More definitions:

- Cohomological dimension:

  G group

  $cd(G) = \sup\{n :$
  $H^n(G, M) \neq 0$ for
  some $\mathbb{Z}G$ module $M\}$.

- Compactly supported cohomology:

  $H_c^*(X, \mathbb{Z})$ is cohom. of

  X: cw complex

  $C_c^*(X)$ = compactly supported cochains.

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e.g. $\tilde{H}_c^i(\mathbb{R}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

- G tor. free hyperbolic, $P_d(G)$ contractible. Then

$$H^i(G, \mathbb{Z}G) \cong H_c^i(P_d(G), \mathbb{Z})$$

$$= \varinjlim \{H_c^i(P_d(G), P_d(G) \setminus Y) : Y \text{ compact}\}$$

$$\cong \check{H}^i(\overline{P_d(G)}, \partial_\infty G).$$

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Theorem 9.13 (Bestvina-Mess '91)

G Gr. hyp, virtually tor. free, then

$$\dim(\partial_\infty G) = vcd(G) - 1$$

$:= cd(H)$

$H \underset{f.i.}{\leq} G$, tor. free.

Theorem 9.14 (B-Mess)

G hyperbolic.

$$H^i(G, \mathbb{Z}G) \cong \check{H}^i(\partial_\infty G, \mathbb{Z}).$$

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Proof hints:

Key:

Theorem 9.15 (B.-M.)

$\overline{P_d(G)}$ is an absolute retract, and

$\partial_\infty G \subset \overline{P_d(G)}$ is a $\mathcal{Z}$-set.

We'll skip definitions here:

ex. $S^n \subset B^{n+1}$.

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long exact sequence for $(\overline{P_d(G)}, \partial_\infty G)$:

$$\check{H}^{i-1}(\overline{P_d(G)}) = 0 \to \check{H}^{i-1}(\partial_\infty G) \to \check{H}^i(\overline{P_d(G)}, \partial_\infty G) \to \check{H}^i(\overline{P_d(G)}) = 0$$

Local connectivity of the boundary

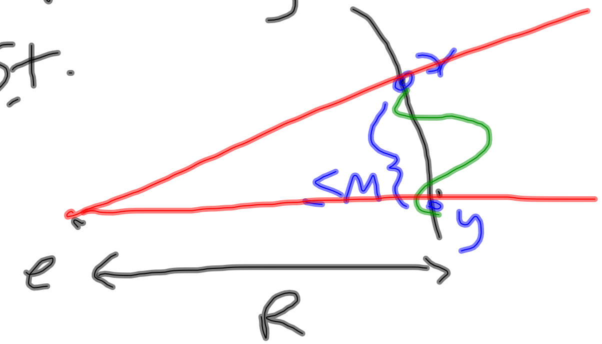
Theorem 9.16 (Bestv.-Mess)

$\partial_\infty G$ has no (global) cut point $\Rightarrow \partial_\infty G$ loc. conn.
(G one ended)

Proof
(1) $\exists$ bi-infinite geodesic within C of any $x \in \Gamma(G)$

② $\forall M$ large $\exists L$

s.t.



$\exists$ path of length $\leq L$ joining such x, y without going into $B(e, R-C)$.

Proof If not, $\exists x_n, y_n$ in $S(e, R_n)$ as above, so that need path length n to join outside $B(e, R-C)$.

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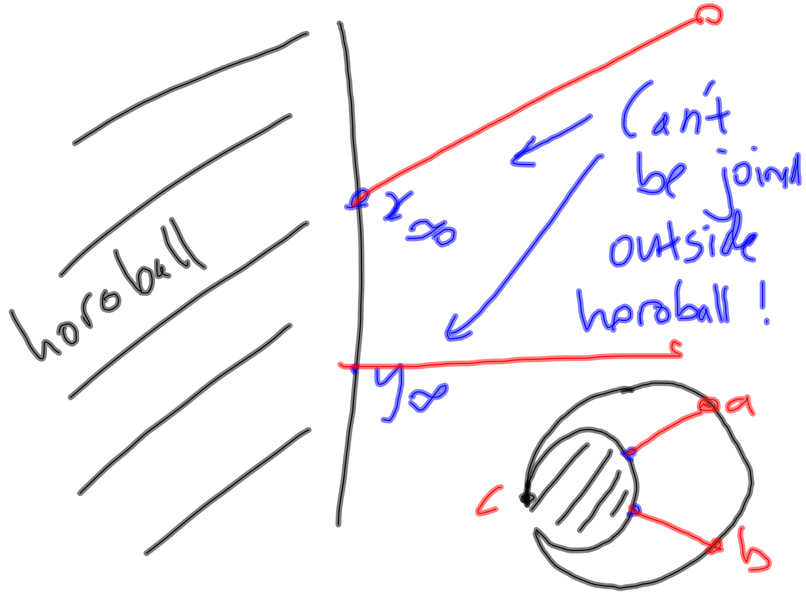
Shift x to base point, take limit



$a, b \in \partial_\infty G$ can't be joined without going through c, which is a cut point ※.

