POINCARÉ PROFILES OF GROUPS AND SPACES

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Abstract. We introduce a spectrum of monotone coarse invariants for metric spaces called Poincaré profiles. The two extremes of this spectrum determine the growth of the space, and the separation profile as defined by Benjamini–Schramm–Timár. In this paper we focus on properties of the Poincaré profiles of hyperbolic spaces, and of groups with polynomial growth. One application is that there is a collection of hyperbolic Coxeter groups, indexed by a countable dense subset of $(1, \infty)$, such that G_s does not coarsely embed into G_t whenever $s < t$.

Contents

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1. Introduction

A monotone coarse invariant of a collection of metric spaces $\mathcal{X}$ is a function Λ from $\mathcal{X}$ to a partially ordered set $(P, \leq)$ with the property that $\Lambda(X) \leq \Lambda(Y)$ whenever there is a coarse embedding of X into Y. The asymptotic dimension and the growth function are natural and well-studied examples of such invariants, and a more recent example is the separation profile of bounded degree graphs introduced by Benjamini–Schramm–Timár [BST12].

Here we will only define the Poincaré profiles of graphs; however, our results naturally extend to compactly generated locally compact groups and Riemannian manifolds with bounded geometry. The majority of the paper is presented in a more general context which includes all of these spaces.

Inspired by work of the first author [Hum17], which gives an equivalent definition of the separation profile in terms of the Cheeger constant, for each $p \in [1, \infty]$ we define the **p-Poincaré constant** of a finite graph Γ with vertex set $V\Gamma$ and edge set $E\Gamma$ to be

$$h^p(\Gamma) = \inf \left\{ \frac{||\nabla f||_p}{||f - f_\Gamma||_p} \ : \ f \in \mathrm{Map}(V\Gamma \to \mathbb{R}), \ f \not\equiv f_\Gamma \right\}$$

where $\nabla f(x) = \max\{|f(x) - f(y)| \ : \ xy \in E\Gamma\}$, $||\cdot||_p$ is the usual p-norm in $\mathbb{R}^{|V\Gamma|}$ and f_Γ is the average $|V\Gamma|^{-1} \sum_{x \in V\Gamma} f(x)$. It is worth noting that for bounded degree graphs $h^p(\Gamma)$ is biLipschitz equivalent to $\lambda_{1,p}(\Gamma)^{\frac{1}{p}}$, where $\lambda_{1,p}(\Gamma)$ denotes the smallest non-zero eigenvalue of the p-Laplacian on Γ (see remark 3.8 below).

Now we define the **L^p-Poincaré profile** of an infinite graph X to be

$$\Lambda^p_X(r) = \sup\left\{|\Gamma|\, h^p(\Gamma) \,:\, \Gamma \leq X,\ |V\Gamma| \leq r\right\}.$$

We consider Poincaré profiles up to the natural order $\lesssim$ where $f \lesssim g$ if there exists a constant C such that $f(r) \leq Cg(Cr + C) + C$ for all r, and $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$. Often, the constant C will depend on p; to emphasise this we will use the notations $\lesssim_p$ and $\simeq_p$.

A lower bound on the L^p-Poincaré profile corresponds to a "p-Poincaré inequality" for functions on a finite subgraph of the corresponding size.[1] Poincaré inequalities have been intensively studied, particularly in the case of balls in doubling metric spaces, see [SC02, HK00] and references therein. For finite graphs, there is a vast literature linking Cheeger constants and spectral gaps to such inequalities when $p = 1, 2$, see [Chu97, SC97]. Discrete Poincaré inequalities on balls in metric spaces have been studied before by, for example, Holopainen–Soardi [HS97] and Gill–Lopez [GL15]. Our approach differs in that we are working in a situation where global Poincaré inequalities do not necessarily hold, where measures need not be doubling, and where we have to consider inequalities on all subsets, not just balls.

Our first important result is that these Poincaré profiles are monotone coarse invariants.

Theorem 1. *Let X, Y be graphs with bounded degree. If there is a regular map $r : VX \to VY$, then for all $p \in [1, \infty]$, $\Lambda^p_X \lesssim_p \Lambda^p_Y$.*

A map $r : VX \to VY$ is said to be **regular** if it is Lipschitz and $\sup_{y \in VY} |r^{-1}(y)| < \infty$. In particular every quasi-isometric or coarse embedding is regular. Thus for each p the Poincaré profile is a well-defined coarse invariant of a finitely generated group G.

1.1. **Extremal cases.**

In the cases $p = \infty$ and $p = 1$ the Poincaré profile is easily understood in terms of the growth and separation profile respectively.

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[1] Technically these Poincaré inequalities are Neumann-type, rather than Dirichlet-type Poincaré inequalities which consider only functions which are 0 on the boundary of the subgraph in the ambient space. Dirichlet-type Poincaré inequalities were introduced in [Cou00, Section 7.2] where they are called Sobolev inequalities (see also [Tes08] for a related notion of L^p-isoperimetric profile). They were especially studied for $p = 1$, where they are equivalent to isoperimetric inequalities, and for $p = 2$, where they govern the asymptotic behaviour of the probability of return of the simple symmetric random walk.

Recall the **growth function** of a graph X: $\gamma_X(k)$ is the maximum number of vertices contained in a closed ball $B(x,k)$ of radius k centred at some vertex $x \in VX$. We define the **inverse growth function**: $\kappa_X(r)$ is the smallest positive k such that $\gamma_X(k) > r$.

At one extreme, $p = \infty$, the Poincaré profile detects inverse growth.

Proposition 2. *Let X be a graph. Then* $\Lambda^\infty_X(r) \simeq \sup_{3 \leq s \leq r} \frac{s}{\kappa_X(s)}$.

From this, we may easily deduce Theorem 1 in the case $p = \infty$. At the other extreme we show that the L^1-Poincaré profile is equivalent to the separation profile, as introduced by Benjamini–Schramm–Timár [BST12]. The perspective we adopt of studying Poincaré profiles up to regular maps is inspired by their observation that separation is monotone under regular maps.

We recall that the separation profile of an infinite graph X may be defined by $\mathrm{sep}_X(r) = \max\{|\Gamma|\, h(\Gamma)\}$ where the maximum is taken over all subgraphs Γ of X with at most r vertices, and $h(\Gamma)$ is the Cheeger constant [Hum17].

Proposition 3. *For every bounded degree graph X, $\Lambda^1_X(r) \simeq \mathrm{sep}_X(r)$.*

Remark 4. *The case of $p = 2$ is also natural, being the largest spectral gap among subgraphs of a given size. The spectral gap can be used to bound mixing times of random walks on the subgraph. A related* spectral profile *was considered by Goel–Montenegro–Tetali* [GMT06].

1.2. **Relating profiles.** The following results are classical, and are likely to be easy exercises for experts; for completeness we present full proofs.

Proposition 5. *Let $1 \leq p \leq q < \infty$. There exists a constant $C = C(p,q)$ such that for every bounded degree graph X and every r we have $\Lambda^p_X(r) \leq C\Lambda^q_X(r)$.*

In the opposite direction we have the following.

Proposition 6. *If Γ is a finite graph and $p \in [1,\infty)$, then $h^p(\Gamma)^p \leq 2^p h^1(\Gamma)$.*

Asymptotically this is sharp for balls in the 3-regular tree, as we will see in section 10. Proposition 5 cannot be extended to the case $q = \infty$ since there are bounded degree graphs containing expanders: combining the above propositions with results in [Hum17] we see that for every $p \in [1,\infty)$, $\Lambda^p_X(r)/r \not\to 0$ as $r \to \infty$ if and only if X contains an expander, while a bounded degree graph Y has at most exponential growth, so always satisfies $\Lambda^\infty_Y(r) \lesssim r/\log(r)$.

1.3. **Polynomial growth.** Gromov's celebrated polynomial growth theorem asserts that every finitely generated group with polynomial growth is virtually nilpotent. Results of Bass–Guivarc'h then show that for every group G of polynomial growth there is an integer d such that $\gamma_G(r) \simeq r^d$ [Gro81, Bas72, Gui73].

Theorem 7. *Let G be a finitely generated group such that $\gamma_G(r) \simeq r^d$. Then for all $p \in [1, \infty]$, $\Lambda^p_G(r) \simeq_p r^{\frac{d-1}{d}}$.*

The above result is new even in the case of separation. To prove the lower bound on $\Lambda^p_X(r)$ we calculate a lower bound on the separation profile using a Poincaré inequality and apply Propositions 3 and 5. For the upper bound we use a general result which holds for any bounded degree graph with finite Assouad–Nagata dimension [Hum17, Theorem 1.5].

Recall that by a classical result of Heintze [Hei74], every simply connected negatively curved homogeneous Riemannian manifold M is isometric to a connected Lie group of the form $N \rtimes \mathbb{R}$ equipped with a left-invariant Riemannian metric, where N is a simply connected nilpotent Lie group and the action of $\mathbb{R}$ on N is contracting. We immediately deduce from Theorem 7 that for every $p \in [1, \infty]$, the L^p-Poincaré profile of such a manifold is bounded from below by $r^{\frac{d-1}{d}}$, where d is the homogeneous dimension of N. As a special case of this we deduce the following lower bounds for rank one symmetric spaces.

Corollary 8. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be a real division algebra, and let $X = \mathbb{H}^m_{\mathbb{K}}$ be a rank-one symmetric space for $m \geq 2$ (and $m = 2$ when $\mathbb{K} = \mathbb{O}$). Then, for all $1 \leq p < \infty$, we have $\Lambda^p_X(r) \gtrsim_p r^{(Q-1)/Q}$ where $Q = (m+1)\dim_{\mathbb{R}} \mathbb{K} - 2$.*

For large p this bound is far from optimal as we will see in the next section.

1.4. **Hyperbolic spaces.** We begin by considering the case of an infinite 3-regular tree.

Theorem 9. *Let T be the infinite 3-regular tree. Then $\Lambda^p_T(r) \simeq_p r^{\frac{p-1}{p}}$, for all $p \in [1, \infty)$.*

Note that when $p = \infty$, $\Lambda^p_T(r) \simeq r/\log(r)$ by Proposition 6.1. Using Theorem 9, together with results of Cornulier–Tessera and Benjamini–Schramm on embeddings of trees into solvable groups with exponential growth and non-amenable groups respectively [dCT08, BS97] we obtain the following corollary.

Corollary 10. *Let G be a finitely generated solvable group with exponential growth or a finitely generated infinite non-amenable group. Then for all $p \in [1,\infty)$, $\Lambda^p_G(r) \gtrsim_p r^{\frac{p-1}{p}}$.*

We continue with a striking result which suggests a connection between conformal dimension of boundaries and a phase transition in the Poincaré profiles of hyperbolic groups.

Our key examples are real hyperbolic spaces, and a collection of Fuchsian groups $G_{m,n} = \langle s_1,\ldots,s_m \mid s_i^n, [s_1,s_2],\ldots,[s_{m-1},s_m],[s_m,s_1]\rangle$, $m \geq 5, n \geq 3$ studied by Bourdon and Bourdon–Pajot via associated Fuchsian buildings $\Delta_{m,n}$ [Bou97, BP99]. When m is even these groups are virtually torsion free, and commensurable to hyperbolic Coxeter groups [NTV16].

Theorem 11. *Let X be either a real hyperbolic space $\mathbb{H}^k_{\mathbb{R}}$ or a Bourdon-Pajot group $G_{m,n}$ and let Q be the Ahlfors regular conformal dimension of the boundary of X. Then*

$$\Lambda^p_X(r) \simeq \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } p < Q \\ r^{\frac{p-1}{p}} & \text{if } p > Q \end{cases}$$

If $p = Q$, then

$$r^{\frac{Q-1}{Q}} \lesssim \Lambda^Q_{G_{m,n}}(r) \lesssim r^{\frac{Q-1}{Q}} \log(r)^{\frac{1}{Q}}, \text{ and}$$
$$\Lambda^Q_{\mathbb{H}^k_{\mathbb{R}}}(r) \simeq r^{\frac{Q-1}{Q}} \log(r)^{\frac{1}{Q}}.$$

In the case of $\Lambda^1_{\mathbb{H}^k_{\mathbb{R}}}$, these sharp bounds for the separation profile appear in [BST12]. It is interesting to note that uniform lattices G in $PSL(2,\mathbb{R})$ satisfy $\Lambda^1_G(r) \simeq \log(r)$ and $\Lambda^p_G(r) \simeq r^{\frac{p-1}{p}}$ for all $p > 1$, while non-uniform lattices H satisfy $\Lambda^1_H(r) \simeq r^{\frac{p-1}{p}}$ for all $p \geq 1$. We have no other examples of this distinction between uniform and non-uniform lattices for any other p or for any groups of higher rank.

We do not define the conformal dimension here, but comment that for real hyperbolic space of dimension k we have $Q = k-1$ and for the groups $G_{m,n}$ we have $Q = 1 + \log(n-1)/\operatorname{arccosh}((m-2)/m)$, which can take a dense set of values in $(1,\infty)$.

So by Theorem 11 we find a new collection of functions which can be obtained as separation profiles of finitely generated groups:

Corollary 12. *There exists a dense subset A of $(0,1)$ such that for all $\alpha \in A$ there is a hyperbolic group G_α with $\operatorname{sep}_{G_\alpha}(r) \simeq r^\alpha$.*

The upper bound in Theorem 11 is obtained by constructing specific functions on the boundary using the abundance of hyperplanes which the above spaces admit. The lower bound in Theorem 11 for real hyperbolic spaces $\mathbb{H}^k_{\mathbb{R}}$ and $p < k-1$ comes from a coarsely embedded copy of $\mathbb{R}^{k-1}$. Alternatively, and for Bourdon–Pajot groups, we use the following more general result.

Theorem 13. *Suppose that X is a visual Gromov hyperbolic graph with a metric $\rho \in \mathcal{C}_X$ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a p-Poincaré inequality. Then for all $q \geq p$, $\Lambda^q_X(r) \gtrsim r^{(Q-1)/Q}$.*

Here a "p-Poincaré inequality" is in the sense of Heinonen–Koskela [HK98], namely an analytic property of the compact metric space $\partial_\infty X$. Such inequalities hold on boundaries of rank-one symmetric spaces, see e.g. [Jer86, HK98, MT10], so we can apply this lower bound to obtain an alternative proof of Corollary 8. For Bourdon–Pajot groups, the Poincaré inequalities are constructed in [BP99].

The sharp lower bound on $\Lambda^{k-1}_{\mathbb{H}^k_{\mathbb{R}}}$ comes from showing a suitable Poincaré inequality on balls in $\mathbb{H}^k_{\mathbb{R}}$. It is interesting to observe that for $p < k-1$, $p = k-1$, and $p > k-1$, the sharp lower bounds on $\Lambda^{k-1}_{\mathbb{H}^k_{\mathbb{R}}}$ are realised by embedded spheres, balls and trees respectively.

Finally, Theorems 9 and 11, together with the embedding theorem of Bonk–Schramm [BS00], imply that for every hyperbolic group G there is some p_0 such that for all $p > p_0$, we have $\Lambda^p_G(r) \simeq r^{\frac{p-1}{p}}$. The relationship between the infimal such p_0 and the conformal dimension of the boundary of G is one of the most intriguing aspects of these profiles.

1.5. **Consequences.** The key purpose of a monotone coarse invariant is to be able to distinguish situations in which one space cannot be coarsely embedded into another. There are few general tools to do this; asymptotic dimension is one and growth (or equivalently, the L^∞-Poincaré profile) is another. Here we present and discuss some results of this form which cannot be obtained by studying growth and/or asymptotic dimension.

Corollary 14. *If there is a coarse embedding of $\mathbb{H}^k_{\mathbb{C}}$ into $\mathbb{H}^l_{\mathbb{R}}$, then $l > 2k$. Likewise, if there is a coarse embedding of $\mathbb{H}^k_{\mathbb{H}}$ into $\mathbb{H}^l_{\mathbb{R}}$, then $l > 4k+2$.*

Proof. Consider Λ^1. $\square$

To prove the analogous result for quasi-isometric embeddings, one can use the conformal dimension of the boundary, however, a coarse embedding does not necessarily induce a well-defined map between boundaries [BR13] so this approach cannot be expected to work. Using asymptotic dimension as an invariant one could only deduce that $l \geq 2k$ in the first case and $l \geq 4k$ in the second.

By [BS00], every hyperbolic group quasi-isometrically embeds into some $\mathbb{H}^k_{\mathbb{R}}$. A natural obstruction to a coarse embedding $G_k \to \mathbb{H}^k_{\mathbb{R}}$ is that the asymptotic dimension of G_k is greater than k. Poincaré profiles provide a different obstruction.

Corollary 15. *For every k there is a hyperbolic group G_k of asymptotic dimension 2 which does not coarsely embed into $\mathbb{H}^k_{\mathbb{R}}$.*

We can take G_k to be a Bourdon–Pajot group $G_{m(k),5}$ for some appropriately chosen $m(k)$ and apply Theorem 11.

It is in general very difficult to prove a statement of the form "a hyperbolic group H is not isomorphic to a subgroup of a hyperbolic group G". One may use torsion or asymptotic dimension in certain cases, here we show that the Poincaré profiles can exclude subgroups when the two methods listed above fail.

Corollary 16. *There exists a collection of (torsion-free) hyperbolic groups $(G_q)_{q\in\mathbb{Q}}$ with asymptotic dimension 2 such that whenever $i < j$ there is no coarse embedding from G_i to G_j. In particular, G_i is not virtually a subgroup of G_j.*

Indeed, the Bourdon–Pajot groups with even m are virtually torsion-free so we may choose the G_q in Corollary 16 to be torsion-free. By results of Gersten, finitely presented subgroups of hyperbolic groups with cohomological dimension 2 (which equals the asymptotic dimension for torsion-free hyperbolic groups [BM91, BL07]) are hyperbolic but not necessarily quasi-convex. The fact that G_j is not a quasi-convex subgroup of G_i is immediate by considering the conformal dimension.

