POINCARÉ PROFILES OF GROUPS AND SPACES

DAVID HUME, JOHN M. MACKAY, AND ROMAIN TESSERA

Abstract. We introduce a spectrum of monotone coarse invariants for metric measure spaces called Poincaré profiles. The two extremes of this spectrum determine the growth of the space, and the separation profile as defined by Benjamini–Schramm–Timár. In this paper we focus on properties of the Poincaré profiles of groups with polynomial growth, and of hyperbolic spaces, where we deduce a striking connection between these profiles and conformal dimension. One application of our results is that there is a collection of hyperbolic Coxeter groups, indexed by a countable dense subset of $(1,\infty)$, such that G_s does not coarsely embed into G_t whenever $s < t$.

Contents

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1. Introduction

A monotone coarse invariant of a collection of metric spaces $\mathcal{X}$ is a function Λ from $\mathcal{X}$ to a partially ordered set $(P, \leq)$ with the property that $\Lambda(X) \leq \Lambda(Y)$ whenever there is a coarse embedding of X into Y. The asymptotic dimension and the growth function are natural and well-studied examples of such invariants, and a more recent example is the separation profile of bounded degree graphs introduced by Benjamini–Schramm–Timár [BST12]. In this paper, we introduce a new family of monotone coarse invariants called the L^p-Poincaré profiles.

We will only define the Poincaré profiles of graphs in the introduction; however, our results naturally extend to compactly generated locally compact groups and Riemannian manifolds with bounded geometry. The majority of the paper is presented in a more general context which includes all of these spaces.

Inspired by work of the first author [Hum17], which gives an equivalent definition of the separation profile in terms of the Cheeger constant, for each $p \in [1, \infty]$ we define the **p-Poincaré constant** of a finite graph Γ with vertex set $V\Gamma$ and edge set $E\Gamma$ to be

$$h^p(\Gamma) = \inf \left\{ \frac{||\nabla f||_p}{||f - f_\Gamma||_p} \ : \ f \in \mathrm{Map}(V\Gamma \to \mathbb{R}), \ f \not\equiv f_\Gamma \right\}$$

where $\nabla f(x) = \max\{|f(x) - f(y)| \ : \ xy \in E\Gamma\}$, $||\cdot||_p$ is the usual p-norm in $\mathbb{R}^{|V\Gamma|}$ and f_Γ is the average $|V\Gamma|^{-1} \sum_{x \in V\Gamma} f(x)$. It is worth noting that for bounded degree graphs $h^p(\Gamma)$ is bi-Lipschitz equivalent to $\lambda_{1,p}(\Gamma)^{\frac{1}{p}}$, where $\lambda_{1,p}(\Gamma)$ denotes the smallest non-zero eigenvalue of the p-Laplacian on Γ (see Remark 3.8 below).

Now we define the **L^p-Poincaré profile** of an infinite graph X to be

$$\Lambda^p_X(r) = \sup\{|\Gamma| \, h^p(\Gamma) \ : \ \Gamma \leq X, \ |V\Gamma| \leq r\}.$$

We consider Poincaré profiles up to the natural order $\lesssim$ where $f \lesssim g$ if there exists a constant C such that $f(r) \leq Cg(Cr + C) + C$ for all r, and $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$. Often, the constant C will depend on p; to emphasise this we will use the notations $\lesssim_p$ and $\simeq_p$.

A lower bound on the L^p-Poincaré profile corresponds to a "p-Poincaré inequality" for functions on a finite subgraph of the corresponding size.[1] Poincaré inequalities have been intensively studied, particularly in the case of balls in doubling metric spaces, see [SC02, HK00]

[1]Technically these Poincaré inequalities are Neumann-type, rather than Dirichlet-type Poincaré inequalities which consider only functions which are 0 on

and references therein. For finite graphs, there is a vast literature linking Cheeger constants and spectral gaps to such inequalities when $p = 1, 2$, see [Chu97, SC97]. Discrete Poincaré inequalities on balls in metric spaces have been studied before by, for example, Holopainen–Soardi [HS97] and Gill–Lopez [GL15]. Our approach differs in that we are working in a situation where global Poincaré inequalities do not necessarily hold, where measures need not be doubling, and where we have to consider inequalities on all subsets, not just balls.

Our first important result is that these Poincaré profiles are monotone coarse invariants.

Theorem 1. *Let X, Y be graphs with bounded degree. If there is a regular map $r : VX \to VY$, then for all $p \in [1, \infty]$, $\Lambda^p_X \lesssim_p \Lambda^p_Y$.*

A map $r : VX \to VY$ is said to be **regular** if it is Lipschitz and $\sup_{y \in VY} |r^{-1}(y)| < \infty$. In particular every quasi-isometric or coarse embedding is regular. Thus for each p the L^p-Poincaré profile is a well-defined coarse invariant of a finitely generated group G.