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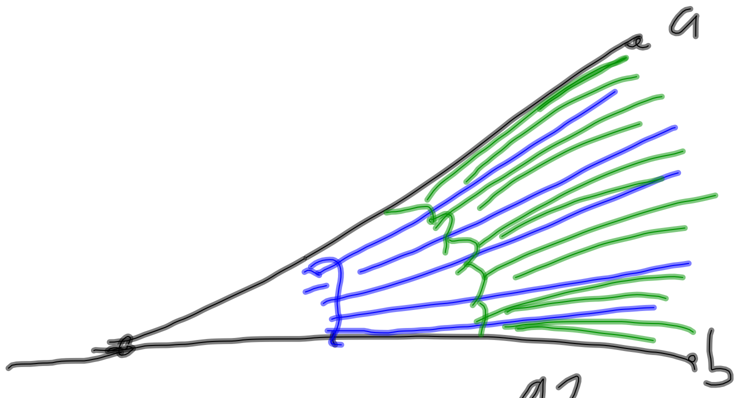
③ local connectivity:
use ② repeatedly:



Theorem 9.17 $\overset{\sim 97}{}$ (Bowditch, Swarup)
G one ended hyp.
$\Rightarrow$ no global cut points
($\Rightarrow \partial_\infty G$ loc. conn.)

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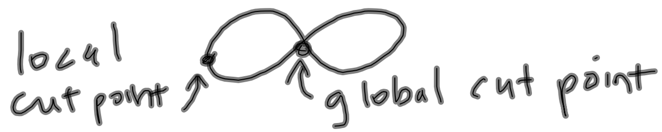
Remark Actually, Bowditch showed a lot more:
local cut points . . .
(i.e. x s.t. $\exists$ open $U \ni x$, U conn., $U \setminus \{x\}$ not conn.)



. . . come in pairs and encode the JSJ decomposition of G (splittings over v. $\mathbb{Z}$ groups)
See also Papasoglu for f.p. G.

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10. Conformal dimension

We saw q.i. type of G

$\Longleftrightarrow$ qM type of $\partial_\infty G$.

Def 10.1 (variation on Pansu '89)

The (Ahlfors regular) conformal dimension of a metric space X is

$$\mathrm{Cdim}(X) = \inf\{\dim_H(Y) : X \overset{qs}{\cong} Y,\ Y \text{ Ahlfors regular}\}.$$

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Remark - Recall X is Ahlfors Q-regular if

$$\mathcal{H}^Q(B(x,r)) \asymp r^Q$$

$\forall x \in X$, $r \leq \operatorname{diam} X$.

- $X \overset{qs}{\cong} Y$ means X, Y are quasi-symmetric

$X \overset{qs}{\cong} Y \Longleftrightarrow X \overset{qM}{\cong} Y$ if X, Y bounded.

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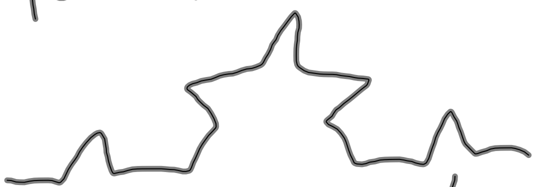
- Why? Try to find best (nice) metric on $\partial_\infty G$.
- e.g. $(X,d) \overset{qm}{\cong} (X, d^\varepsilon)$ $\varepsilon \in (0,1]$,

$$\dim_H(X, d^\varepsilon) = \frac{1}{\varepsilon} \dim_H(X,d)$$

ex $([0,1], d^{\log 3/\log 4})$



$$\dim_H = \frac{\log 4}{\log 3}.$$

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So can make $\dim_H$ of a visual metric on $\partial_\infty G$ as large as we want.

We try to make it as small as possible.

Prop 10.2 X Ahlfors Q-reg.

- $\dim_{top}(X) \le \mathrm{Cdim}(X) \le Q$
- $X \overset{qs}{\cong} Y \Rightarrow \mathrm{Cdim}(X) = \mathrm{Cdim}(Y)$.
- $\mathrm{Cdim}(\partial_\infty G)$, G hyp. is well defined.