Remark 17. *By a recent result of Pansu* [Pan16]*, if a hyperbolic group H coarsely embeds into a hyperbolic group G, then the "L^p-cohomological dimension" of H is less than or equal to the conformal dimension of the boundary of G. In the cases of the Bourdon–Pajot buildings and rank one symmetric spaces, these two numbers turn out to coincide. This provides an alternative proof of Corollaries 14, 15 and 16.*

1.6. **Structure of the paper.** The paper splits roughly into three parts. The first part introduces Poincaré profiles as monotone coarse invariants. After introducing our notations and fixing the class of metric measure spaces under consideration, we present the more general definition of Poincaré constants in Section 3 and explain some basic properties. We then introduce Poincaré profiles and prove Theorem 1 in Sections 4 and 5 respectively.

The second part deals with relationships between Poincaré profiles. The descriptions of extremal profiles (Propositions 2 and 3) are proved in Section 6, and the dependence on p (Propostions 5 and 6) is discussed in Section 7.

The final part is dedicated to calculating profiles using the technology developed in the rest of the paper.

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2. Notation and framework

We first introduce notation to be used throughout the paper.

Suppose $f, g : S \to [0,\infty)$ where $S = \mathbb{N}$ or $S = [0,\infty)$. We write $f \preceq_{u,v,\dots} g$ if there exists a constant $C > 0$ depending only on $u, v, \dots$ such that $f(x) \leq Cg(x)$ for all $x \in S$. If $f \preceq_{u,v,\dots} g$ and $g \preceq_{u,v,\dots} f$ then we write $f \asymp_{u,v,\dots} g$. We drop the subscripts if the constants are understood.

We write $f \lesssim_{u,v,\dots} g$ if there exists a constant $C > 0$ depending only on $u, v, \dots$ such that $f(x) \leq Cg(Cx + C) + C$ for all $x \in S$; similarly, we write $f \simeq_{u,v,\dots} g$ if $f \lesssim_{u,v,\dots} g$ and $g \lesssim_{u,v,\dots} f$.

Given a subset A of a metric space (X, d) and some $M \geq 0$ we define the closed M-neighbourhood of A to be

$$[A]_M = \{x \in X \,:\, d(x, A) \leq M\}.$$

Given a point $x \in X$ and $r \geq 0$ we denote by $B(x, r)$ the closed metric ball of radius r centered at x.

Let (Z, ν) be a measure space with positive finite measure. We denote the averaged integral by

$$⨍_Z f d\nu = \frac{1}{\nu(Z)} \int_Z f d\nu.$$

Given a function $f \in L^p(X, \mu)$, another measure μ' such that $f \in L^p(X, \mu')$ and a measurable subset $Z \subseteq X$ we write

$$||f||_{p,\mu'} = \left(\int_X |f(z)|^p \, d\mu'(z)\right)^{\frac{1}{p}} \quad \text{and}$$

$$||f||_{Z,p} = \left(\int_Z |f(z)|^p \, d\mu(z)\right)^{\frac{1}{p}}.$$

The L^∞ norms $||\cdot||_{\infty,\mu'}$ and $||\cdot||_{Z,\infty}$ are defined analogously.

Given a graph $\Gamma = (V\Gamma, E\Gamma)$ and a subset $A \subset V\Gamma$, the full (or induced) subgraph of Γ with vertex set A is the graph with vertex set A and edge set $\{xy \in E : x, y \in A\}$.

The purpose of the remainder of this section is to introduce the class of spaces we will consider in this paper.

Definition 2.1. A **metric measure space** is a triple (X, d, μ) where μ is a locally finite Borel measure on a complete, separable metric space (X, d).

The key examples are: graphs of bounded degree, Riemannian manifolds with bounded geometry and compactly generated locally compact groups, so we will make the following standing assumptions.

We will assume throughout the paper that any metric measure space (X, d, μ) satisfies the following properties:

- X has **bounded packing on large scales**[2]: if there exists $r_0 \geq 0$ such that for all $r \geq r_0$, there exists $K_r > 0$ such that
$$\forall x \in X,\ \mu(B(x, 2r)) \leq K_r \mu(B(x, r)).$$
We then say that X has **bounded packing on scales** $\geq r_0$.
- X is k-**geodesic** for some $k > 0$: for every pair of points $x, y \in X$ there is a sequence $x = x_0, \dots, x_n = y$ such that $d(x_{i-1}, x_i) \leq k$ for all i and $d(x, y) = \sum_{i=1}^n d(x_{i-1}, x_i)$.

Up to rescaling the metric and/or the measure we will assume that X is 1-geodesic and has bounded packing on scales $\geq r_0 = 1$.

A subspace $Z \subset X$ is always assumed to be **1-thick** (a union of closed balls of radius 1), so in particular it has positive measure. We equip Z with the subspace measure and the induced 1-distance
$$d(z, z') = \inf \left\{ \sum_{i=1}^{n} d(z_{i-1}, z_i) \right\}$$
where the infimum is taken over all sequences $z = z_0, \dots, z_n = z'$, such that each $z_i \in Z$ and $d(z_{i-1}, z_i) \leq 1$.

Note that (as in the case of a disconnected subgraph) the induced 1-distance will take values in $[0, \infty]$.

Remark 2.2. In the case of (the vertex set of) a bounded degree graph X, d is the shortest path metric and μ is the (vertex) counting measure. Subspaces Z are (vertex sets of) 1-thick subgraphs equipped with the vertex counting measure and their own shortest path metric (the induced 1-distance).

[2]If $r_0 = 0$, then we simply say that X has bounded packing.

In a locally compact group G with compact generating set K, we equip G with a Haar measure (which is unique up to scaling) and the word metric $d = d_K$.

The reason for working with thick sets is justified by the following easy lemma (see [Tes08, Lemma 8.4]).

Lemma 2.3. *Assume X has bounded packing on scales $\geq r_0$, and let $A \subset X$ be r-thick for some $r \geq r_0$. Then for all $u > 0$,*

$$\mu([A]_u) \preceq_u \mu(A).$$

3. Poincaré constants

Let (X, d) be a metric space and let $a > 0$. Given a measurable function $f : X \to \mathbb{R}$, we define its **upper gradient at scale** a to be

$$|\nabla_a f|(x) = \sup_{y,y' \in B(x,a)} |f(y) - f(y')|.$$

Remark 3.1. We have slightly modified the notation from [Tes08], where the upper gradient was referred to as the "local norm of the gradient" and was denoted by $|\nabla f|_a$. The changes in this paper are for brevity; in what follows $\||\nabla_a f|\|_p$ will simply be denoted by $\|\nabla_a f\|_p$.

Definition 3.2. Let (Z, d, ν) be a metric measure space with finite measure and fix a scale $a > 0$. We define the **L^p-Poincaré constant at scale** a of Z to be

$$h^p_a(Z) = \inf_f \frac{\|\nabla_a f\|_p}{\|f\|_p},$$

where the infimum is taken over all $f \in L^p(Z, \nu)$ such that $f_Z := ⨍_Z f d\nu = 0$ and $f \not\equiv 0$. We adopt the convention that $h^p_a(Z) = 0$ whenever $\nu(Z) = 0$.

Before continuing we list some basic properties of the Poincaré constant.

Lemma 3.3. *Let (Z, d, ν) be a metric measure space with finite measure.*

(i) Let $\theta : (0, \infty) \to (0, \infty)$ be a non-decreasing function, and let (Z', d', ν') be a metric measure space such that $(Z, \nu) = (Z', \nu')$, and $d'(z_1, z_2) \leq \theta(d(z_1, z_2))$ for all z_1, z_2. Then for all $a > 0$,

$$h^p_a(Z) \leq h^p_{\theta(a)}(Z').$$

(ii) Let (Z', d', ν') be a metric measure space where $(Z, d) = (Z', d')$ and there exists some $M \geq 1$ such that $M^{-1}\nu(A) \leq \nu'(A) \leq M\nu(A)$ for every measurable $A \subseteq Z$. Then for all $a > 0$,

$$h_a^p(Z') \leq 2M^{2/p} h_a^p(Z).$$

Proof. Part (i) is immediate. For part (ii), let $f : X \to \mathbb{R}$ be a measurable function such that $⨍ f d\mu = 0$ and let $m = ⨍ f d\nu'$. We see that

$$\|f\|_{p,\nu} \leq 2\|f - m\|_{p,\nu} \leq 2M^{1/p}\|f - m\|_{p,\nu'}$$

The first inequality above is the $C = -m$ case of inequality (3.5) proved in Lemma 3.4. On the other hand

$$\|\nabla f\|_{p,\nu} \leq M^{1/p}\|\nabla f\|_{p,\nu'},$$

so we are done. □

To obtain a sensible definition of the L^p-Poincaré constant it is necessary to only consider functions whose average is zero and to choose a notion of gradient. In both cases there are multiple ways to do this.

3.1. **Choice of average.** Given a measure space (Z, ν) with finite positive measure, there are multiple ways to define the "average" of a measurable function $f : (Z, \nu) \to \mathbb{R}$:

(1) the **average** $f_Z = ⨍_Z f d\nu$,
(2) a **median** m_f: any value such that $\nu(\{f < m_f\}) \leq \nu(Z)/2$ and $\nu(\{f > m_f\}) \leq \nu(Z)/2$,
(3) a p-**energy minimizer**: any value c_p such that $\inf_c ||f - c||_p$ is attained for $c = c_p$.

There is a simple comparison between the average and any energy minimizer, so choosing (1) or (3) gives comparable Poincaré constants.

Lemma 3.4. *Let (Z, ν) be a measure space with finite positive measure, and let $f : (Z, \nu) \to \mathbb{R}$ be a measurable function. For every $p \in [1, \infty)$ we have $||f - c_p||_p \leq ||f - f_Z||_p \leq 2\,||f - c_p||_p$.*

Proof. For any $C \in \mathbb{R}$ we have

$$\begin{aligned}
\|f - f_Z\|_p &\leq \|f + C\|_p + \|C + f_Z\|_p \\
&= \|f + C\|_p + \nu(Z)^{1/p}\left|C + \frac{1}{\nu(Z)}\int_Z f(z)d\nu(z)\right| \\
&\leq \|f + C\|_p + \nu(Z)^{-1+1/p}\int_Z |C + f(z)|d\nu(z) \\
&\leq \|f + C\|_p + \nu(Z)^{-1+1/p}\|C + f\|_p\|1\|_{p/(p-1)} \\
&= 2\|f + C\|_p.
\end{aligned} \tag{3.5}$$

In addition, if $C = c_p$, then $\|f - c_p\|_p \leq \|f - f_Z\|_p$ by definition. $\square$

In the case of $p = 1$, this lemma combines with the following to show that taking either averages or medians will yield comparable Poincaré constants.

Lemma 3.6. *Let (Z, ν) be a measure space with finite positive measure ν and let $f : Z \to \mathbb{R}$ be a measurable function. Then a value c is a 1-energy minimizer c_1 of f if and only if it is a median m_f.*

Proof. For $c' > c$, a calculation gives:

$$\|f - c'\|_1 - \|f - c\|_1 = (c' - c)\big(\nu(\{f \leq c\}) - \nu(\{f \geq c'\})\big) + \int_{\{c<f<c'\}} (c + c' - 2f) d\nu. \tag{3.7}$$

If c minimizes $\|f - c\|_1$, (3.7) gives

$$0 \leq (c' - c)\big(\nu(\{f \leq c\}) - \nu(\{f \geq c'\})\big) + (c' - c)\nu(\{c < f < c'\}).$$

Letting $c' \to c$, we get $\nu(\{f > c\}) \leq \nu(\{f \leq c\})$. The same argument applied to $-f$ gives $\nu(\{f < c\}) \leq \nu(\{f \geq c\})$, so c is a median of f.

Conversely, if c is a median for f, (3.7) gives

$$\begin{aligned}
&\|f - c'\|_1 - \|f - c\|_1 \\
&\quad = (c' - c)\left(\nu(\{f \leq c\}) - \nu(\{f > c\})\right) \\
&\qquad\quad + (c' - c)\nu(\{c < f < c'\}) + \int_{\{c<f<c'\}} (c + c' - 2f) d\nu. \\
&\quad \geq (c' - c)(\tfrac{1}{2}\nu(Z) - \tfrac{1}{2}\nu(Z)) + \int_{\{c<f<c'\}} (2c' - 2f) \geq 0,
\end{aligned}$$

so increasing c cannot lower $\|f - c\|_1$. The same argument applied to $-f$ gives that the median c is also a minimizer for $\|f - c\|_1$. $\square$

Remark 3.8. For Γ a finite graph of constant degree d, $\lambda_{1,p}(\Gamma)$, the first eigenvalue of the p-Laplacian on Γ, may be calculated to be the infimum of $\left(\sum_{xy \in E\Gamma} |f(x) - f(y)|^p\right) / \left(\sum_{x \in V\Gamma} |f(x) - c_p(f)|^p d\right)$ over all non-constant f with $c_p(f)$ the energy minimizer of f (see [Bou12]). Thus by Lemma 3.4 we have $\lambda_{1,p}(\Gamma) \asymp h^p(\Gamma)^{1/p}$.

3.2. **Comparison with Lipschitz gradient.** Classical Poincaré inequalities on balls in $\mathbb{R}^n$ involve the L^p-norms of the usual gradient vector ∇f. For general metric spaces this makes no sense, but it is possible to define an analogue of the point-wise norm $|\nabla f|$. Given this, one can define what it means for a metric measure space to satisfy a Poincaré inequality in this infinitesimal sense (see Section 11).

Let (Z, d, ν) be a metric measure space with finite (positive) measure. We define the Lipschitz gradient to be

$$\mathrm{Lip}_x(f) = \limsup_{h \to 0} \sup_{y \in B(x,h)} \frac{|f(x) - f(y)|}{h}.$$

Given a metric space (Z, d) we can define

$$h^p_{\mathrm{Lip}}(Z) = \inf \frac{||\mathrm{Lip}_x(f)||_p}{||f||_p}$$

where the infimum is taken over all non-constant Lipschitz functions $f : Z \to \mathbb{R}$ with average 0.

Following §10.2 and §10.3 from [Tes08], one can show that—under suitable assumptions on a metric measure space—the Poincaré constant relative to the Lipschitz norm (for Lipschitz functions) is equivalent to the Poincaré constant with respect to the gradient at some fixed scale $\alpha > 0$.

Here, we will focus on one direction (the only one required in the paper, namely in the proof of Theorem 11.1) which relies solely on a bounded packing assumption:

Proposition 3.9. *Let (Z, d, ν) be a metric measure space with finite measure ≥ 1, let $a > 0$ and let $C \in \mathbb{N}$. Assume that for all $x \in Z$, $\nu(B(x, 2a)) \leq C\nu(B(x, a/2))$. Then,*

$$h^p_{\mathrm{Lip}}(Z) \preceq_{C,a,p} h^p_a(Z).$$

Proof. We first need the following lemma:

Lemma 3.10. *Assume $h^p_a(Z) \leq 1/8$. Let $(P_x)_x$ be a family of probability measures on Z, such that P_x is supported in $B(x, a)$ for every $x \in Z$. Then there exists $f \in L^\infty$ such that*

$$\frac{\|\nabla_a f\|_p}{\|Pf - (Pf)_Z\|_p} \leq 4h^p_a(Z),$$

where $Pf(x) := \int f dP_x$.

Proof. We start with f with average 0, $f_Z = 0$, such that

$$\frac{\|\nabla_a f\|_p}{\|f\|_p} \leq 2h^p_a(Z) \leq \frac{1}{4}.$$

Observe that

$$\|f - Pf\|_p \leq \|\nabla_a f\|_p \leq \tfrac{1}{4}\|f\|_p,$$

from which we deduce that

$$|(Pf)_Z| = \left| \fint_Z Pf \right| = \left| \fint_Z Pf - f \right| \leq \frac{\|f - Pf\|_p}{\nu(Z)^{1/p}} \leq \frac{\|f\|_p}{4\nu(Z)^{1/p}}.$$

So $\|(Pf)_Z\|_p \leq \frac{1}{4}\|f\|_p$ and then we deduce by the triangle inequality that

$$\|Pf - (Pf)_Z\|_p \geq \frac{\|f\|_p}{2}. \qquad \square$$

The rest of the proof of the proposition is similar to that of [Tes08, Theorem 10.9]. For the convenience of the reader we sketch it. Define a 1-Lipschitz map $\theta : Z \times Z \to \mathbb{R}_+$ by $\theta(x, y) = d(y, B(x, a)^c)$. For $U \subset Z$ write

$$P_x(U) = \int_U \frac{\theta(x, y)}{K(x)} d\nu(y),$$

where $K(x) = \int_{B(x,a)} \theta(x, z) d\nu(z)$. Note that $K(x) \asymp_C \nu(B(x, a))$, and that by assumption, $\nu(B(x, a)) \asymp_C \nu(B(y, a))$ as soon as $d(x, y) \leq a/2$. Since θ is 1-Lipschitz with respect to x, we see that for all $f \in L^\infty(Z)$,

$$\mathrm{Lip}_x(Pf) \preceq_C |\nabla_a f|.$$

Note that if $h^p_a(Z) > \frac{1}{8}$, then the statement of the proposition follows trivially. Hence we can assume that $h^p_a(X) \leq \frac{1}{8}$. By Lemma 3.10 we deduce that there exists some function f such that

$$\frac{\|\mathrm{Lip}_x(Pf)\|_p}{\|Pf - (Pf)_Z\|_p} \preceq_C h^p_a(Z).$$

Hence the proposition follows. $\square$

4. Poincaré profiles for metric measure spaces

Our goal in this section is to generalise the Poincaré profile to the class of metric measure spaces defined in Section 2.

