The Rips complex $P_R(X)$ is the simplicial complex with vertex set X, and an n-simplex $\{x_0, \dots, x_n\} \Longleftrightarrow \operatorname{diam}\{x_0, \dots, x_n\} \leq R$.

- If G f.g. group. $P_R(G) = P_R(G, d_S)$.

Mar 1-14:00

Prop 9.2 G hyp $\Longrightarrow P_R(G)$ contractible for R large enough.

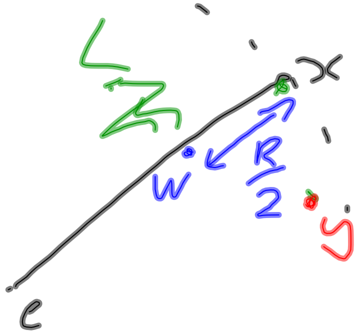
Proof

- By Whitehead's theorem, suffices $\pi_n(P_R(G)) = 0$ $\forall n \geq 1$.
- Suffices every finite subcomplex $L \subset P_R(G)$ is homotopic to a point.
- **Case 1** $L \subset B(e, \frac{R}{2})$. All L lies in some simplex.

Mar 1-14:03

Case 2 $\exists x \in L^{(0)}$ s.t. $d(x,e) > \frac{R}{2}$ and maximal.

Then



Define $f : L^{(0)} \to P_R(G)$

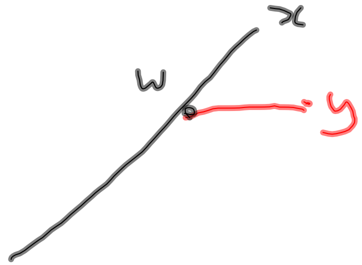
$$f(z) = \begin{cases} w & \text{if } z = x \\ z & \text{otherwise} \end{cases}$$

Want to extend f to L.

If $y \in L^{(0)}$, $d(y,x) \leq R$

then $d(y,w) \leq \frac{R}{2} + C < R$ for $R \gg 0$.

Mar 1-14:08



So $L \simeq f(L)$ in $P_R(G)$.

L finite, so continue until in Case 1. $\square$

Cor 9.3 G hyp $\Rightarrow$ fin. pres.

Proof G f.g. then

G is fin. pres

$\Longleftrightarrow$ $P_R(G)$ is simply conn. for R large enough. (Ex.) $\square$

Mar 1-14:13

Cor 9.4 G hyp. then $\exists$ simplicial complex P, action $G \curvearrowright P$ that is

(a) simplicial, cocompact, finite stabilisers

(b) P is fin. dim, contractible, locally finite.

(c) $G \curvearrowright P^{(0)}$ free and transitive.

In particular,

(1) If G torsion free then $G \backslash P$ is a $K(G,1)$ so G finite cohomological dim, type F_n for all n.

Mar 1-14:19

(2) G virtually tor. free $\Rightarrow$ finite virtual cohom. dim.

(3) $H^*(G, \mathbb{Q})$ is finite dim.

(4) Type FP_∞ (Brown)

Remark Open Q:

Is every Gr. hyp group v. torsion free?

$\Leftrightarrow$ Is every Gr. hyp. group residually finite? (Kapovich-Wise '00).

Mar 1-14:23

Def 9.5 G fin. gen.

Volume entropy of G (with respect to fin. gen. set S) is $h(G,S) =$

$$\limsup_{r\to\infty} \frac{1}{r} \log |B_G(e,r)|$$

$$\leq \log(2|S|-1)$$

Def 9.6 X metric space $A \subset X$. $\alpha \geq 0$

Then the Hausdorff α-measure of A is

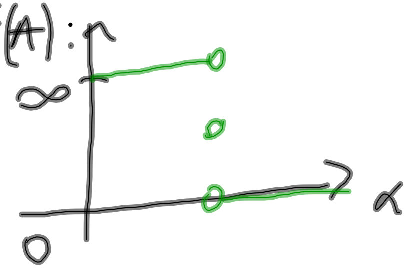
Mar 1-14:28

$$\mathcal{H}^\alpha(A) = \lim_{\delta\to 0} \inf \left\{ \sum \operatorname{diam}(E_i)^\alpha : A \subset \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) < \delta \right\}$$

The Hausdorff dimension of A is

$$\dim_H A = \sup\{\alpha : \mathcal{H}^\alpha(A) = \infty\}$$
$$= \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\}.$$

Graph of $\mathcal{H}^\alpha(A)$:



Mar 1-14:31

Ex $X=\mathbb{R}^n$ then

$$\mathcal{H}^\alpha(\text{open set}) = \begin{cases} 0 & \alpha > n \\ c\mathcal{L}(\text{set}) & \alpha = n \\ \infty & \alpha < n \end{cases}$$

(proportional to Lebesgue meas.)