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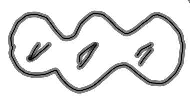
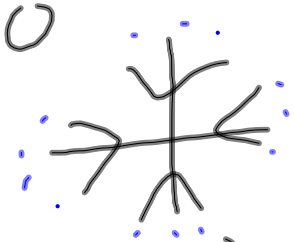
Ex

· $\mathrm{Cdim}(\partial_\infty \pi_1($  $)) = 1$

S^1, usual metric is a visual metric

· $\mathrm{Cdim}\left(\partial_\infty \pi_1\left(\begin{smallmatrix}\text{closed}\\ \text{hyp}\end{smallmatrix} M^n\right)\right) = n-1$

· $\mathrm{Cdim}(\partial_\infty F_2) = 0$



$\partial_\infty F_2$ is a Cantor set (with positive $\dim_H > 0$)

but $\partial_\infty F_2$ has a visual metric that is an ultrametric:

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$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

and d^ε is a metric $\forall \varepsilon \in (0,\infty)$

so let $\varepsilon \to \infty$

$$\mathrm{Cdim}(\partial_\infty F_2) \leq \inf_\varepsilon \dim_H(\partial_\infty F_2, d^\varepsilon) \leq \lim_{\varepsilon\to\infty} \frac{1}{\varepsilon} \dim_H(\partial_\infty F_2, d) = 0.$$

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In fact: G infinite

- $\mathrm{Cdim}(\partial_\infty G) = 0$

$\iff \mathrm{Cdim}(\partial_\infty G) < 1$

$\iff G$ is v. $\mathbb{Z}$ (Cdim attained) or v. free (not attained)

(Uses Cor 6.13)

- $\mathrm{Cdim}(\partial_\infty G) = 1$, attained

$\Rightarrow G \curvearrowright \mathbb{H}^2$ disc. cocpt. isom. (Cor 8.8).

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Prop. 10.3: Z compact Ahlfors Q-regular.

- $\mathcal{E}$ a family of connected sets, $\mathrm{diam}(E) \geq c > 0$ $\forall E \in \mathcal{E}$.

- ν prob. measure on $\mathcal{E}$ s.t. $\exists C, \alpha > 0$ and $\forall B(z,r) \subset Z$, $r \leq \mathrm{diam}(Z)$,

$$\nu(\{E \in \mathcal{E} : E \cap B(z,r) \neq \emptyset\}) \leq C r^\alpha$$

Then: $\mathrm{Cdim}(Z) \geq 1 + \frac{\alpha}{Q - \alpha}$.

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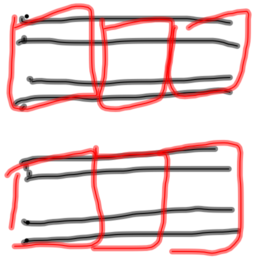
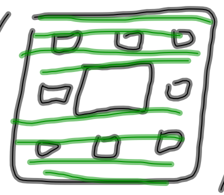
Ex $C = \frac{1}{3}$ Cantor set.

$$\mathrm{Cdim}(C \times [0,1]) = 1 + \frac{\log 2}{\log 3}.$$



as $\alpha = \frac{\log 2}{\log 3}$

$Q = 1 + \alpha$

Ex Cdim() $\geq 1 + \frac{\log 2}{\log 3}$

$\leq \dim_H($  $) = \frac{\log 8}{\log 3}$.

Open Q.

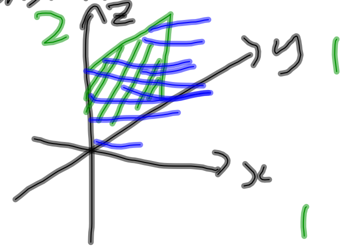
Ex $\mathrm{Cdim}(\partial_\infty \mathbb{C}H^2) = 4$.

CAT(-1) space,
curvature $\in [-4,-1]$

$\partial_\infty \mathbb{C}H^2 \overset{\text{homeo}}{\cong} S^3$,

but looks like $\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}$

with "Carnot metric" of Hausdorff dimension 4.



Lemma gives Cdim ≥ 4.

e.g. $\pi_1\left(\underset{\text{closed}}{\mathbb{H}^4/\Gamma}\right) \overset{q.i.}{\not\simeq} \pi_1\left(\underset{\text{closed}}{\mathbb{CH}^2/\Gamma}\right)$

(See Pansu '89 Annals.)

Proof of Prop. (Pansu, Bourdon, Gromov) (modulo lemmas)

- Suppose $f: Z \to Z'$ is qs (think qM), $\dim_H(Z') < \beta = \frac{Q}{Q-\alpha}$.
- For any $\varepsilon > 0$, $\exists \{B_i' = B(x_i', r_i')\}$

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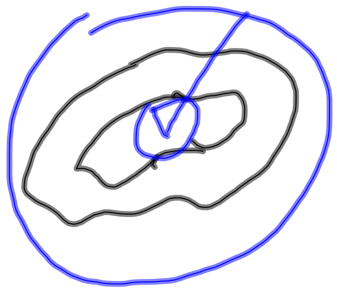
So that

- $\{5B_i'\}$ covers Z',
- $\sum r_i'^{\beta} < \varepsilon$
- $\{B_i'\}$ disjoint.