Definition 4.1. Let (X, d, μ) be a metric measure space satisfying our standing assumptions, and fix some number $a \geq 2$. We define the **L^p-Poincaré profile** $\Lambda^p_{X,a}(r)$ of X at scale a to be the supremum of $\mu(A)h^p_a(A)$ over all subspaces $A \subset X$ satisfying $\mu(A) \leq r$. If no such subspace exists, define $\Lambda^p_{X,a}(r) = 0$.

Recall that by assumption, we only consider 1-thick subsets of X to be subspaces.

We first prove that the Poincaré profile does not actually depend on the choice of a.

Proposition 4.2. *Assume that (Z, d, ν) is a finite metric measure space. Then for all $a \geq 2$ and all $p \in [1, \infty)$ we have*

$$h^p_a(Z) \asymp_a h^p_2(Z).$$

Proof. We claim that for any $t \geq 0$,

$$\nu\left(\{|\nabla_a f| \geq t\}\right) \preceq_a \nu\left(\left\{|\nabla_2 f| \geq \frac{t}{5a}\right\}\right), \tag{4.3}$$

and $\nu(\{|\nabla_2 f| \geq t\}) \leq \nu(\{|\nabla_a f| \geq t\})$. Together these inequalities immediately imply the proposition. The second inequality is obvious. Let $z \in Z$, and let $x, y \in B(z, a)$. Then one can easily check that our standing assumption implies that there exists a 1-path $x = x_0, \ldots, x_n = y$ within $B(z, a)$ such that $n \leq 5a$. By the triangle inequality, this means that for at least one $1 \leq i \leq n$, $|f(x_i) - f(x_{i-1})| \geq \frac{1}{5a}|f(x) - f(y)|$. Now for all $z' \in B(x_i, 1)$ this implies that $|\nabla_2 f|(z') \geq \frac{1}{5a}|f(x) - f(y)|$. Hence there is a 1-thick subset which is $2a$-dense in the set $\{|\nabla_a f| \geq t)\}$ on which $|\nabla_2 f|(z') \geq \frac{t}{5a}$. Thus, the left-hand inequality in (4.3) follows from Lemma 2.3. □

Corollary 4.4. *Assume that (X, d, μ) satisfies our standing assumptions. Then for all $a, a' \geq 2$ and all $p \in [1, \infty)$ we have*

$$\Lambda^p_{X,a} \asymp_{a,a'} \Lambda^p_{X,a'}.$$

Moreover, by Lemma 3.3, choosing a biLipschitz equivalent metric and/or measure does not affect the L^p-Poincaré profile $\Lambda^p_{X,a}$ for sufficiently large a (up to $\simeq$). In particular this means that for a compactly generated locally compact group, the L^p-Poincaré profile does not depend on the choice of Haar measure or on the choice of compact generating set.

In light of Corollary 4.4, we now refer to Λ^p_X as the L^p-Poincaré of X, without the need to specify a scale.

5. Regular maps and large scale equivalence

The goal of this section is to prove Theorem 1. Firstly, we formally introduce the notion of a coarse regular map and prove that the definition coincides with regular maps for bounded degree graphs.

5.1. Regular maps.

We introduce a natural generalisation of a regular map between graphs suited to the context of metric measure spaces. In this section we show that Poincaré profiles are monotone non-decreasing under coarse regular maps.

Definition 5.1. A map $F : (X, d, \mu) \to (X', d', \mu')$ is called **coarsely regular** if it satisfies the following properties:

(i) F is coarse Lipschitz: there exists an increasing function ρ_+ such that for all $x, y \in X$,

$$d(F(x), F(y)) \leq \rho_+(d(x, y));$$

(ii) F is coarsely measure preserving: there exists δ_0 such that for all $\delta \geq \delta_0$ and for all (1-thick) subspaces $A \subset X$,

$$\mu([A]_\delta) \asymp_\delta \mu'([F(A)]_\delta) \asymp_\delta \mu([F^{-1}(F(A))]_\delta).$$

The *parameters* of F are the constant δ_0 as well as the function ρ_+.

Remark 5.2. Coarse regular maps between spaces with bounded packing on large scales are stable under composition.

In applications, coarsely regular maps often are embeddings of the following kind.

Definition 5.3. A coarse regular map $F : (X, d, \mu) \to (Y, d, \nu)$ is called a **large-scale embedding** if it is also a coarse embedding; there exists a function ρ_- such that $\lim_{t\to\infty} \rho_-(t) = \infty$ and for all $x, y \in X$,

$$\rho_-(d(x, y)) \leq d(F(x), F(y)).$$

If, in addition, $[F(X)]_C = Y$ for some $C \geq 0$ (in other words, if F is a coarse equivalence), then F is called a **large-scale equivalence**.

It is easy to see that the relation "there exists a large scale equivalence from X to Y" is an equivalence relation among metric measure spaces.

Lemma 5.4. *Let X, X' be simplicial graphs of bounded degree equipped with the shortest path metrics and vertex counting measures. A map $F : VX \to VX'$ is regular in the sense of* [BST12] *if and only if it is coarsely regular as a map between metric measure spaces.*

Proof. If F is regular, then by definition there exists a constant K such that F is K-Lipschitz, the image of every set of measure m has measure at most m and the pre-image of every set of measure m has measure at most Km. Since X and X' have bounded degree, F is coarsely regular.

Suppose F is coarsely regular, then it is $\rho_+(1)$-Lipschitz. Fix some suitable δ_0, let $x' = F(x) \in F(VX)$ and notice that the (1-thick) subspace $A = [x]_1$ satisfies

$$\left|F^{-1}(x')\right| \leq \left|[F^{-1}(F(A))]_\delta\right| \preceq_\delta |[A]_\delta| \leq |[x]_{\delta+1}| \preceq_\delta 1.$$

Thus, F is regular. □

The following proposition is the main goal of this section, and will be proved in §5.2.

Proposition 5.5. *Let $F : X \to X'$ be a coarsely regular map between metric measure spaces which satisfy our standing assumptions. Then for all $p \in [1, \infty)$,*

$$\Lambda^p_X \lesssim_p \Lambda^p_{X'}.$$

Theorem 1 follows immediately from Lemma 5.4 and this proposition. Note that by Proposition 4.2 it suffices to prove $\Lambda^p_{X,a} \lesssim_p \Lambda^p_{X',a'}$. for some $a, a' \geq 2$.

An important consequence of Proposition 5.5 is the following.

Proposition 5.6. *Let G and H be compactly generated locally compact groups, and let $\phi : H \to G$ be a proper continuous morphism (i.e.* $\ker \phi$ *is compact and $\phi(H)$ is a closed subgroup). We assume that both G and H are equipped with left-invariant Haar measures and word metrics with respect to some compact symmetric generating sets. Then, for all $p \in [1, \infty)$, $\Lambda^p_H \lesssim_p \Lambda^p_G$. If $\phi(H)$ is co-compact then $\Lambda^p_H \simeq_p \Lambda^p_G$.*

Proof. The morphism ϕ is a large-scale embedding hence it is coarsely regular. If $\phi(H)$ is co-compact then ϕ is a large-scale equivalence. The result then follows from Proposition 5.5. □

5.2. **Proof of Theorem 1.** The argument behind the proof is as follows: given a coarse regular map $F : X \to X'$ which is ρ_+-coarse Lipschitz and a subspace $Z \subseteq X$, we define $M = \max\{\rho_+(1), \delta_0\}$ and build metric measure space discretizations Y of Z and Y' of the 1-thick subspace $[[F(Z)]_M]_1$. By the definition of a coarse regular map and Lemma 2.3,

$$\mu_X(Z) \asymp_M \mu_X([Z]_M) \asymp_M \mu_{X'}([[F(Z)]_M]_1).$$

We then show that the process of taking a discretization yields spaces with equal measure and comparable Poincaré constants, and finally prove that Y and Y' have comparable Poincaré constants.

The first step of the proof consists in constructing discretizations of our spaces. We fix some $b \geq M$ (which we refer to as the *discretization parameter*). We let $Y \subset Z$ be a maximal $3b$-separated subset of Z. By maximality Z is covered by the union of balls $\bigcup_{y \in Y} B(y, 9b)$. We pick (measurably) a set A_y for each $y \in Y$ with the following properties:

- $B(y, b) \subset A_y \subset B(y, 9b)$;
- $(A_y)_{y \in Y}$ forms a measurable partition of Z.

We equip Y with the subspace distance and the measure $\nu_Y(y) = \nu(A_y)$. Let $\pi : Z \to Y$ be defined by "$\pi(z)$ is the only $y \in Y$ such that $z \in A_y$". Note that π is surjective, and a right-inverse of the inclusion $j : Y \to Z$. Moreover, $\pi^{-1}(y) = A_y$ for every $y \in Y$.

Remark 5.7. Observe that the choice of b ensures that Y has bounded packing at all scales ≥ 0, and that both π and j are large-scale equivalences. In particular, if Y' is a similar discretization of $[[F(Z)]_M]_1$, then $\Psi = \pi' \circ F \circ j$ is a coarse regular map. Moreover, if one chooses the discretization parameter b' large enough, then Ψ is surjective.

Our next goal is to compare the Poincaré constant of a subspace with that of its discretization.

Lemma 5.8. *Let (Z,d,ν) be a metric measure space with finite measure. Suppose (Y,d,ν_Y) is a discretization (with parameter $b \geq 2$) of Z as above. Then for all $a \geq b$,*

$$h^p_a(Y) \lesssim_a h^p_{20a}(Z), \quad \text{and} \quad h^p_a(Z) \leq h^p_{20a}(Y).$$

Proof. Let $f \in L^\infty(Z)$ be such that $⨍_Z f d\nu = 0$. We define $\phi \in \ell^\infty(Y)$ by $\phi(y) = ⨍_{A_y} f d\nu$. Clearly $⨍_Y \phi d\nu_Y = 0$ and $\|\phi \circ \pi\|_{Z,p} = \|\phi\|_{Y,p}$. Write $f(z) = \phi(\pi(z)) + ⨍_{A_{\pi(z)}}(f(z) - f(w))d\nu(w)$. Then

$$\begin{aligned}
\|f\|_{Z,p} &\leq \|\phi \circ \pi\|_{Z,p} + \left(\int_Z \left| ⨍_{A_{\pi(z)}} (f(z) - f(w))\, d\nu(w) \right|^p d\nu(z)\right)^{1/p} \\
&\leq \|\phi\|_{Y,p} + \left(\int_Z ⨍_{A_{\pi(z)}} |f(z) - f(w)|^p\, d\nu(w) d\nu(z)\right)^{1/p} \\
&\leq \|\phi\|_{Y,p} + \left(\int_Z |\nabla_{10a} f|(z)^p\right)^{1/p} \\
&= \|\phi\|_{Y,p} + \|\nabla_{10a} f\|_p \ .
\end{aligned}$$

On the other hand, it is immediate from the definitions that $|\nabla_a \phi|(y) \leq |\nabla_{20a} f|(z)$ for all $z \in A_y$.

We now prove the first inequality. If $h^p_{20a}(Z) \leq \frac{1}{2}$, then for any $\epsilon \in (0, 1/6)$ we can find f as above so that

$$\frac{2}{3} \geq \frac{1}{2} + \epsilon \geq h^p_{20a}(Z) + \epsilon \geq \frac{\|\nabla_{20a} f\|_p}{\|f\|_p} \geq \frac{\|\nabla_{20a} f\|_p}{\|\phi\|_p + \|\nabla_{20a} f\|_p}.$$

Thus $\|\nabla_{20a} f\|_p \leq 2\|\phi\|_p$ and

$$h^p_{20a}(Z) + \epsilon \geq \frac{\|\nabla_a \phi\|_p}{3\|\phi\|_p} \geq \frac{1}{3} h^p_a(Y).$$

Since ϵ was arbitrary, $h^p_a(Y) \leq 3h^p_{20a}(Z)$. Moreover, it is easy to see that $h^p_a(Y) \lesssim_a 1$ (a much more general statement is proved in Proposition 7.1), so if $h^p_{20a}(Z) \geq \frac{1}{2}$, then $h^p_a(Y) \lesssim_a h^p_{20a}(Z)$.

The other direction is easier: given $\psi \in \ell^\infty(Y)$, such that $⨍_Y \psi d\nu_Y = 0$ we define $g = \sum_{y \in Y} \psi(y) 1_{A_y}$, where 1_{A_y} denotes the characteristic function of A_y. We clearly have $⨍ g d\nu = 0$ and $\|g\|_p = \|\psi\|_p$. Hence we are left with comparing the gradients.

$$\begin{aligned}\|\nabla_r g\|_p^p &= \sum_Y \nu(A_y) \fint_{A_y} \sup_{z',z''\in B(z,a)} |g(z')-g(z'')|^p d\nu(z) \\ &\leq \sum_Y \nu(A_y) \sup_{z',z''\in B(y,10a)} |g(z')-g(z'')|^p \\ &\leq \sum_Y \nu_Y(y) \sup_{y',y''\in B(y,20a)\cap Y} |\psi(y')-\psi(y'')|^p \\ &= \|\nabla_{20a}\psi\|_p^p. \end{aligned}$$ □

Now we compare the Poincaré constants of discrete spaces related by a sufficiently nice surjective coarse regular map.

Lemma 5.9. *Let $\pi : (Y, d, \nu) \to (Y', d, \nu')$ be a map between two discrete metric measure spaces with finite (non-degenerate) measures, and assume that:*

- *π is surjective;*
- *$\nu(\pi^{-1}(y')) = \nu'(y')$.*

Then for all $a \geq 0$ and C such that $d(y, z) \leq a$ implies $d(\pi(y), \pi(z)) \leq Ca$, we have

$$h_a^p(Y) \leq h_{Ca}^p(Y').$$

Proof. Let $f' \in \ell^\infty(Y')$ such that $\int f' d\mu' = 0$ and let $f = f' \circ \pi$. Clearly, we have $\int f d\mu = 0$, and

$$\|f\|_p = \|f'\|_p.$$

Moreover, for every $y \in Y$, if $y_1, y_2 \in B(y, a)$ then $\pi(y_1), \pi(y_2) \in B(\pi(y), Ca)$. So a straightforward computation shows that

$$\|\nabla_a f\|_p \leq \|\nabla_{Ca} f'\|_p.$$ □

As a result we obtain a version of Proposition 5.5 in the uniformly discrete case.

Corollary 5.10. *Let $\Psi : (Y, d, \nu) \to (Y', d, \nu')$ be a surjective coarse regular map between uniformly discrete spaces, which have bounded packing at any scale. Then, for all $a > 0$, there exists $C > 0$ such that*

$$h_a^p(Y) \preceq_a h_{Ca}^p(Y').$$

Proof. The assumptions imply that Ψ is coarse Lipschitz, surjective, and such that $\nu(\Psi^{-1}(y')) \asymp_a \nu'(y')$. Hence the corollary follows from Lemmas 3.3(ii) and 5.9: if we push the measure ν forward with Ψ to obtain a measure $\Psi_*\nu$ on Y' we have

$$h_a^p(Y, \nu) \preceq h_{Ca}^p(Y', \Psi_*\nu) \preceq h_{Ca}^p(Y', \nu').$$ □

Combining these results we are in a position to prove Proposition 5.5.

Proof of Proposition 5.5. Let Z be a 1-thick subspace of X and define $Z' = [[F(Z)]_M]_1$ where $M = \max\{\delta_0, \rho_+(1)\}$. Then Z' is a 1-thick subspace of X' and $\mu(Z) \asymp_M \mu'(Z')$. Let b, b' be sufficiently large that the discretizations Y of Z and Y' of Z' satisfy the hypotheses of Lemma 5.8 for some suitable $a = a(b, b') \geq 2$ and so that $\Psi = \pi' \circ F \circ j$ is surjective. Note that b and b' may be chosen independently of the choice of subspace Z of X, hence a does not depend on Z.

Applying Corollary 5.10 we see that there exists a constant C depending only on a such that $h^p_a(Y) \preceq_a h^p_{Ca}(Y')$. Now, by Lemma 5.8 $h^p_a(Z) \preceq_{a,M} h^p_{C'a}(Z')$ where a, M, C' do not depend on Z.

Thus

$$\begin{aligned} \Lambda^p_{X,a}(r) &= \sup\{\mu(Z)h^p_a(Z) \,:\, \mu(Z) \leq r\} \\ &\lesssim_{a,M} \sup\{\mu'(Z')h^p_{C'a}(Z') \,:\, Z' = [[F(Z)]_M]_1,\ \mu(Z) \leq r\} \\ &\leq \Lambda^p_{X',C'a}(Mr). \end{aligned}$$

We conclude using Corollary 4.4. □

6. Extremal profiles: growth and separation

6.1. **Growth and the L^∞-Poincaré profile.** In this section we give the proof of Proposition 2. Recall our standing assumptions: a metric measure space (X, d, μ) is 1-geodesic and has bounded packing at scales $\geq r_0 = 1$. Recall also that the inverse growth function of X, is defined by letting $\kappa(n)$ be the infimal s such that there exists a ball $B \subset X$ of radius s with measure $> n$. By assumption subspaces are 1-thick and equipped with a 1-geodesic metric.

Proposition 6.1. *Let (X, d, μ) be a metric measure space with infinite diameter and let $a \geq 2$. Then*

$$\Lambda^\infty_{X,a}(r) \simeq_a \sup\left\{\frac{s}{\kappa_X(s)} \,:\, 3 \leq s \leq r\right\}.$$

In all our applications, the function $\sup\left\{\frac{s}{\kappa_X(s)} \,:\, 3 \leq s \leq r\right\}$ will be equivalent to $\frac{r}{\kappa_X(r)}$ but in general this may not be the case. The proof requires a lemma.