(More in §10)

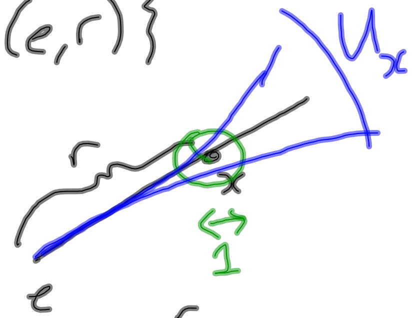
Prop 9.7 G hyp, $\varepsilon > 0$, d_ε is a visual metric on $\partial_\infty G$ with param. ε, then $\dim_H(\partial_\infty G, d_\varepsilon) \le \frac{1}{\varepsilon} h(G,S)$.

Proof

If $\alpha > \alpha' > \frac{1}{\varepsilon} h(G,S)$

$\exists C$ s.t. $|B_G(e,r)| \le C e^{\alpha' \varepsilon r}$.

Let $\mathcal{U}_r$ be cover $\{U_x : x \in S(e,r)\}$



$\operatorname{diam}(U_x) \le C e^{-\varepsilon r}$

Then $\sum_{\mathcal{U}_r} (\operatorname{diam} U_x)^\alpha \le (Ce^{-\varepsilon r})^\alpha C e^{\varepsilon r \alpha'} \lesssim e^{-\varepsilon r(\alpha - \alpha')} \to 0$ as $r \to \infty$.

i.e. $\dim_H \le \alpha$. □

In fact

Theorem 9.8 (Coornaert '93)

$\exists$ Ahlfors Q-regular measure μ on $(\partial_\infty G, d_\varepsilon)$ with $Q = \frac{1}{\varepsilon} h(G,S)$.

i.e. $\dim_H(\partial_\infty G, d_\varepsilon) = Q$.

Rmk Ahlfors Q-regular

$\Rightarrow \forall r \leq \operatorname{diam}(\partial_\infty G, d_\varepsilon)$

$\mu(B(x,r)) \asymp r^Q$.

(Fact: then $\mathcal{H}^Q \asymp \mu$.).

Cor 9.9 $\dim \partial_\infty G < \infty$

Proof $\dim \partial_\infty G < \dim_H(\partial_\infty G, d_\varepsilon)$ $\square$.

Mar 1-14:43

Prop 9.10 Can define compactification (G hyp.)

$\overline{G} = G \cup \partial_\infty G$

(or $\overline{P(G)} = P(G) \cup \partial_\infty G$)

and $\overline{G}$ is a compact, metrizable, fin. dim space.

Proof

$$G \leftrightarrow \left\{ \begin{array}{l} \gamma : [0,\infty) \to \Gamma(G) : \\ \gamma(0) = e,\ \exists n \text{ s.t. } \\ \gamma|_{[0,n]} \text{ is geodesic,} \\ \gamma|_{[n,\infty)} \equiv \gamma(n). \end{array} \right\}.$$

Mar 1-14:48

Then $G \cup \partial_\infty G$ is compact.

Has uniform structure so metrizable.

$$\dim \overline{G} \le \dim(\Gamma(G)) + \dim(\partial_\infty G) + 1.$$

$\square$

Recall:

Def 9.11 X top. space.

$\mathcal{U}, \mathcal{V}$ open covers.

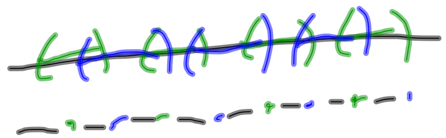
- Say $\mathcal{V}$ refines $\mathcal{U}$ if $\forall V \in \mathcal{V}\ \exists U \in \mathcal{U}$ s.t. $V \subseteq U$.

Mar 1-14:52

- Nerve of $\mathcal{U}$ is the simp. complex $N(\mathcal{U})$ with vertex set $\mathcal{U}$ and an n-simplex $\{U_0, \dots, U_n\}$ $\iff U_0 \cap \dots \cap U_n \neq \emptyset$.

Def 9.12 X has $\dim(X) \le n$ $\iff$ every cover $\mathcal{U}$ of X has a refinement $\mathcal{V}$ s.t. $\dim N(\mathcal{V}) \le n$.

Ex $\mathbb{R}$:



Mar 1-14:56

· If $\mathcal{V}$ refines $\mathcal{U}$

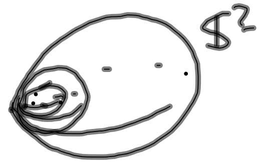
obvious map $N(\mathcal{V}) \to N(\mathcal{U})$

get $H^*(N(\mathcal{U})) \to H^*(N(\mathcal{V}))$

Čech cohomology is

$$\check{H}^*(X) = \varinjlim H^*(N(\mathcal{U})).$$

This is nice(r): X:



then $H^3(X) \neq 0$

but $\check{H}^3(X) = 0$

$\check{H}^2(X) = \bigoplus_{\mathbb{N}} \mathbb{Z}$.

Mar 1-14:59

Mar 1-15:03