(Use definition of $\dim_H$ and 5B-lemma.)

- $\exists H > 0$, balls $B_i = B(x_i, r_i)$ s.t. $B_i \subset f^{-1}(B_i') \subset f^{-1}(5B_i') \subset HB_i$



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Let $\chi_i : \mathcal{E} \to \{0,1\}$

$$\chi_i(E) = \begin{cases} 1 & f(E) \cap 5B_i' \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$\exists c' > 0$

$\cdot \forall E \in \mathcal{E}$

$$0 < c' \leq \operatorname{diam} f(E) \leq \sum_i 10 r_i' \chi_i(E)$$

$$\Rightarrow \frac{c'}{10} \leq \int_{\mathcal{E}} \sum_i r_i' \chi_i(E) \, d\nu(E)$$

$$\leq \sum_i r_i' \underbrace{\int_{\mathcal{E}} \chi_i(E) \, d\nu(E)}_{\substack{\nu\{E \in \mathcal{E} : f(E) \cap 5B_i' \neq \emptyset\} \\ \leq \nu\{E \in \mathcal{E} : E \cap HB_i \neq \emptyset\} \\ \leq C(Hr_i)^\alpha}}$$

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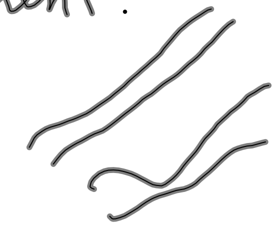
So $\frac{c'}{10} \leq \sum_i r_i' r_i^\alpha$

$$\frac{1}{\beta} + \frac{\alpha}{Q} = 1$$

$$\leq \underbrace{\Big(\sum_i r_i'^\beta\Big)^{1/\beta}}_{< \varepsilon^{1/\beta}} \underbrace{\Big(\sum_i r_i^Q\Big)^{\alpha/Q}}_{\leq C \mathcal{H}^Q(Z) < \infty}$$

※ for ε small enough. $\square$

<u>Remark</u> This is a kind of "modulus" argument.



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Theorem 10.4 (Bonk-Kleiner '05)

G hyperbolic, $\partial_\infty G \overset{\text{homeo}}{\cong} S^2$.

$\mathrm{Cdim}(\partial_\infty G)$ is attained

$\Rightarrow G \curvearrowright \mathbb{H}^3$ disc., cocpt isom.

Rmk i.e. Cannon's conjecture is equivalent to $\partial_\infty G \overset{\text{homeo}}{\cong} S^2$ then $\mathrm{Cdim}(\partial_\infty G)$ is attained.

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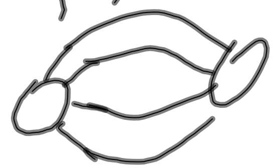
Proof ideas:

1. (Keith-Laakso '03)
   $\mathrm{Cdim}(X)$ attained
   $\Rightarrow$ some "tangent" to X has a curve family of positive modulus

2. Dynamics of $G \curvearrowright \partial_\infty G$
   $\partial_\infty G$ is "Loewner"
   Geometry is controlled by modulus of families of curves



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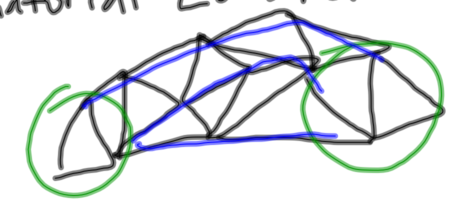
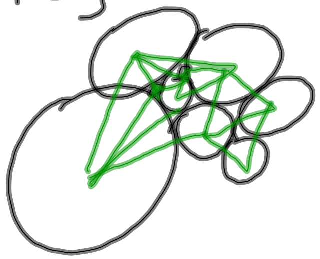
3. Approximate $\partial_\infty G$ by a graph which combinatorially equivalent to a triangulation of S^2. On graph "combinatorial Loewner"



4. Andreev-Koebe-Thurston: realize graph by circle packing:



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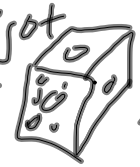
This gives $f_i : \partial_\infty G \to S^2$ on the vertex set, use Loewner to get that $\lim_{i\to\infty} f_i = f : \partial_\infty G \to S^2$ q.s.

$\square$

Remark: Cdim$(\partial_\infty G)$ not always attained (Bourdon-Pajot when $\partial_\infty G \cong$ )

"Then ℓ_p cohomology seems to be interesting. (Bourdon-Kleiner arXiv yesterday)

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