Lemma 6.2. *Let Z be a subspace of X with diameter m and let $a \geq 2$. Then $h^\infty_a(Z) \leq \frac{4a}{m}$, and if every $y, z \in Z$ can be joined by a 1-path of length $\leq 2m$ then $h^\infty_a(Z) \geq \frac{1}{2m}$.*

Proof. Choose $x, y \in Z$ such that $d(x, y) \geq m - \delta$, and define $f(z) = d(x, z)$. It is clear that $f(x) = 0$ and $f(y) \geq m - \delta$, so $||f - f_Z||_\infty \geq \frac{m-\delta}{2}$, while $||\nabla_a f||_\infty \leq 2a$ by the triangle inequality. Thus $h_a^\infty(Z) \leq \frac{4a}{m-\delta}$ for all $\delta > 0$.

For the second inequality, fix $\delta > 0$ and let $f \in L^\infty(Z)$ satisfy $\inf_{z\in Z} f(z) = 0$. Choose y, z so that $(f(z) - f(y)) + \delta \geq \sup_{z\in Z} |f(z)| = ||f||_\infty$.

By our hypothesis there exists a sequence of points $y = z_0, \ldots, z_k = z$ such that $k \leq 2m$ and $d(z_i, z_{i+1}) \leq 1$ for all i. Therefore, $\nabla_a f(z_i) \geq \frac{1}{2m}(||f||_\infty - \delta)$ for some i, so $||\nabla_a f||_\infty \geq \frac{1}{2m}(||f||_\infty - \delta)$. Since we have $||f - f_Z||_\infty \leq ||f||_\infty$, letting $\delta \to 0$, we see that $h_a^\infty(Z) \geq \frac{1}{2m}$. □

Proof of Proposition 6.1. The upper bound on $\Lambda^\infty_{X,a}(r)$ follows immediately from Lemma 6.2. Indeed, if $\mu(Z) \leq r$ then

$$\mu(Z)h_a^\infty(Z) \lesssim_a \frac{\mu(Z)}{\mathrm{diam}(Z)} \leq \frac{\mu(Z)}{\kappa(\mu(Z))}.$$

We now prove the lower bound.

Let $s \geq 2$ and choose $x_s \in X$ such that $\mu(B(x_s, s)) \geq \frac{1}{2}\gamma_X(s)$. Define Z_s to be the 1-thick subspace $[B(x_s, s-1)]_1$. By Lemma 2.3 there is a constant C (which does not depend on s) such that $\mu(Z_s) \leq \mu(B(x_s, s)) \leq \gamma_X(s) \leq C\mu(Z_s)$.

By Lemma 6.2 $h_a^\infty(Z_s) \in [\frac{1}{4s}, \frac{2a}{(s-1)}]$, so $\mu(Z_s)h_a^\infty(Z_s) \asymp_a \frac{\gamma_X(s)}{s}$.

Let r be sufficiently large that $\kappa_X(r) \geq 3$, and let $3 \leq s \leq r$. Repeating the above argument, we see that $\gamma_X(s)/\gamma_X(s-1)$ has a uniform upper bound. Thus,

$$\Lambda^\infty_{X,a}(r) \succeq_a \frac{\gamma_X(s-1)}{s-1} \succeq_a \frac{\gamma_X(s)}{s}.$$

As this can be done for all such s, the lower bound holds. □

6.2. **Separation profiles of metric measure spaces.** We wish to extend the Cheeger constant definition of separation [Hum17] to the setting of metric measure spaces (X, d, μ) which are 1-geodesic and has bounded packing at scales ≥ 1.

Given a subspace $A \subset X$ (which as usual we assume is 1-thick and equipped with the induced measure and induced 1-geodesic metric) we define the **boundary at scale** $a \geq 2$ of A to be

$$\partial_a A = [A]_a \cap [A^c]_a$$

with the usual notation $A^c = X \setminus A$. For clarity, given a subspace Z of X and $A \subset Z$, we also define the boundary at scale a of A in Z to be $\partial_a^Z A = Z \cap \partial_a(A)$.

Definition 6.3. Let (Z, d, ν) be a metric measure space, where $\nu(Z)$ is finite and let $a \geq 2$. We define the **Cheeger constant at scale** a of Z to be

$$h_a(Z) = \inf \left\{ \frac{\nu(\partial_a \Omega)}{\nu(\Omega)} \,:\, \nu(\Omega) \leq \frac{\nu(Z)}{2} \right\}.$$

Let (X, d, μ) be a metric measure space. We define the function $\mathrm{sep}_{X,a}(r) = \sup\{\mu(Z) h_a(Z)\}$, where the supremum is taken over all (1-thick) subspaces $Z \subseteq X$ with $\mu(Z) \leq r$, and is 0 if no such subspaces exist.

Remark 6.4. If Γ is a finite graph of bounded degree D then the boundary at scale a has comparable size to the vertex boundary, so the usual (vertex) Cheeger constant $h(\Gamma)$ satisfies $h(\Gamma) \asymp_{a,D} h_a(\Gamma)$. As a result, if X is an infinite graph of bounded degree D, then $\mathrm{sep}_{X,a} \asymp_{a,D} \mathrm{sep}_X$, where sep_X is the usual separation function for graphs. (See [Hum17, Propositions 2.2, 2.4].)

6.3. **Comparing Cheeger and L^1-Poincaré constants.** Our next goal is to prove Proposition 3. Along the way we will also prove Proposition 6.

Proposition 6.5. *Let (X, d, μ) be a metric measure space and let $a \geq 2$. Then*

$$\frac{1}{2} \mathrm{sep}_{X,a} \leq \Lambda^1_{X,a} \leq \mathrm{sep}_{X,a}\,.$$

We prove this by comparing the Cheeger constant and the L^1-Poincaré constant.

Proposition 6.6. *Let (X, d, μ) be a metric measure space. The following co-area formula holds for every non-negative measurable function $f : X \to \mathbb{R}$.*

$$\int_X |\nabla_a f|(x) d\mu(x) = \int_{\mathbb{R}_+} \mu\left(\partial_a\{f > t\}\right) dt \tag{6.7}$$

Proof. For every measurable subset $A \subset X$, we have

$$\mu(\partial_a A) = \int_X |\nabla_a 1_A|(x) d\mu(x). \tag{6.8}$$

Thus, (6.7) follows by integrating over X the following local equalities

$$|\nabla_a f|(x) = \int_{\mathbb{R}_+} |\nabla_a 1_{\{f>t\}}|(x) dt. \tag{6.9}$$

It remains to show that these equalities hold for all $x \in X$.

Notice that $|\nabla_a 1_{\{f>t\}}(x)| = 1$ if and only if there exists $y, y' \in B(x, a)$ with $f(y) > t$ and $f(y') \leq t$. In particular, $|\nabla_a 1_{\{f>t\}}(x)|$

equals one for $t \in (\inf_{B(x,a)} f, \sup_{B(x,a)} f)$ and equals zero for $t \notin [\inf_{B(x,a)} f, \sup_{B(x,a)} f]$. Hence,

$$\int_{\mathbb{R}_+} |\nabla_a 1_{\{f>t\}}|(x)dt = \sup_{B(x,a)} f - \inf_{B(x,a)} f = |\nabla_a f|(x),$$

which proves (6.9). $\square$

Using this co-area formula we can prove the required relation between $h_a(Z)$ and $h_a^1(Z)$.

Proposition 6.10. *Let (Z, d, ν) be a metric measure space with finite positive measure ν and let $a \geq 2$. Then*

$$h_a^1(Z) \leq h_a(Z) \leq 2h_a^1(Z).$$

Proof. Let $\Omega \subset Z$ such that $\nu(\Omega) \leq \nu(Z)/2$. We deduce from (6.8) that

$$\|\nabla_a f\|_1 = \nu(\partial_a \Omega),$$

where $f = 1_\Omega$. On the other hand,

$$\|f - f_Z\|_1 = \nu(\Omega)\left(1 - \frac{\nu(\Omega)}{\nu(Z)}\right) + (\nu(Z) - \nu(\Omega))\left(\frac{\nu(\Omega)}{\nu(Z)}\right) \geq \nu(\Omega).$$

Hence $h_a^1(Z) \leq h_a(Z)$.

By Lemmas 3.4 and 3.6, for each $\delta > 0$ we may choose $f \in L^1(Z, \nu)$ (with median 0) such that

$$\frac{\|\nabla_a f\|_1}{\|f\|_1} \leq 2h_a^1(Z) + \delta.$$

Let $f_+ = \max\{f, 0\}$ and $f_- = \min\{f, 0\}$. For any $s, s', t, t' > 0$ if $\frac{s+s'}{t+t'} \leq C$ then $\frac{s}{t} \leq C$ or $\frac{s'}{t'} \leq C$. Since $\|f\|_1 = \|f_-\|_1 + \|f_+\|_1$ and $\|\nabla_a f\|_1 = \|\nabla_a f_+\|_1 + \|\nabla_a f_-\|_1$, we deduce that up to replacing f by $-f$, we have

$$\frac{\|\nabla_a f_+\|_1}{\|f_+\|_1} \leq 2h_a^1(Z) + \delta.$$

Hence using (6.7) and the fact that

$$\|f_+\|_1 = \int_{\mathbb{R}_+} \nu(\{f > t\})dt,$$

we conclude that there exists some $t \geq 0$ such that the subset $\Omega_t = \{f > t\}$ satisfies

$$h_a(Z) \leq \frac{\nu(\partial_a \Omega_t)}{\nu(\Omega_t)} \leq 2h_a^1(Z) + \delta.$$

This proves the second inequality. $\square$

Proof of Proposition 6: The first half of the above proof can easily be adapted to prove that $2^{1-p}h_a^p(Z)^p \leq h_a(Z)$. Hence, $h_a^p(Z)^p \leq 2^p h_a^1(Z)$. $\square$

7. Dependency on p

One trivial upper bound can always be put on Poincaré constants.

Proposition 7.1. *Let (Z,d,ν) be a metric measure space with $\nu(Z)$ finite. Assume there is no $z \in Z$ with $\nu(\{z\}) > \frac{2}{3}\mu(Z)$. For all $p \in [1,\infty)$ and all $a \geq 2$, $h_a^p(Z) \leq 2 \cdot 3^{\frac{1}{p}}$.*

Proof. Let $Y \subset Z$ satisfy $\frac{1}{3}\nu(Z) \leq \nu(Y) \leq \frac{2}{3}\nu(Z)$ and let f be the characteristic function of Y. It is an easy exercise to verify that such a subset Y exists.

Then $||f - f_Y||_p^p \geq \frac{\nu(Z)}{3 \cdot 2^p}$ and $||\nabla_a f||_p^p \leq \nu(Z)$, thus $h_a^p(Z) \leq 2 \cdot 3^{\frac{1}{p}}$. $\square$

Equipped with this we are now able to study the relationship between different Poincaré profiles of the same space and prove Proposition 5.

Proposition 7.2. *Let (Z,d,ν) be a metric measure space with $\nu(Z)$ finite. Assume there is no $z \in Z$ with $\nu(\{z\}) > \frac{2}{3}\nu(Z)$. Then for all $1 \leq p \leq q < \infty$ and all $a \geq 2$,*

$$h_a^q(Z) \succeq_{p,q} h_a^p(Z).$$

For all metric measure spaces (X,d,μ) (where μ is possibly infinite), and all $1 \leq p \leq q < \infty$,

$$\Lambda_X^q \succeq_{p,q} \Lambda_X^p.$$

Proof. Our goal is to prove that for any function $g : Z \to \mathbb{R}$, there is a function $f : Z \to \mathbb{R}$ such that

$$\frac{||\nabla_a g||_q}{||g - g_Z||_q} \succeq_{p,q} \frac{||\nabla_a f||_p}{||f - f_Z||_p} \geq h_a^p(Z).$$

Taking the infimum over all g would then yield the desired result. From this, we see that it suffices to consider all functions g which satisfy the upper bound $||\nabla_a g||_q \leq 6\, ||g - g_Z||_q$ given by Proposition 7.1. By (3.5) we have that for all $C \in \mathbb{R}$, $6\, ||g - g_Z||_q \leq 12\, ||g - C||_q$.

For $a \in \mathbb{R}$ and $p \geq 1$, write $\{a\}^p = \operatorname{sign}(a)|a|^p$. For each C, define $f^C : Z \to \mathbb{R}$ by $f^C(z) = \{g(z) + C\}^{q/p}$, for some $C \in \mathbb{R}$. Since f_Z^C is a continuous function of C, we fix C so that $f_Z^C = 0$. Set $f = f^C$.

For each $z \in Z$ let $\overline{(g+C)}_a(z) = \sup\{|g(z') + C| \,:\, d(z,z') \leq a\}$.

By the mean value theorem (see e.g. Matoušek [Mat97, Lemma 4]), for every $s,t \in \mathbb{R}$ and $\alpha \geq 1$,

$$|\{s\}^\alpha - \{t\}^\alpha| \leq \alpha(|s|^{\alpha-1} + |t|^{\alpha-1})|s-t|.$$

For each $z \in Z$ we apply this to $s = g(x) + C, t = g(y) + C, \alpha = \frac{q}{p}$ for all pairs of points $x, y \in B(z, a)$ and see that

$$|\nabla_a f|(z) \leq \frac{2q}{p} \overline{(g+C)}_a(z)^{\frac{q-p}{p}} |\nabla_a g|(z).$$

By the definition of ∇_a, $\overline{(g+C)}_a(z) \leq |g(z) + C| + |\nabla_a g|(z)$, so taking pth powers and integrating, we see that

$$\begin{aligned}
&||g+C||_q^q\, h_a^p(Z)^p = ||f||_p^p\, h_a^p(Z)^p = ||f - f_Z||_p^p\, h_a^p(Z)^p \\
&\leq \int_Z |\nabla_a f|(z)^p d\nu \\
&\leq \left(\frac{2q}{p}\right)^p \int_Z (|g(z) + C| + |\nabla_a g|(z))^{q-p} |\nabla_a g|(z)^p d\nu \\
&\overset{(\star)}{\leq} \left(\frac{2q}{p}\right)^p 2^{q-p} \left(\int_Z |g(z) + C|^{q-p} |\nabla_a g|(z)^p d\nu + ||\nabla_a g||_q^q \right) \\
&\overset{(\dagger)}{\leq} \frac{2^q q^p}{p^p} \left(||g+C||_q^{q-p} ||\nabla_a g||_q^p + 12^{q-p} ||g+C||_q^{q-p} ||\nabla_a g||_q^p \right) \\
&\preceq_{p,q} ||g+C||_q^{q-p} ||\nabla_a g||_q^p,
\end{aligned}$$

where $(\star)$ follows from $(s+t)^\alpha \leq 2^\alpha(s^\alpha + t^\alpha)$ for any $s, t, \alpha > 0$, and $(\dagger)$ follows from Hölder's inequality and $||\nabla_a g||_q \leq 12\,||g+C||_q$. Rearranging, taking pth roots, and applying (3.5) we have

$$||g - g_Z||_q \leq 2\,||g+C||_q \preceq_{p,q} \frac{||\nabla_a g||_q}{h_a^p(Z)}. \qquad \square$$

Remark 7.3. There are graphs X of bounded degree containing expanders, and by Propositions 7.2 and 3,

$$\Lambda_X^p(r_n) \succeq_p \Lambda_X^1(r_n) \asymp \mathrm{sep}_X(r_n) \succeq r_n$$

on some unbounded subsequence (r_n) [Hum17], but $\Lambda_X^\infty(r) \simeq r/\log(r)$ by Proposition 2, so one should not expect universal constants (independent of p, q) in the above proposition.

8. Poincaré profiles of groups with polynomial growth

The goal of this section is to prove the lower bound in Theorem 7.

Given a compactly generated locally compact group group G, with compact symmetric generating set K, let $d = d_K$ be the associated word metric and let μ be a left-invariant Haar measure. We refer to the triple (G, d, μ) as a **metric measure CGLC group**. By Lemma 3.3 and Corollary 4.4, the L^p-Poincaré profile of G is well-defined (up to $\simeq$).

Theorem 8.1. *Let (G, d, μ) be a metric measure CGLC group. If there exists some $m > 0$ such that $\gamma(r) \asymp r^m$, then for every $p \in [1, \infty]$, $\Lambda^p_G(r) \gtrsim_p r^{\frac{m-1}{m}}$.*

Note that the $p = \infty$ case follows immediately from Proposition 6.1. Moreover, by Proposition 7.2 $\Lambda^p_G \gtrsim_p \Lambda^1_G$ for all $p \in [1, \infty)$. Using Proposition 6.5 we see that Theorem 8.1 follows from

Theorem 8.2. *Let (G, d, μ) be a metric measure CGLC group. If $\mu(B(1, r)) \asymp r^m$ then for all a, r sufficiently large, there is a subset B_r of $B(1, r)$ with measure at least $\frac{1}{2}\mu(B(1, r))$ satisfying $h_a(B_r) \succeq_a r^{-1}$.*

This theorem will be our goal for the section. The proof is in three parts: the first part gives a general Poincaré inequality satisfied by any compactly generated locally compact group. Secondly we refine this inequality for groups with polynomial growth. In the third part we use this Poincaré inequality (specifically in the L^1 setting) to obtain lower bounds on the Cheeger constant at scale a of metric balls.

8.1. **A Poincaré inequality.** Poincaré inequalities are well known to hold for groups with polynomial growth, see for example [SC02]. In this subsection we present a generalisation of [Kle10, Theorem 2.2] (attributed to Saloff-Coste and explicitly appearing in the L^2 case in [DSC93]) to compactly generated locally compact groups in our framework. The proof below is also similar in nature to [HK00, Proposition 11.17] which is attributed to Varapoulos [Var87].

Theorem 8.3. *Let (G, d, μ) be a metric measure CGLC group. Let $\Delta : G \to \mathbb{R}$ be the modular function on G; i.e., for $U \subset G$ and $g \in G$, $\mu(Ug) = \mu(U)\Delta(g)$. Define $\Delta(K) = \sup_{g \in K} \Delta(g)$.*

For any $p \geq 1$, $a \geq 1$, for any metric ball $B = B(x_0, R)$ of radius R and any function $f \in L^1(G)$ we have the following:

$$\int_B |f(x) - f_B|^p \, d\mu(x) \leq \frac{(2R)^p \mu(2B)\Delta(K)^{2R}}{\mu(B)} \int_{3B} |\nabla_a f| (x)^p d\mu(x),$$

where for $\lambda > 0$, $\lambda B = B(x_0, \lambda R)$.

Proof. We may assume $x_0 = e$. Recall that

$$|\nabla_a f| (x) = \sup \{|f(y) - f(z)| : y, z \in B(x, a)\}.$$

If $a \leq a'$ then $|\nabla_a f| (x) \leq |\nabla_{a'} f| (x)$ so it suffices to prove the result above for $a = 1$.

For every $z \in 2B$, we choose a geodesic $\gamma_z : \{0, 1, \ldots, k\} \to G$ with $\gamma_z(0) = e$ and $\gamma_z(k) = z$.

For $x, y \in B(R)$, let $z = x^{-1}y$, and let $|\gamma_z| = k$ be the length of the corresponding path. Then by the triangle and Hölder's inequality,

$$|f(x) - f(y)|^p \leq \left| \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i)) \right|^p \leq |\gamma_z|^{p-1} \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i))^p.$$

For fixed $z \in 2B$, consider the map $F : (x, i) \mapsto (x\gamma_z(i), i)$. This is clearly injective, so $(x, i) \mapsto x\gamma_z(i)$ is at most $2R$-to-1, and

$$\begin{aligned}
\int_B \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i))^p d\mu(x) &= \sum_{i=1}^{|\gamma_z|} \int_B |\nabla_1 f|(x\gamma_z(i))^p d\mu(x) \\
&= \sum_{i=1}^{|\gamma_z|} \int_{B\cdot\gamma_z(i)} |\nabla_1 f|(x)^p \Delta(\gamma_z(i)^{-1}) d\mu(x) \\
&\leq 2R \sup_{g\in 2B} \Delta(g) \int_{3B} |\nabla_1 f|(x)^p d\mu(x).
\end{aligned}$$

Since $2B = K^{2R}$ we have $\sum_{g\in 2B} \Delta(g) = \Delta(K^{2R}) \leq \Delta(K)^{2R}$, so

$$\begin{aligned}
\int_B |f - f_B|^p \, d\mu &\leq \int_B \left| \int_B |f(x_1) - f(x_2)| \frac{d\mu(x_2)}{\mu(B)} \right|^p d\mu(x_1) \\
&\leq \frac{1}{\mu(B)} \int_{B\times B} |f(x_1) - f(x_2)|^p \, d\mu(x_1) d\mu(x_2) \\
&\leq \frac{(2R)^{p-1}}{\mu(B)} \int_{x\in B} \int_{z\in 2B} \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i))^p d\mu(z) d\mu(x) \\
&\leq \frac{(2R)^p \Delta(K)^{2R}}{\mu(B)} \int_{z\in 2B} \int_{x\in 3B} |\nabla_1 f|(x)^p d\mu(x) d\mu(z) \\
&\leq \frac{(2R)^p \Delta(K)^{2R} \mu(2B)}{\mu(B)} \int_{3B} |\nabla_1 f|(x)^p d\mu(x). \qquad \square
\end{aligned}$$

8.2. **CGLC groups with polynomial growth.** We begin by refining the above Poincaré inequality.

Lemma 8.4. *If* $\liminf_{r\to\infty} \frac{1}{r} \log(\mu(B(1, r))) = 0$*, then* G *is unimodular.*

Proof. Suppose G is not unimodular, then there exists some $g \in G$ such that $\Delta(g) > 1$. Since Δ is multiplicative, there is some $k \in K$ with $\Delta(k) > 1$.

Now, for each n, $Kk^n \subseteq B(1, n+1)$, so $\mu(B(1, n+1)) > \Delta(k)^n \mu(K)$, and therefore $\liminf_{r\to\infty} \frac{1}{r} \log(\mu(B(1, r))) > 0$. $\square$

From this we obtain the following refinement of a special case of Theorem 8.3.

Corollary 8.5. *If G has polynomial growth then there exists a constant C such that, for any $p \geq 1$ and $a \geq 1$, for any metric ball $B = B(x_0, R)$ of radius R and any function $f \in L^p(G)$ we have the following:*

$$\int_B |f(x) - f_B|^p \, d\mu(x) \leq CR^p \int_{3B} |\nabla_a f| (x)^p d\mu(x). \tag{8.6}$$

Using this refined Poincaré inequality (specifically the case $p = 1$) we will now present a proof of Theorem 8.2 via a series of lemmas. The goal is to prove that any subset A of B such that both $A \cap B$ and $A^c \cap B$ have measure proportional to B must have large boundary inside B. It is not sufficient to apply the Poincaré inequality (8.6) to the characteristic function of A inside B as we cannot distinguish the contribution coming from the boundary of A in B with that coming from the boundary of B in X. The solution is to apply the Poincaré inequality (8.6) "deep inside" B.

From this we will show that there is a large subset of B with sufficiently large Cheeger constant. This step is modelled on ideas from [Hum17] relating the cut size and Cheeger constant definitions of separation.

Definition 8.7. Let X be a metric space, let $x \in X$, and let $r, s \in \mathbb{R}_+$ with $s > r$. The (r, s)-**corona** around x is the set $C_{r,s}(x) = B(x, s) \setminus B(x, r)$.

Lemma 8.8. *Let (G, d, μ) be a metric measure CGLC group with $\gamma(r) \asymp r^m$. For each $\delta \in (0, 1)$ there exists some $\epsilon > 0$ such that for every $x \in G$ and r sufficiently large, we have $\mu(C_{r,(1+\epsilon)r}(x)) \leq \delta\mu(B(x, r))$.*

Proof. By [Tes07, Lemma 24], there exist constants $\alpha, \beta > 0$ independent of r such that $\mu\left(C_{r-s,r}(x)\right) \geq \alpha\mu(C_{r,r+s}(x))$ for every $x \in G$ whenever $4\beta < s \leq r$.

Let $\epsilon' \in (0, 1)$ and for each $0 \leq i \leq k = \lfloor -\log_2 \epsilon' \rfloor$, let $b_i = \mu(C_{(1-2^i\epsilon')r,r})$.

By construction $b_i \geq (1 + \alpha)b_{i-1}$ for all $i \leq k$, so $b_k \geq (1 + \alpha)^{k-1}b_1$.

Fix $\delta \in (0, 1)$. If $\mu(C_{r,(1+\epsilon')r}) > \delta\mu(B(x, r))$, then $\mu(C_{r,(1+\epsilon')r}) \geq \delta b_k \geq \delta(1 + \alpha)^{k-1}b_1$. But, by [Tes07, Lemma 24], $b_1 \geq \alpha\mu(C_{r,(1+\epsilon')r})$, so $\alpha\delta(1 + \alpha)^{k-1} \leq 1$.

Thus $k \leq \log_{1+\alpha}(\frac{1}{\alpha\delta}) + 1$, which implies that

$$\epsilon' \geq \epsilon_{\alpha,\delta} := \frac{\alpha\delta}{4\log_{1+\alpha}(2)}.$$

The conclusion of the lemma holds for all $\epsilon < \epsilon_{\alpha,\delta}$. $\square$

Lemma 8.9. *Let (G, d, μ) be a metric measure CGLC group with $\gamma(r) \asymp r^m$. There exist constants $r_0, \epsilon, k > 0$ such that the following holds for all $r \geq r_0$.*

For any $A \subset B(x,r)$ with $\frac{1}{4}\gamma(r) \leq \mu(A) \leq \frac{1}{2}\gamma(r)$, there exists a point $w \in B(x,r)$ such that $B(w, 3\epsilon r) \subset B(x,r)$, and such that $\mu(B(w,\epsilon r) \cap A) \geq k\gamma(r)$ and $\mu(B(w,\epsilon r) \cap A^c) \geq k\gamma(r)$.

Proof. By Lemma 8.8, for all r sufficiently large, and ϵ sufficiently small the corona $C_{(1-3\epsilon)r,r}(x)$ has size $< \frac{1}{10}\gamma(r)$ for every $x \in X$. Now fix $k > 0$ such that $\gamma(\epsilon r) \geq \frac{80}{3}k\gamma(r)$ for all $r \geq r_0$. Applying Lemma 8.8 with $\delta = \frac{k}{2}$ we deduce that there exists a $\epsilon' > 0$ such that for all r sufficiently large (and at least $(\epsilon\epsilon')^{-1}$),

$$\gamma(\epsilon r + 1) - \gamma(\epsilon r) \leq \frac{k}{2}\gamma(\epsilon r) \leq \frac{k}{2}\gamma(r). \tag{8.10}$$

Since $\mu(A \cap B(x,(1-3\epsilon)r)) \geq \frac{3}{20}\gamma(r)$, there exists a point $y \in B(x,(1-3\epsilon)r))$ such that $\mu(A \cap B(y,\epsilon r)) \geq \frac{3}{20}\gamma(\epsilon r) \geq 2k\gamma(r)$. Similarly, there is some $z \in B(x,(1-3\epsilon)r))$ such that $\mu(A^c \cap B(z,\epsilon r)) \geq 2k\gamma(r)$. Now, by our choice of k, for every $v \in B(x,(1-3\epsilon)r)$,

$$\max\{\mu(A \cap B(v,\epsilon r)), \mu(A^c \cap B(v,\epsilon r))\} \geq \tfrac{1}{2}\gamma(\epsilon r) \geq 2k\gamma(r).$$

Since $y, z \in B(x,(1-\epsilon)r))$ there is a sequence $y = v_0, v_1, \ldots, v_l = z$ such that $d(v_{i-1}, v_i) = 1$, $l \leq 2r$ and $\{v_i\} \subset B(x,(1-\epsilon)r))$. By (8.10), we see that the measure of the symmetric difference of $B(v_i,\epsilon r)$ and $B(v_{i+1},\epsilon r)$ is at most $k\gamma(r)$ for all i.

Choose i maximal such that $\mu(A \cap B(v_i,\epsilon r)) \geq 2k\gamma(r)$. If $i = l$ then we choose $w = v_l$ and the proof is complete. If $i < l$ then $\mu(A^c \cap B(v_{i+1},\epsilon r)) \geq 2k\gamma(r)$, but since the symmetric difference of $B(v_i,\epsilon r)$ and $B(v_{i+1},\epsilon r)$ has measure at most $k\gamma(r)$, we see that $\mu(A^c \cap B(v_i,\epsilon r)) \geq k\gamma(r)$ and we set $w = v_i$. $\square$

With this lemma we can show that large subsets of balls have large boundaries inside the ball.

Proposition 8.11. *Let (G, d, μ) be a metric measure CGLC group with $\gamma(r) \asymp r^m$. For every a sufficiently large, there exists a constant $k = k(a)$ such that for every ball B of radius r, and any subspace $A \subset B(x,r)$ with $\frac{1}{4}\gamma(r) \leq \mu(A) \leq \frac{1}{2}\gamma(r)$, we have $\mu(\partial_a^B A) \geq kr^{m-1}$.*

Proof. Let $A \subset B(x,r) = B$ be such that $\frac{1}{4}\mu(B) \leq \mu(A) \leq \frac{1}{2}\mu(B)$. By Lemma 8.9 there exists some $w \in B(x,(1-3\epsilon)r)$ such that $\mu(B(w,\epsilon r) \cap A)$, $\mu(B(w,\epsilon r) \cap A^c) \geq k\mu(A)$. Applying the Poincaré inequality (8.6) with $p = 1$ to the characteristic function $\mathbf{1}_A$ on the ball $B(w, \epsilon r)$ we see that

$$\tfrac{1}{2}k\mu(A) \leq C\epsilon r\mu(\partial_a^{B(w,3\epsilon r)}A).$$

Since $B(w, 3\epsilon r) \subseteq B$ we deduce that there exists a constant $k' > 0$ (independent of r) such that

$$\mu(\partial_a^B A) \geq \frac{k'}{r}\mu(B). \qquad \square$$

The last step in this argument ensures that there is a large subset of the ball with suitable Cheeger constant at scale a. This is a generalisation of a similar result for graphs presented in [Hum17].

Proposition 8.12. *Let (X, d, μ) be a metric measure space such that $\inf_{x\in X} \mu(B(x, 1)) = c > 0$, and let $a, r \geq 2$. If there exists a constant $\lambda = \lambda(a, r) \leq \frac{1}{4}$ and a ball $B = B(x, r)$ of radius r, such that for any subspace $A \subset B$ with $\frac{1}{4}\mu(B) \leq \mu(A) \leq \frac{1}{2}\mu(B)$, we have $\mu(\partial_a^B A) \geq \lambda\mu(A)$, then there exists some 1-thick subspace B' of B such that $\mu(B') \geq \frac{1}{2}\mu(B)$ and $h_a(B') \geq \frac{\lambda}{2}$.*

Proof. Fix B as above. Given any subset A_0 of B such that $\mu(A_0) \leq \frac{1}{2}\mu(B)$ and $\mu(\partial_a A_0) < \frac{\lambda}{2}\mu(B)$, we have $\mu(A_0) < \frac{1}{4}\mu(B)$ by assumption.

Let m be the supremum of the measures of all subsets A_0 satisfying the above, and let A_1 be such a subset with measure at least $m - \frac{2c}{\lambda}$. Define $A' = [B \setminus [A_1]_a]_1 \subseteq B \setminus A_1$. Note that A' is 1-thick and $\mu(A') \geq \mu(B) - \mu(A_1) - \mu(\partial_a A_1) \geq \frac{5}{8}\mu(B)$.

We wish to show that $h_a(A') \geq \lambda/2$. Suppose for a contradiction that there exists some subset $E \subset A'$ with $\mu(E) \leq \frac{1}{2}\mu(A')$ and $\mu(\partial_a^{A'} E) < \frac{\lambda}{2}\mu(E)$. Since $\partial_a^B E \subseteq \partial_a^{A'} E \cup \partial_a^B A_1$, we have $\mu(\partial_a^B(E \cup A_1)) < \frac{\lambda}{2}(\mu(E) + \mu(A_1))$. From this we deduce that $\mu(E) + \mu(A_1) > \frac{1}{2}\mu(B)$, or, if this is not the case, then $\mu(E) \leq \frac{2c}{\lambda}$ by the choice of A_1.

In the second case we are done: $\partial_a^{A'} E$ contains a ball of radius 1, so $c \leq \mu(\partial_a^{A'} E) < \frac{\lambda}{2}\mu(E) \leq c$ which is a contradiction.

Otherwise $\mu(E \cup A_1) \in (\frac{1}{2}\mu(B), \frac{3}{4}\mu(B))$ so $E' = B \setminus (E \cup A_1)$ satisfies $\mu(E') \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B))$ and $\mu(\partial_a^B E') = \mu(\partial_a^B(E \cup A_1)) < \frac{\lambda}{2}\mu(B)$, which is also a contradiction. $\square$

Proof of Theorem 8.2. This follows immediately from Propositions 8.11 and 8.12 with $\lambda = k/r$. $\square$

9. Upper bounds and large-scale dimension

The goal of this section is to obtain upper bounds on the Poincaré profiles of a metric measure space which is finite dimensional in the sense of the definition below. In doing so, we will prove that the lower bound for groups of polynomial growth in section 8 is sharp.

Definition 9.1. Let (X, d, μ) be a metric measure space. We say X has **measurable dimension at most** n ($\mathrm{mdim}(X) \leq n$) if, for all $r \geq 0$ we can write $X = X_0 \cup \cdots \cup X_n$ and decompose each $X_i = \bigcup X_{ij}$ so that each X_{ij} is 1-thick, $\sup(\mu(X_{ij})) < \infty$ and $d(X_{ij}, X_{ij'}) \geq r$ whenever $j \neq j'$.

If $\mathrm{mdim}(X) \leq n$ we define the function $\gamma_n(r)$ to be the infimal value of $\sup(\mu(X_{ij})) + 1$ taken over all decompositions of X satisfying the above hypotheses.

Notice that $\gamma_n(r)$ is non-decreasing as a function of r.

A simple comparison can be made with asymptotic dimension when the metric measure space has **bounded geometry**: for all $r \geq 0$ there exists some C_r such that $\mu(B(x, r)) \leq C_r$ for all $x \in X$.

Lemma 9.2. *Let (X, d, μ) be a metric measure space with bounded geometry. Then the asymptotic dimension of X is at least* $\mathrm{mdim}(X)$.

Proof. Suppose $asdim(X) \leq n$. This implies that for all $r \geq 0$ one can decompose $X = X_0' \cup \ldots \cup X_n'$ and further decompose each $X_i' = \bigcup X_{ij}'$ so that $\sup \left\{\mathrm{diam}(X_{ij}')\right\} = K_r < \infty$ and $d(X_{ij}, X_{ij'}) \geq r + 2$ whenever $j \neq j'$.

Define $X_{ij} = \bigcup_{y \in X_{ij}'} B(y, 1)$. Each X_{ij} is 1-thick, it has diameter at most $L = K_r + 2$ and $d(X_{ij}, X_{ij'}) \geq r$ whenever $j \neq j'$. Since X has bounded geometry, $\mu(X_{ij}) \leq C_L$ for all i, j. □

Lemma 9.3. *Let (X, d, μ) and (Y, d', μ') be metric measure spaces and suppose Y has bounded packing at scales ≥ 1. If there exists a coarsely regular map $F : X \to Y$, then* $\mathrm{mdim}(X) \leq \mathrm{mdim}(Y)$. *Moreover, for all suitable n we have $\gamma_n^X \lesssim_n \gamma_n^Y$.*

Proof. Suppose $\mathrm{mdim}(Y) \leq n$. Then for all $r \geq 0$ one can write $Y = \bigcup_{i=0}^n \bigcup_j Y_{ij}^r$ where each Y_{ij}^r is 1-thick, $\mu'(Y_{ij}^r) \leq C$ for some C and all i, j, and $d'(Y_{ij}^r, Y_{ij'}^r) > \rho_+(r + 2)$ whenever $j \neq j'$.

Let $X_{ij}^r = [F^{-1}(Y_{ij}^r)]_1$. By Definition 5.1(i), $d(X_{ij}^r, X_{ij'}^r) > r$ whenever $j \neq j'$, and by (ii) $\mu(X_{ij}^r) \asymp \mu'([Y_{ij}^r]_1) \preceq \mu'(Y_{ij}^r)$ by Lemma 2.3. □

Remark 9.4. One can remove the assumption that Y has bounded packing at scales ≥ 1 by removing the assumption that each X_{ij} is 1-thick in the definition of measurable dimension.

Proposition 9.5. *Let (X, d, μ) be a metric measure space with $\mu(X) = \infty$ and measurable dimension at most n. For all $\delta > 0$,*

$$\Lambda_X^p(r) \lesssim_n \sup \left\{\gamma_n(t + \delta)/t \,:\, \gamma_n(t) \leq r/(4n + 4)\right\}.$$

Proof. If γ_n is bounded then μ is bounded, which is a contradiction.

Choose $s > 4(n+1)\gamma_n(0)$ and assume $\mu(A) = s \leq r$. Fix $\delta > 0$ and find t so that $4(n+1)\gamma_n(t) \leq \mu(A) \leq 4(n+1)\gamma_n(t+\delta)$. Select a decomposition of X into sets X^t_{ij} as above where $\mu(X^t_{ij}) \leq \gamma_n(t)$ for all i, j.

Then there exists some i such that $\mu(A \cap X_i) \geq \frac{1}{n+1}\mu(A) \geq 4\gamma_n(t)$. Without loss of generality, assume $i = 0$. Choose J so that

$$X'_0 := \bigcup_{j\in J} X^t_{0j} \quad \text{satisfies} \quad \frac{\mu(A)}{4(n+1)} \leq \mu(A \cap X'_0) \leq \frac{\mu(A)}{2(n+1)}.$$

Set $X''_0 = X^t_0 \setminus X'_0$ and let $f_t : A \to \mathbb{R}$ be the function $f(x) = \frac{1}{t}\min\{t, d_X(x, X'_0)\}$.

Now f_t is $\frac{1}{t}$-Lipschitz, so $\int_A |\nabla_2 f|^p \leq \frac{2^p}{t^p}\mu(A)$. Since f takes values in $[0, 1]$ and has value 0 on X'_0 and value 1 on X''_0 each of measure $\geq \mu(A)/4(n+1)$, we see that $\int_A |f - f_R|^p \, d\mu(x) \geq (\frac{1}{2})^p \frac{1}{4(n+1)}\mu(A)$.

Thus, $h^p_a(A) \leq \frac{4}{t}(n+1)^2 \preceq_n \frac{1}{t}$. As this holds for every measurable $A \subset X$ of finite measure the result follows. $\square$

Remark 9.6. Under nice circumstances, for instance when a space X has a cobounded isometry group, and finite asymptotic dimension where the K_r can be bounded by an affine function of r (sometimes called linearly controlled or asymptotic Assouad–Nagata dimension), the function $\gamma_n(s_r + \delta)/s_r$ is equivalent (up to $\simeq$) to $r/\kappa(r)$ where κ is the inverse growth function. This is easily deduced from the argument in the proof of Proposition 6.1.

Proof of Theorem 7. Let (G, d, μ) be a CGLC metric measure group with $\mu(B(1, r)) \asymp r^m$. Such groups have finite asymptotic Assouad–Nagata dimension ([Bre14, Theorem 1.2] and [HP13, Theorem 5.5]), so by Proposition 9.5, $\Lambda^p_G(r) \lesssim r^{\frac{m-1}{m}}$ for all $p \geq 1$. The lower bound is proved in Theorem 8.1. $\square$

Example 9.7. As another example, for X equal to the product of two 3-regular trees we have $\Lambda^p_X(r) \simeq r/\log(r)$ for all $p \in [1, \infty]$: The case $p = \infty$ follows immediately from Proposition 2. By [BST12, Theorem 3.1] and Proposition 6.5, the lower bound holds when $p = 1$, so the lower bound for general p follows from Proposition 7.2. For the upper bound, X has exponential growth, a cobounded isometry group, and asymptotic Assouad–Nagata dimension 2, so by Proposition 9.5, $\Lambda^p_X(r) \lesssim r/\log(r)$ for all $p \geq 1$.

10. Trees

In this section, we calculate the Poincaré profile for regular trees.

Theorem 10.1. *Let T be the infinite 3-regular tree. Then for every $p \in [1,\infty)$, $\Lambda^p_T(r) \asymp_p r^{(p-1)/p}$.*

For $p = 1$ this is immediate from [BST12]. This theorem immediately implies the following corollary for groups admitting quasi-isometric embeddings of such trees.

Corollary 10.2. *If (G, d, μ) is a CGLC measure group which is non-amenable, non-unimodular, or is compact-by-solvable and has exponential growth, then for any $p \geq 1$, $\Lambda^p_G(r) \gtrsim_{G,p} r^{(p-1)/p}$.*

In this section, for a graph X, and a function $f : VX \to \mathbb{R}$, we define $|\nabla f| : EX \to \mathbb{R}$ as $|\nabla f|(e) = |f(x) - f(y)|$ where $e \in EX$ has endpoints $x, y \in VX$. If X has maximum vertex degree d then for each $p \geq 1$,

$$\|\nabla_2 f\|_p \asymp_d \|\nabla f\|_p = \left(\sum_{e \in EX} |\nabla f|(e)^p \right)^{1/p}.$$

A key step in proving Theorem 10.1 is to reduce to an etimate on complete graphs in the spirit of, for example, Spielman [Spi15, Section 4.7].

Proposition 10.3. *For any $r \in \mathbb{N}, r \geq 2$ and $p \in [1,\infty)$, letting K_r denote the complete graph on r vertices, we have*

$$r^{1/p} \leq \inf \left\{ \frac{\|\nabla f\|_p}{\|f - f_{K_r}\|_p} \ : \ f : VK_r \to \mathbb{R}, f \not\equiv f_{K_r} \right\} \preceq_p r^{1/p}.$$

Proof. Let $f : VK_r \to \mathbb{R}$ be any non-constant function on K_r. Then

$$\begin{aligned}
\|f - f_{K_r}\|_p^p &= \sum_x \left| f(x) - \frac{1}{r} \sum_y f(y) \right|^p \\
&\leq \frac{1}{r^p} \sum_x \left(\sum_y |f(x) - f(y)| \right)^p \\
&\leq \frac{1}{r^p} \sum_x \left(\sum_y |f(x) - f(y)|^p \right) r^{p-1} \\
&= r^{-1} \|\nabla f\|_p^p.
\end{aligned}$$

This proves the first inequality; the second can be seen by considering a function which is 1 and -1 on one vertex each, and zero everywhere else. $\square$

Proof of Theorem 10.1. First we show the upper bound, which is relatively simple.

Suppose $A \subset T$ is a graph of size $|A| = r$; we can find a vertex x so that on deleting this vertex, all remaining connected components have size $\leq r/2$. Group these components into sets U, V of size $\in [r/4, 3r/4]$. Let $f : A \to [-1, 1]$ be identically -1 on U, 1 on V and 0 on x.

Clearly $\|f - f_A\|_p^p \geq \frac{1}{4}r$, and since ∇f is only non-zero on edges adjacent to x, $\|\nabla f\|_p^p \leq 3$. Thus $h^p(A) \leq (12/r)^{1/p}$ and

$$\Lambda^p(r) = \sup_{|A| \leq r} |A| h^p(A) \leq 12 r^{(p-1)/p}.$$

Second, we show the lower bound.

For any $r > 0$ there exists a ball $B = B(x_0, t) \subset T$ of size $\asymp 2^t \asymp r$, so we can assume $r = |B|$ and it then suffices to show that $h^p(B) \succeq |B|^{-1/p}$, with constant independent of B.

Let K_r be the complete graph on r vertices. Suppose that a non-constant function $f : B \to \mathbb{R}$ is given. Consider f as a function on the complete graph $K_r = K_{|B|}$. In light of Proposition 10.3, to show that $h^p(B) \succeq |B|^{-1/p}$, it suffices to show that

$$\sum_{e \in EB} |\nabla f(e)|^p \geq \frac{1}{2|B|^2} \sum_{x,y \in B} |f(x) - f(y)|^p,$$

for then $h^p(B) \succeq |B|^{-2/p} r^{1/p} \succeq |B|^{-1/p}$.

Now for each $x, y \in B$, let γ_{xy} be the simple path in T joining x to y. Observe that $|f(x) - f(y)| \leq \sum_{e \in \gamma_{xy}} |\nabla f(e)|$.

For each $e \in EB$, let N_e be the number of such simple paths that pass through e. Observe that $N_e \asymp 2^t \cdot 2^{t - d(x_0, e)}$, where $d(x_0, e)$ is the distance from the centre of the ball to the edge e.

Using Hölder's inequality, we have

$$\begin{aligned}
\sum_{x,y \in B} |f(x) - f(y)|^p &\leq \sum_{x,y \in B} \left(\sum_{e \subset \gamma_{xy}} |\nabla f(e)| \right)^p \\
&= \sum_{x,y \in B} \left(\sum_{e \subset \gamma_{xy}} |\nabla f(e)| N_e^{-1/p} N_e^{1/p} \right)^p \\
&\leq \sum_{x,y \in B} \left(\sum_{e \subset \gamma_{xy}} |\nabla f(e)|^p N_e^{-1} \right) \left(\sum_{e \subset \gamma_{xy}} N_e^{1/(p-1)} \right)^{p-1}
\end{aligned}$$

For each simple path, $N_e^{1/(p-1)}$ takes values in (two) geometric series, with ratio depending only on p and maximum value $\preceq (2^{2t})^{1/(p-1)} \asymp |B|^{2/(p-1)}$, and so the sum inside the second parentheses above is also

$\preceq |B|^{2/(p-1)}$. Thus,

$$\sum_{x,y\in B} |f(x)-f(y)|^p \preceq \sum_{x,y\in B}\sum_{e\in\gamma_{xy}} |\nabla f(e)|^p N_e^{-1}|B|^2$$
$$\leq 2|B|^2 \sum_{e\in B} |\nabla f(e)|^p,$$

and so we are done. □

11. Lower bounds for hyperbolic spaces with boundary Poincaré inequalities

In this section we find lower bounds on Poincaré profiles for hyperbolic groups whose boundaries admit Poincaré inequalities in the sense of Heinonen and Koskela. We will apply these results to rank 1 symmetric spaces, and a family of hyperbolic buildings introduced and studied by Bourdon and Pajot.

Suppose a metric space (Z,ρ) is **Ahlfors Q-regular**, i.e. there is a measure μ on Z so that for every ball $B(z,r)$ in Z with $r \leq \operatorname{diam}(Z)$, we have $\mu(B(z,r)) \asymp r^Q$. (We may take μ to be the Hausdorff Q-measure on Z.) For $p,q\geq 1$, we say (Z,ρ) admits a **(q,p)-Poincaré inequality** (with constant $L\geq 1$) if for every Lipschitz function $f: Z\to\mathbb{R}$ and every ball $B(z,r)\subset Z$,

$$\left(⨍_{B(z,r)} |f-f_{B(z,r)}|^q \, d\mu\right)^{1/q} \leq Lr\left(⨍_{B(z,Lr)} (\operatorname{Lip}_x f)^p \, d\mu(x)\right)^{1/p},$$

where for $U\subset Z$, $f_U = ⨍_U f\,d\mu = \frac{1}{\mu(U)}\int_U f\,d\mu$, and

$$\operatorname{Lip}_x f = \limsup_{r\to 0} \sup_{y\in B(x,r)} \frac{|f(y)-f(x)|}{r}.$$

If $q=1$, we say Z admits a **p-Poincaré inequality**. By Hölder's inequality, if Z admits a p-Poincaré inequality, it admits a q-Poincaré inequality for all $q\geq p$. Moreover, since Z is doubling, it will admit (q,q)-Poincaré inequalities for all $q\geq p$ by [HK00, Theorem 5.1].

Theorem 11.1. *Suppose that X is a visual Gromov hyperbolic graph with a metric $\rho\in\mathcal{C}_X$ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a p-Poincaré inequality. Then for all $q\geq p$, $\Lambda^q_X(r)\gtrsim r^{1-1/Q}$.*

Proof. Consider $\partial_\infty X$ with the metric ρ, which admits a p-Poincaré inequality with some constant $L\geq 1$. As a consequence, $\partial_\infty X$ is connected (in fact, quasiconvex).

Following Bourdon–Pajot [BP03, Section 2.1], we ensure that $Z = (\partial_\infty X,\rho)$ has diameter $1/2$ by rescaling, and define a graph Γ which approximates Z: Γ has vertex set $\{z_t^i : t \in \mathbb{N}, 1 \leq i \leq k(t)\}$ where for each $t \in \mathbb{N}$, $\Gamma_t = \{z_t^1, \ldots, z_t^{k(t)}\}$ is a maximal e^{-t}-separated net in Z. To each z_t^i we associate a ball $B(z_t^i, e^{-t}) \subset Z$, and we join z_t^i and z_u^j by an edge if and only if $|t-u| \leq 1$ and $B(z_t^i, e^{-t}) \cap B(z_u^j, e^{-u}) \neq \emptyset$.

By Bourdon–Pajot [BP03, Proposition 2.1, Corollary 2.4], Γ, with the path metric d, is a bounded degree hyperbolic graph which is quasi-isometric to X, and so it suffices to show the separation bound for Γ.

We now consider the sequence $Z_t = (Z, \rho_t, \mu_t)$ of metric measure spaces, where $\rho_t = e^t \rho$, and $\mu_t = e^{Qt}\mu$. Note that $\mu_t(Z_t) \asymp e^{Qt}$. We deduce from the Poincaré inequality satisfied by Z that Z_t satisfies for all Lipschitz function f on Z_t, for all $q \geq p$

$$\left(\fint |f - f_{Z_t}|^q \, d\mu_t\right)^{1/q} \preceq e^t \left(\fint (\mathrm{Lip}_x f)^q \, d\mu_t(x)\right)^{1/p},$$

and therefore that

$$h^q_{\mathrm{Lip}}(Z_t) \succeq e^{-t}.$$

By Proposition 3.9, this implies that

$$h^q_1(Z_t) \succeq e^{-t}.$$

Now equip Γ_t with the counting measure and the distance induced from its inclusion in Z_t. Since Γ_t is a maximal 1-separated subset of Z_t, we can find a partition

$$Z_t = \bigsqcup_{\gamma \in \Gamma_t} A_\gamma,$$

where

$$B_{\rho_t}(\gamma, \tfrac{1}{2}) \subset A_\gamma \subset B_{\rho_t}(\gamma, 3).$$

By the Ahlfors regularity of Z_t, $\mu(A_\gamma) \asymp 1$. Hence by Lemmas 5.8 and 3.3(ii), we deduce that

$$h^q_{10}(\Gamma_t, \rho_t) \succeq e^{-t}.$$

In order to conclude, we need to show that there exists a constant C such that two vertices $x, y \in \Gamma_t$ such that $\rho_t(x,y) \leq 10$ satisfy $d(x,y) \leq C$ (where $d(x,y)$ is their distance in Γ). Indeed, that will show that

$$h^q_C(\Gamma_t, d) \succeq e^{-t},$$

and since $|\Gamma_t| \asymp e^{Qt}$,

$$\Lambda^q_\Gamma(r) \succeq r^{1-1/Q}.$$

By [BP03, Lemma 2.2], for $x, y \in \Gamma$ corresponding to balls $B_x, B_y \subset Z$, $e^{-(x|y)} \asymp \mathrm{diam}(B_x \cup B_y)$, where $(x|y)$ denotes the Gromov product with respect to the base point z_1^1. For $x, y \in \Gamma_t$, we have $(x|y)$ equal to $t - \frac{1}{2}d(x,y)$ up to a uniform additive error, and $\mathrm{diam}(B_x \cup B_y) \asymp e^{-t} + \rho(x,y)$, so

$$e^{-t}e^{\frac{1}{2}d(x,y)} \asymp \mathrm{diam}(B_x \cup B_y) \asymp e^{-t} + \rho(x,y).$$

Thus, $\rho(x,y) \leq 10e^{-t}$ implies that $d(x,y) \preceq 1$, which completes the proof of Theorem 11.1. □

12. Upper bounds for hyperbolic spaces with hyperplanes

In this section we present an approach to finding upper bounds on the L^p-Poincaré profiles of hyperbolic spaces.

Definition 12.1. Let (X,d) be a metric space and $x_0 \in X$. For $C \geq 1$, a subset $A \subseteq X$ is said to be a **C-asymptotic shadow of** x_0 if, for every $x \in A$ there is a C-quasi-geodesic ray $\gamma_x : [0,\infty) \to X$ with $\gamma_x(0) = x_0$ and $d(\gamma_x(r_x)) = x$ for some r_x, and $\gamma_x[r_x,\infty) \subseteq A$. (Recall that a C-quasi-geodesic ray is a (C,C)-quasi-isometric embedding of $[0,\infty)$.)

Our hypotheses are as follows:

(1) (X,d,μ) is a δ-hyperbolic geodesic metric measure space, and it is **visual**: there exists $x_0 \in X$ and $C \geq 0$ so that every $x \in X$ belongs to a C-quasi-geodesic ray $\gamma : [0,\infty) \to X$ with $\gamma(0) = x_0$.
(2) There exists a constant $h(X) > 0$ (called the volume entropy) and a constant $C \geq 0$ such that for every $R > 0$, $h(X)R - C \leq \log_e(\mu(B_R(x_0))) \leq h(X)R + C$.
(3) There is a visual metric ρ on $\partial_\infty X$ based at $x_0 \in X$ with visibility parameter $\epsilon > 0$; i.e., $\rho(\cdot,\cdot) \asymp \exp(-\epsilon(\cdot|\cdot)_{x_0})$, where $(\cdot|\cdot)_{x_0}$ denotes the Gromov product with respect to x_0.

Let $\mathrm{Isom}_\mu(X)$ be the group of μ-preserving isometries of X. In the case that μ is non-zero, and that $\mathrm{Isom}_\mu(X)$ contains a finitely generated group acting properly and cocompactly on X, these properties all hold [Coo93, Theorem 7.2], and $(\partial_\infty X, \rho)$ is Ahlfors Q-regular with $Q = \frac{1}{\epsilon}h(X)$.

We have one more hypothesis, which only need hold for large a, where $a \geq 2$ is the constant of thickness in Definition 4.1.

(4) There exist constants $\kappa, N, C > 0$ such that for any a-thick subspace Z of X with measure at least N, there is some $\psi \in \mathrm{Isom}_\mu(X)$, and there exist two measurable subsets $H^\pm$ of X which are C-asymptotic shadows of x_0, and satisfy the inequalities $\rho(\partial_\infty H^+, \partial_\infty H^-) \geq \kappa$, $\mu(\psi(Z) \cap H^+) \geq \kappa\mu(Z)$ and $\mu(\psi(Z) \cap H^-) \geq \kappa\mu(Z)$.

All of these conditions are satisfied, for example, by $\mathbb{H}^k_{\mathbb{R}}$ with its usual metric and measure, with $h(\mathbb{H}^k_{\mathbb{R}}) = k-1$, and ρ the Euclidean metric on $\partial_\infty \mathbb{H}^k_{\mathbb{R}} = \mathbb{S}^{k-1}$; for condition (4) see later in this section.

Proposition 12.2. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be proper hyperbolic metric measure spaces admitting cobounded measure-preserving isometry groups such that there exists a coarse regular map $F : X \to Y$ which is also a quasi-isometry. If condition (4) holds for X, then it also holds for Y.*

Proof. Fix $x_0 \in X$. Let $F : X \to Y$ be a (K, C)-quasi-isometry, and let $F^{-1} : Y \to X$ be a C-coarse inverse of R. Set $y_0 = F(x_0)$ and fix a visual metric ρ_Y on Y. Let Z' be a $Ka + C$-thick subspace of Y and let $Z'' = [F^{-1}(Y)]_a$. Since r is coarse regular, if Z' has sufficiently large measure in Y then $\mu_X(Z'') \geq N$. Now Z'' is a-thick, so there is some $\psi \in \mathrm{Isom}_{\mu_X}(X)$ and two measurable subsets $H^\pm_X$ of X which are D-asymptotic shadows of x_0, and satisfy the inequalities $\rho_X(\partial_\infty H^+_X, \partial_\infty H^-_X) \geq \kappa$, $\mu(\psi(Z'') \cap H^+_X) \geq \kappa\mu(Z'')$ and $\mu(\psi(Z'') \cap H^-_X) \geq \kappa\mu(Z'')$.

For each $\psi \in \mathrm{Isom}_{\mu_X}(X)$ we may choose $\phi_\psi \in \mathrm{Isom}_{\mu_Y}(Y)$ such that $\psi(F^{-1}(\phi_\psi^{-1} y_0))$ is contained in a ball centred at x_0 of uniformly bounded radius (at most L, independent of ψ). The maps $\psi^{-1} \circ F \circ \phi_\psi : Y \to X$ are all (K, C)-quasi-isometric coarse regular maps and all map x_0 uniformly close to y_0. Fixing a visual metric ρ_Y on Y such that $\rho_Y(\cdot,\cdot) \asymp \exp(-\epsilon'(\cdot,\cdot)_{y_0})$ for some $\epsilon' > 0$, we see that the quasi-symmetry q from $\partial_\infty X \to \partial_\infty Y$ induced by $\psi^{-1} \circ F \circ \phi_\psi$ has the property that $\rho_Y(q(\partial_\infty H^+_X), q(\partial_\infty H^-_X)) \geq \kappa' > 0$ for some uniform κ' (again, independent of ψ).

Define $H^\pm_Y$ to be the set of all points within distance A of a geodesic ray connecting y_0 to a point in $q(\partial_\infty H^\pm_X)$. These sets are clearly A-asymptotic shadows of y_0. If A is sufficiently large—in terms of K, C and the hyperbolicity constant of Y—then the Morse lemma implies that $F(H^\pm_X) \subseteq H^\pm_Y$. Since F is coarse regular, there is some $\kappa'' > 0$ such that $\mu_Y(Z' \cap H^\pm_Y) \geq \kappa''\mu_Y(Z')$ and we are done. $\square$

Theorem 12.3. *Suppose X satisfies conditions (1)–(4) above for some fixed $\delta, C, \epsilon, \kappa, N$ and set $Q = h(X)/\epsilon$. Then we have the following bounds on Λ^p_X:*

$$\Lambda^p_{X,a}(r) \lesssim_{\delta,C,\kappa,N} \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } p < Q, \\ r^{\frac{p-1}{p}} \log(r)^{\frac{1}{p}} & \text{if } p = Q, \\ r^{\frac{p-1}{p}} & \text{if } p > Q. \end{cases}$$

Proof. Let $x_0 \in X$ and $a \geq 2$ be fixed so that (4) holds. Let Z be an a-thick subspace of X of sufficiently large finite measure (to be determined later). Apply (4) to move Z; without loss of generality we may assume that $\psi = id$. Let $H^{\pm}$ be the corresponding C-asymptotic shadows of x_0.

Define $\partial_\infty \phi : (\partial_\infty X, \rho) \to [0,1]$ by

$$\partial_\infty \phi(z) = \min\{1, \max\{0, \tfrac{3}{\kappa}\rho(z, \partial_\infty H^-) - 1\}\};$$

this is a $\frac{3}{\kappa}$-Lipschitz function so that $\partial_\infty \phi$ is zero on $[\partial_\infty H^-]_{\kappa/3}$ and one on $[\partial_\infty H^+]_{\kappa/3}$.

We choose a function $\phi : X \to [0,1]$ by setting $\phi(x) = \partial_\infty \phi(\eta)$ where $\eta \in \partial_\infty X$ is the endpoint of some C-quasi-geodesic $\gamma_x : [0,\infty) \to X$ with $\gamma_x(0) = x_0$ and $\gamma_x(t) = x$ for some t. Regardless of the choices made in defining this function we have the following control: for any $x, y \in X$ with $d(x,y) \leq C'$ there exists $K = K(\delta, C, C', \rho, \kappa)$ so that

$$|\phi(x) - \phi(y)| \leq K \exp(-\epsilon d(x, x_0)). \tag{12.4}$$

By a similar argument, there exists $L > 0$ so that if $d(x, x_0) \geq L$ and $x \in H^-$ then the endpoint η of γ_x used to define $\phi(x)$ satisfied $\rho(\eta, \partial_\infty H^-) \leq \kappa/3$, and so $\phi(x) = 0$. Likewise, if $x \in H^+$ and $d(x, x_0) \geq L$ then $\phi(x) = 1$.

By assuming that $\mu(Z)$ is greater than $\frac{2}{\kappa}\mu(B(x_0, L))$, we know—by assumption (4)—that $\mu(Z \cap H^- \setminus B(x_0, L))$ and $\mu(Z \cap H^+ \setminus B(x_0, L))$ are both $\geq \frac{\kappa}{2}\mu(Z)$. Switching the roles of $H^{\pm}$ if necessary, we assume $\phi_Z \geq 1/2$ and so

$$||\phi - \phi_Z||_{Z,p}^p \geq |\phi_Z|^p \mu(Z \cap H^- \setminus B(x_0, L)) \geq 2^{-p-1}\kappa\mu(Z). \tag{12.5}$$

We now bound $||\nabla_a \phi||_{B,p}$ on the ball $B = B(x_0, r)$. Since we have $\mu(B(x_0, R)) \asymp \exp(h(X)R)$, (12.4) gives

$$||\nabla_a \phi||_{B,p}^p \preceq_{K,\kappa,p} \int_{t=0}^{r} \exp(h(X)t)\exp(-p\epsilon t)dt. \tag{12.6}$$

We now consider the three cases for p separately.

Case 1, $p > Q$**:** Equation (12.6) gives that $||\nabla_a \phi||_{X,p}^p$ is bounded by some constant D only depending on K, κ and p, so (12.5) gives $h_a^p(Z) \preceq_{K,\kappa,p} \mu(Z)^{-1/p}$ for any subspace Z and the case $p > Q$ follows.

Case 2, $p < Q$**:** The function ϕ is no longer a p-Dirichlet, but its gradient is well-behaved. Indeed, (12.4) gives $|\nabla_a \phi|(x) \preceq \exp(-\epsilon d(x, x_0))$, so

$$\|\nabla_a \phi\|_{B,p}^p \preceq \|\exp(-\epsilon d(\cdot, x_0))\|_{Z,p}^p. \tag{12.7}$$

Now we wish to put an upper bound on $||\nabla_a\phi||^p_{Z,p} \,/\, ||\phi-\phi_Z||^p_{Z,p}$, which by (12.5) is bounded by $||\nabla_a\phi||^p_{Z,p} \,/\mu(Z)$ up to a uniform multiplicative error. Thus by (12.7) it suffices to maximize $||\exp(-\epsilon d(\cdot,x_0))||^p_{Z,p}$ among all sets Z with the same measure.

But, up to a uniform multiplicative error, $||\exp(-\epsilon d(\cdot,x_0))||^p_{Z,p}$ is maximised when $d(\cdot,x_0)$ is minimised as a function from Z to $\mathbb{R}$. Clearly this occurs when Z is a metric ball centred at x_0.

By (12.6), for $Z=B(x_0,r)$,

$$||\nabla_a\phi||^p_{Z,p} \preceq \exp(h(X)r)\cdot\exp(-p\epsilon r) \asymp \mu(Z)\cdot\mu(Z)^{-p/Q}, \tag{12.8}$$

thus $h^p_a(Z) \preceq \mu(Z)^{-1/Q}$ and the bound on $\Lambda^p_{X,a}(\mu(Z))$ follows.

Case 3, $p=Q$**:** If $p=Q$ then the same argument as in Case 2 shows that inequality (12.8) is maximised for a metric ball, so

$$||\nabla_a\phi||^p_{Z,p} \preceq \int_{t=0}^{r} \exp(h(X)t)\exp(-p\epsilon t)dt = r \asymp \log(\mu(Z)),$$

so $h^p_a(Z) \preceq \log(\mu(Z))^{1/p}\cdot\mu(Z)^{-1/p}$ and thus the bound on Λ^p_X for $p=Q$ follows. □

The following is an immediate consequence of Proposition 12.2 and Theorem 12.3.

Corollary 12.9. *Let G be a compactly generated locally compact hyperbolic group admitting a geometric action on a space X which satisfies (1)-(4) above. Set $Q=\inf\{h(Y)/\epsilon_Y\}$ where the infimum is taken over all spaces Y admitting a geometric G action and all visibility parameters ϵ_Y of Y. Then*

$$\Lambda^p_{X,a}(r) \lesssim_{\delta,C,\kappa,N} \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } p<Q, \\ r^{\frac{p-1}{p}} & \text{if } p>Q. \end{cases}$$

If the infimum can be realised, then

$$\Lambda^Q_X(r) \lesssim_{\delta,C,\kappa,N} r^{\frac{Q-1}{Q}} \log(r)^{\frac{1}{Q}}.$$

12.1. **Helly's Theorem.** In the next two sections we use Helly's theorem for CAT(0) spaces in order to show that the hypotheses of Theorem 12.3 are satisfied for Bourdon–Pajot buildings and real hyperbolic spaces. The version suitable for our needs is the following variation on a result of Ivanov [Iva14].

Theorem 12.10 (Ivanov)**.** *Let X be a uniquely geodesic space of compact topological dimension $k<\infty$ (for example, a CAT(0) space of geometric dimension k). Let $\mathcal{H}$ be a (possibly infinite) collection of* *closed convex subsets of* X*, with the property that there exists a compact convex set* $Y \subset X$ *so that for any* $H_1, \ldots, H_{k+1} \in \mathcal{H}$ *we have* $Y \cap H_1 \cap \cdots \cap H_{k+1} \neq \emptyset$*. Then* $\bigcap_{H \in \mathcal{H}} H \supset \bigcap_{H \in \mathcal{H}} H \cap Y \neq \emptyset$*.*

Proof. If not, then for any $y \in Y$ there exists $H_y \in \mathcal{H}$ with $y \notin H_y$. Since Y is compact, for some $y_1, \ldots, y_m$ we have that $\{X \setminus H_{y_i}\}_{i=1,\ldots,m}$ is a finite subcover of the open cover $\{X \setminus H_y\}_{y \in Y}$ of Y. By assumption, any $k+1$ of the finite collection of convex sets $\{Y, H_{y_1}, \ldots, H_{y_m}\}$ have non-empty intersection (in Y), and so Helly's Theorem [Iva14, Theorem 1.1] implies that there exists $y \in Y \cap H_{y_1} \cap \cdots \cap H_{y_m} \neq \emptyset$. This is a contradiction, since y is not covered by $\{X \setminus H_{y_i}\}_{i=1,\ldots,m}$. □

12.2. **Bourdon-Pajot buildings.** We recall that a **halfspace** of a metric space is a convex set with convex complement.

Bourdon and Pajot [BP99] showed that a family of Fuchsian buildings earlier studied by Bourdon [Bou97] have boundaries that admit 1-Poincaré inequalities.

Definition 12.11. Let $m \geq 5, n \geq 3$ be given. Let R be the regular, right-angled hyperbolic polygon with m sides. Let $I = I_{mn}$ be the Fuchsian building where the chambers are isometric to R, each edge is adjacent to n copies of R, and the vertex links are copies of the complete bipartite graph with n, n vertices.

The group

$$\Gamma_{mn} = \langle s_1, \ldots, s_m \mid s_i^n, [s_i, s_{i+1}]\, \forall i \rangle,$$

where indices are modulo m, acts cellularly and geometrically on I_{mn}. By [BP99, Theorem 1.1], $\partial_\infty \Gamma_{mn} = \partial_\infty I_{mn}$ carries an Ahlfors Q_{mn}-regular metric which admits a 1-Poincaré inequality, where $Q_{mn} = 1 + \log(n-1)/\operatorname{arccosh}((m-2)/m) \in (1, \infty)$.

The apartments in I_{mn} are each copies of the hyperbolic plane tiled by right-angled regular m-gons. As such, they have separation at least $\log(r)$; the boundary geometry lets us find much larger lower bounds.

Theorem 12.12. *Given* $m \geq 5, n \geq 3$*, and* $p \in [1, \infty)$*,*

$$\Lambda^p_{I_{mn}}(r) \simeq_p \begin{cases} r^{1-1/Q_{mn}} & \text{if } p < Q_{mn} \\ r^{1-1/p} & \text{if } p > Q_{mn} \end{cases},$$

and if $p = Q_{mn}$ *then*

$$r^{1-1/Q_{mn}} \lesssim \Lambda^{Q_{mn}}_{I_{mn}}(r) \lesssim r^{1-1/Q_{mn}} \log(r)^{1/Q_{mn}}.$$

Proof. The lower bounds follow from Theorem 11.1 for $p \leq Q_{mn}$ and Corollary 10.2 for $p \geq Q_{mn}$.

The upper bounds will follow from Theorem 12.3 once we have established that the hypotheses of that theorem hold. First we consider $I = I_{mn}$ with its usual CAT(-1) metric, a basepoint o in the centre of a chamber, and boundary metric ρ with visual exponent $\epsilon = 1$ with respect to o. We let μ be the counting measure on the points in the centre of chambers of I (which can be identified with elements of Γ_{mn}).

Our aim is to show that (4) holds.

Following Bourdon [Bou97, 2.4.A], we define *tree-walls* in I: for any edge e in I the tree-wall T through W is the union of all geodesics in I which contain e. This tree T is contained in the 1-skeleton of I and can be described locally as follows: if e has endpoints e_-, e_+, connect e to the edges in I which meet e_+ and correspond to vertices not adjacent to e in the link of e_+; do likewise for e_-, and continue. The set T is a totally geodesic subset of I so that $I \setminus T$ has n connected components which are each open and convex.

Viewed differently, for any chamber R let $\pi_R : I \to A \cong \mathbb{H}^2_{\mathbb{R}}$ be a standard projection of I onto an apartment A of I, which is a copy of $\mathbb{H}^2_{\mathbb{R}}$ tiled by regular right-angled m-gons. For any hyperbolic geodesic in the 1-skeleton of A, its inverse image under π_R is a disjoint union of tree-walls.

Each vertex $v \in I$ is adjacent to n^2 chambers in I. Let $M \geq 3$ be the number of tree-walls which meet these chambers.

Let $Z \subset I$ be a finite collection of points which are in the centres of chambers and assume Z is of sufficiently large size. Let Y be the intersection of all closed half-spaces of I that contain Z, where a closed half-space is the closure of a connected component of the complement of a tree-wall. The geometry of I gives that Y is bounded (consider the image of Z in the projection to any apartment), and since each closed half-space is closed and convex, Y is compact and convex.

Let $\mathcal{H}$ be the collection of closed half-spaces of I which contain $> \frac{M}{M+1}|Z|$ elements of Z. Clearly, any 3 of the sets in $\mathcal{H}$ have non-trivial intersection in $Z \subset Y$, so by Helly's theorem 12.10, $Z_{CH} = \bigcap_{H \in \mathcal{H}} H \neq \emptyset$. This set Z_{CH} is convex and consists of closed chambers of I.

Not all the chambers in Z_{CH} are adjacent to a single vertex, for then ∂Z_{CH} meets at most M tree-walls; each of these has $< |Z|/(M+1)$ points of Z in their half-spaces not containing Z_{CH}. Thus $|Z| \leq |Z_{CH}| + \frac{M}{M+1}|Z|$, a contradiction for $|Z|$ sufficiently large.

Let R be a chamber in Z_{CH}, with image $R_1 = \pi_R(R) \subset A \cong \mathbb{H}^2_{\mathbb{R}}$; we can assume that the centre of R_1 is in the centre of $\mathbb{H}^2_{\mathbb{R}}$ in the disc model. Since not all the chambers of Z_{CH} are adjacent to a single vertex, we may assume that there exist chambers $R_2, R_3 \subset \pi_R(Z_{CH}) \subset A$ adjacent to R_1 but with $\overline{R_2} \cap \overline{R_3} = \emptyset$. Let γ_2, γ_3 be the geodesic lines in A that separate R_1 from R_2 and R_3 respectively, and let $T_2 \subset \pi_R^{-1}(\gamma_2)$ and $T_3 \subset \pi_R^{-1}(\gamma_3)$ be the corresponding tree-walls containing $\overline{R_1} \cap \overline{R_2}$ and $\overline{R_1} \cap \overline{R_3}$ respectively.

Let C_2, C_3 be the connected components of $\mathbb{H}^2_{\mathbb{R}} \setminus (\gamma_2 \cup \gamma_3)$ containing R_2, R_3, respectively. Observe that $\partial_\infty C_2, \partial_\infty C_3$ are at a distance $\geq \kappa_1$ in $\mathbb{S}^1 = \partial_\infty \mathbb{H}^2_{\mathbb{R}}$, where $\kappa_1 > 0$ is independent of R_2, R_3 and depends only on m.

Now by construction T_2 (and likewise T_3) was not bounding a half-space in $\mathcal{H}$. Thus at least one of the $(n-1)$ half-spaces $I \setminus T_2$ not containing R has at least $\frac{|Z|}{(n-1)(M+1)}$ of the points of Z in it. Let H^- be one such half-space, and let H^+ be a corresponding half-space for T_3.

After moving the centre of R to $o \in I$ using the appropriate element of Γ_{mn}, we have that $\rho(\partial_\infty H^-, \partial_\infty H^+) \geq \kappa_2 > 0$ where $\kappa_2 = \kappa_2(\kappa_1)$.

This completes the proof of (4) for the standard metrics on I and $\partial_\infty I$ coming from the CAT(-1) condition. However, we do not have the right volume entropy $h(I)$.

To get the sharp bound on conformal dimension, Bourdon uses a word metric on Γ_{mn} and a visual metric ρ' on $\partial_\infty \Gamma_{mn}$ with a visual exponent ϵ_{mn} so that the volume entropy $h(\Gamma_{mn})/\epsilon_{mn} = Q_{mn}$ [Bou97, Lemma 3.1.4]. Thus hypotheses (1)–(3) of Theorem 12.3 are satisfied with the correct value of Q_{mn}.

It remains to check (4) for the word metric on Γ_{mn} with μ' be the counting measure on Γ_{mn}. However, the word metric on Γ_{mn} is quasi-isometric to the metric induced by that of I, and so our proof above of (4) for I implies that (4) also holds for Γ_{mn}. $\square$

12.3. **Real hyperbolic spaces.** In this section we calculate the Poincaré profiles of real hyperbolic spaces. The case $p = 1$ is dealt with by [BST12, Proposition 4.1] and Proposition 6.5.

Theorem 12.13. *Let $k \geq 2$ and $p \in [1, \infty)$, then*

$$\Lambda^p_{\mathbb{H}^k_{\mathbb{R}}}(r) \simeq \begin{cases} r^{\frac{k-2}{k-1}} & \text{if } p < k-1 \\ r^{\frac{p-1}{p}} \log(r)^{\frac{1}{p}} & \text{if } p = k-1 \\ r^{\frac{p-1}{p}} & \text{if } p > k-1 \end{cases}$$

The lower bound for $p < k-1$ follows from the fact that each $\mathbb{H}^k_{\mathbb{R}}$ contains a coarsely embedded copy of $\mathbb{R}^{k-1}$ as a horosphere. Real hyperbolic spaces admit an embedded 3-regular tree, so the lower bound for $p > k-1$ follows from Corollary 10.2.

Our first goal is to show that the hypotheses of Theorem 12.3 hold for $X = \mathbb{H}^k_{\mathbb{R}}$. Hypotheses (1)–(3) hold for $h(X) = 1, \epsilon = 1$ and ρ the Euclidean metric on the sphere $\partial_\infty \mathbb{H}^k_{\mathbb{R}} = \mathbb{S}^{k-1}$, so it remains to verify condition (4).

Inspired by the arguments presented in [BST12], we first show that finite measure thick subsets of real hyperbolic spaces have "medians".

We equip real hyperbolic spaces $\mathbb{H}^k_{\mathbb{R}}$ with their usual CAT(-1) metric d and associated measure μ.

Lemma 12.14. *(Centrepoint theorem) Let $a > 0$ and let $X = \mathbb{H}^k_{\mathbb{R}}$. There exists a constant $c = c(k, a) > 0$ such that for any a-thick subset Z of X with finite measure, there is a point $x \in X$ such that for any half-space H of X containing x, we have $\mu(H \cap Z) \geq c\mu(Z)$.*

Proof. By assumption $Z = \bigcup_{i\in I} B(z_i, a)$ for some $\{z_i\}_{i\in I} \subset Z$. Let Z' be an $2a$-separated $4a$-net in $\{z_i : i \in I\}$. It follows that $|Z'| \asymp_a \mu(Z)$ since $|Z'|\, \mu(B(z_i, a)) \leq \mu(Z) \leq |Z'|\, \mu(B(z_i, 5a))$ for some (any) z_i.

Let Y be a large closed (convex) ball containing Z'. Let $\mathcal{Z}$ be the set of all closed half-spaces of X containing more than $\frac{k}{k+1} |Z'|$ of the points in Z'. Thus the intersection of any $k + 1$ of the sets in $\mathcal{Z}$ has non-empty intersection with Y.

Applying Helly's theorem 12.10, and the fact that X has geometric dimension k, there exists some $x \in \bigcap_{H\in\mathcal{Z}} H$. Thus for any half-space $H \subset X$ with $|H \cap Z'| > \frac{k}{k+1} |Z'|$ we have $x \in H$.

It is an easy exercise to see that x is contained in every half-space H such that $|Z' \cap H| > \frac{k}{k+1} |Z'|$ if and only if every half-space H containing x satisfies $|Z' \cap H| > \frac{1}{k+1} |Z'|$.

Let H be a half-space containing x and let $Z'_H = Z' \cap H$. It is clear that $\mu(B(z, r) \cap H) \geq \frac{1}{2}\mu(B(z, r))$ for any $z \in Z'_H$ and any $r \geq 0$, so

$$\mu(Z \cap H) \geq \frac{\mu(B(z, a))}{2(k + 1)} |Z'| \asymp_{k,a} \mu(Z). \qquad \square$$

We can use a measure-preserving isometry to move such a centrepoint x to the origin $o \in \mathbb{H}^k_{\mathbb{R}}$ in the Poincaré ball model, and now show that hypothesis (4) of Theorem 12.3 is satisfied.

Lemma 12.15. *There exist constants $\kappa, C > 0$ so that for any $Z \subset X = \mathbb{H}^k_{\mathbb{R}}$ with Z a-thick, and $o \in X$ a centrepoint of Z, there exist C-asymptotic shadows of o denoted by $H^-, H^+ \subset X$ so that we have $\rho(\partial_\infty H^-, \partial_\infty H^+) \geq \kappa$ and that $\mu(Z \cap H^-), \mu(Z \cap H^+) \geq \kappa\mu(Z)$.*

Proof. Fix $a > 0$ and $c = c(k, a) > 0$ the constants from Lemma 12.14.

Let $H \subset \mathbb{H}^k$ be a hyperplane containing o, and let $\alpha > 0$. We denote by H^α the union of all two-sided geodesics passing through o and with end points in the α-neighborhood of the boundary $\partial_\infty H \subset \partial_\infty \mathbb{H}^k$.

We start with an argument inspired by the proof of [BST12, Proposition 4.1]. Consider for every $r > 0$ the sphere $S_r = \{x \in \mathbb{H}^k,\ d(x,o) = r\}$ equipped with its Riemannian measure ν_r. Note that

$$\nu_r(S_r \cap H^\alpha) = \eta(\alpha)\nu_r(S_r)$$

for some increasing function η satisfying $\lim_{\alpha\to 0} \eta(\alpha) = 0$. We now fix $\alpha > 0$ so that $\eta(\alpha) \leq \frac{c}{2}$.

Recall that hyperplanes passing through o are characterized by their normal vector at o, and therefore are parametrized by the projective space P^{k-1}. We consider the Lebesgue probability measure ν on P^{k-1}. Given $\theta \in P^{k-1}$ we define H_θ to be the hyperplane through o with normal vector θ. Recall that $Z \subset X$ is a measurable subset of finite measure, so

$$\int_{P^{k-1}} \nu_r(A \cap H_\theta^\alpha \cap S_r) d\nu(\theta) = \nu_r(A \cap S_r)\frac{\nu_r(S_r \cap H^\alpha)}{\nu_r(S_r)} = \nu_r(A \cap S_r)\eta(\alpha).$$

Integrating over r, we deduce that

$$\int_{P^{k-1}} \mu(A \cap H_\theta^\alpha) d\nu(\theta) = \mu(A)\eta(\alpha),$$

and so for some hyperplane H_Z we have $\mu(A \cap H_Z^\alpha) \leq \mu(Z)\eta(\alpha) \leq \frac{c}{2}\mu(Z)$.

Let H^-, H^+ be the two connected components of the complement of H_Z^α; these are convex and asymptotic shadows of o, and satisfy $\mu(H^- \cap Z), \mu(H^+ \cap Z) \geq \frac{c}{2}\mu(Z)$. Moreover, $\rho(\partial_\infty H^-, \partial_\infty H^+) \geq 2\alpha$. □

Consequently, by Theorem 12.3 we have all the upper bounds in the statement of Theorem 12.13. To complete its proof, we must show the lower bound of $\Lambda^{k-1}_{\mathbb{H}^k}(r) \gtrsim r^{(k-2)/(k-1)} \log^{1/(k-1)}(r)$. We use the following combinatorial model of (a sector of) a ball in $\mathbb{H}^k$: Let V be the usual grid with vertex set $\mathbb{Z}^{k-1}$ and unit edges in the coordinate directions. For $i \in \mathbb{N}_0$, let V_i be the full subgraph of V with vertex set $\{0, \dots, 2^i - 1\}^{k-1}$. Let Γ be the disjoint union of all $V_i, i \in \mathbb{N}_0$, and connect the layers as follows: for each i, connect $\mathbf{a} \in V_i$ to the 2^{k-1} vertices $2\mathbf{a} + \{0,1\}^{k-1}$ in V_{i+1}. Given $t \in 2\mathbb{N}$, let B_t be the full subgraph of Γ containing layers $t/2+1, t/2+2, \dots, t$. We think of the edges in each layer as 'horizontal', and those between layers as 'vertical'.

We can also consider the covering of $\mathbb{H}^{k-1}$ by dyadic cuboids in the upper half-plane model; Γ is then seen as part of the nerve of this covering. Therefore Γ quasi-isometrically embeds into $\mathbb{H}^{k-1}$.

Each $|B_t| \asymp 2^{(k-1)t}$, so our desired bound follows from the following proposition:

Proposition 12.16. *For each $k \geq 2, t \in \mathbb{N}$ we have $h^{k-1}(B_t) \succeq 2^{-t}t^{1/(k-1)}$, with constant independent of t.*

Proof. Given a function $f : B_t \to \mathbb{R}$, i.e. a function on VB_t, let $|\nabla f| : EB_t \to \mathbb{R}$ be the corresponding edge gradient with norm $\|\nabla f\|_{k-1}$. Our goal is to show, with $p = k-1$,

$$\|f - f_{B_t}\|_p \preceq \|\nabla f\|_p 2^t t^{-1/p}. \tag{12.17}$$

Suppose that for each $x, y \in B_t$ we have chosen an edge path γ_{xy} joining x to y. For each edge $e \in EB_t$, denote by N_e the number of paths γ_{xy} through e. Then, similarly to the proofs in Section 10, we have, for $p > 1$,

$$\begin{aligned}
\|f - f_{B_t}\|_p^p &= \sum_x \left| f(x) - \frac{1}{|B_t|} \sum_y f(y) \right|^p \\
&\leq \frac{1}{|B_t|} \sum_x \sum_y |f(x) - f(y)|^p \\
&\leq \frac{1}{|B_t|} \sum_x \sum_y \left(\sum_{e \in \gamma_{xy}} |\nabla f|(e) N_e^{-1/p} N_e^{1/p} \right)^p \\
&\leq \frac{1}{|B_t|} \sum_x \sum_y \left(\sum_{e \in \gamma_{xy}} |\nabla f|(e)^p N_e^{-1} \right) \left(\sum_{e \in \gamma_{xy}} N_e^{1/(p-1)} \right)^{p-1}.
\end{aligned}$$

Provided we have, independent of x and y, a bound

$$\left(\sum_{e \in \gamma_{xy}} N_e^{1/(p-1)} \right)^{p-1} \leq \frac{C}{t} |B_t| \, 2^{tp}, \tag{12.18}$$

we continue to get the desired result

$$\|f - f_{B_t}\|_p^p \leq \frac{C2^{tp}}{t} \sum_x \sum_y \sum_{e \in \gamma_{xy}} |\nabla f|(e)^p N_e^{-1} = \frac{C2^{tp}}{t} \|\nabla f\|_p^p. \tag{12.19}$$

If $p = 1$, i.e. $k = 2$, the same argument works with (12.18) replaced by $\|N_e\|_\infty \leq \frac{C2^t}{t}|B_t|$.

It remains to make our choice of paths γ_{xy} and find a suitable bound (12.18). We do so by taking the standard canonical paths in $\mathbb{Z}^{k-1}$ which change each coordinate in order, and 'spreading them out' a little through the layers.

We first describe the canonical paths in each layer $V_i = \{0, \ldots, 2^i - 1\}^{k-1}$: given $\mathbf{a}, \mathbf{b} \in V_i$, let $\beta_{\mathbf{a},\mathbf{b}}$ be the path which changes the first coordinate of $\mathbf{a}$ until it matches that of $\mathbf{b}$, then changes the second coordinate, and so on. It is easy to see that this path has length $\leq (k-1)2^i$, and that each edge is covered by at most 2^{ik} such paths.

Now for each $x \in B_t$, we make a choice of layer $L_x \in \{t/2+1, \ldots, t\}$ as follows: identify x with $\mathbf{a} = (a_j) \in V_i$ for some i. Let $L_x = 1 + t/2 + ((\sum_j a_j) \bmod (i - t/2))$. Essentially, in each V_i the choice of layer cycles through all available layers $\leq i$.

Given $x, y \in B_t$, let $p_x, p_y \in V_{L_x}$ be points connected vertically to x, y respectively. Define the path γ_{xy} as follows: (1) travel from x vertically to p_x; (2) follow the canonical path horizontally in V_{L_x} from p_x to p_y; (3) travel vertically from p_y to y.

We now estimate (12.18), when $x \in V_i$ and $y \in B_t$, for $k > 2$. As γ_{xy} goes vertically from x to $p_x \in V_{L_x}$, N_e grows exponentially to size $\asymp \frac{1}{t}2^{(t-L_x)(k-1)} \cdot |B_t| \asymp \frac{1}{t}2^{(2t-L_x)(k-1)}$. As γ_{xy} travels horizontally in V_{L_x}, it follows a path of length $\preceq 2^{L_x}$ where each N_e has value at most

$$\left(\tfrac{1}{t}2^{(t-L_x)(k-1)}\right) \cdot 2^{L_x k} \cdot 2^{(t-L_x)(k-1)}.$$

In the final vertical path, N_e decreases geometrically from $\asymp \frac{1}{t}|B_t| \cdot 2^{(t-L_x)(k-1)} \asymp \frac{1}{t}2^{(2t-L_x)(k-1)}$. Applying these observations to (12.18), we see that

$$\begin{aligned}
&\left(\sum_{e \in \gamma_{xy}} N_e^{1/(k-2)}\right)^{k-2} \\
&\preceq \left(2 \cdot \left(\tfrac{1}{t}2^{(2t-L_x)(k-1)}\right)^{1/(k-2)} + 2^{L_x}\left(\tfrac{1}{t}2^{2(t-L_x)(k-1)+L_x k}\right)^{1/(k-2)}\right)^{k-2} \\
&\preceq \tfrac{1}{t}2^{2t(k-1)}\left(2^{-L_x(k-1)/(k-2)} + \left(2^{L_x(k-2)-2L_x(k-1)+L_x k}\right)^{1/(k-2)}\right)^{k-2} \\
&\preceq \tfrac{1}{t}2^{2t(k-1)} \asymp \tfrac{1}{t}|B_t|2^{tp}.
\end{aligned}$$

In the case $k = 2$, $p = 1$, we again have

$$\|N_e\|_\infty \preceq \max\left\{\tfrac{1}{t}2^{(2t-L_x)(k-1)}, \tfrac{1}{t}2^{2(t-L_x)(k-1)+L_x k}\right\} = \tfrac{1}{t}2^{2t} \asymp \tfrac{1}{t}|B_t|2^{tp},$$

completing the proof. □

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