1.1. **Extremal cases.** In the cases $p = \infty$ and $p = 1$ the Poincaré profile is easily understood in terms of the growth and separation profile respectively.

Recall the **growth function** of a graph X: $\gamma_X(k)$ is the maximum number of vertices contained in a closed ball $B(x, k)$ of radius k centred at some vertex $x \in VX$. We define the **inverse growth function**: $\kappa_X(r)$ is the smallest positive k such that $\gamma_X(k) > r$.

At one extreme, $p = \infty$, the Poincaré profile detects inverse growth.

Proposition 2. *For any bounded degree graph X, $\Lambda^\infty_X(r) \simeq \sup\limits_{1 \leq s \leq r} \dfrac{s}{\kappa_X(s)}$.*

From this, we may easily deduce Theorem 1 in the case $p = \infty$. At the other extreme we show that the L^1-Poincaré profile is equivalent to the separation profile, as introduced by Benjamini–Schramm–Timár [BST12]. The perspective we adopt of studying Poincaré profiles up to regular maps is inspired by their observation that separation is monotone under regular maps.

We recall that the separation profile of an infinite graph X may be defined by $\mathrm{sep}_X(r) = \max \{|\Gamma|\, h(\Gamma)\}$ where the maximum is taken over all subgraphs Γ of X with at most r vertices, and $h(\Gamma)$ is the Cheeger constant [Hum17].

Proposition 3. *For any bounded degree graph X, $\Lambda^1_X(r) \simeq \mathrm{sep}_X(r)$.*

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the boundary of the subgraph in the ambient space. See Remark 4.2 for more details.

Remark 4. The case of $p = 2$ is also natural, being the largest spectral gap among subgraphs of a given size. The spectral gap can be used to bound mixing times of random walks on the subgraph. A related *spectral profile* was considered by Goel–Montenegro–Tetali [GMT06].

1.2. **Relating profiles.** The following results are classical, and are likely to be easy exercises for experts; for completeness we present full proofs.

Proposition 5. *Let $1 \leq p \leq q < \infty$. There exists a constant $C = C(p, q)$ such that for every bounded degree graph X and every r we have $\Lambda^p_X(r) \leq C\Lambda^q_X(r)$.*

In the opposite direction we have the following.

Proposition 6. *If Γ is a finite graph and $p \in [1, \infty)$, then $h^p(\Gamma)^p \leq 2^p h^1(\Gamma)$.*

Asymptotically this is sharp for balls in the 3-regular tree, as we will see in section 10. Proposition 5 cannot be extended to the case $q = \infty$ since there are bounded degree graphs containing expanders: combining the above propositions with results in [Hum17] we see that for every $p \in [1, \infty)$, $\Lambda^p_X(r)/r \not\to 0$ as $r \to \infty$ if and only if X contains an expander, while a bounded degree graph Y has at most exponential growth, so always satisfies $\Lambda^\infty_Y(r) \lesssim r/\log(r)$.

1.3. **Polynomial growth.** Gromov's celebrated polynomial growth theorem asserts that every finitely generated group with polynomial growth is virtually nilpotent. Results of Bass–Guivarc'h then show that for every group G of polynomial growth there is an integer d such that $\gamma_G(r) \simeq r^d$ [Gro81, Bas72, Gui73].

Theorem 7. *Let G be a finitely generated group such that $\gamma_G(r) \simeq r^d$. Then for all $p \in [1, \infty]$, $\Lambda^p_G(r) \simeq_p r^{\frac{d-1}{d}}$.*

When G is virtually abelian and $p = 1$ this follows from [BST12]; in all other cases it is new. To prove the lower bound on $\Lambda^p_G(r)$ we calculate a lower bound on the separation profile using a Poincaré inequality and apply Propositions 3 and 5. For the upper bound we use a general result, Proposition 9.5, which holds for any bounded degree graph with finite Assouad–Nagata dimension (cf. [Hum17, Theorem 1.5]).

Recall that by a classical result of Heintze [Hei74], every simply connected negatively curved homogeneous Riemannian manifold M is isometric to a connected Lie group of the form $N \rtimes \mathbb{R}$ equipped with a left-invariant Riemannian metric, where N is a simply connected nilpotent Lie group and the action of $\mathbb{R}$ on N is contracting. We immediately deduce from Theorem 7 that for every $p \in [1, \infty]$, the L^p-Poincaré profile of such a manifold is bounded from below by $r^{\frac{d-1}{d}}$, where d is the homogeneous dimension of N. As a special case of this we deduce the following lower bounds for rank one symmetric spaces.

Corollary 8. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be a real division algebra, and let $X = \mathbb{H}^m_{\mathbb{K}}$ be a rank-one symmetric space for $m \geq 2$ (and $m = 2$ when $\mathbb{K} = \mathbb{O}$). Then, for all $1 \leq p < \infty$, we have $\Lambda^p_X(r) \gtrsim_p r^{(Q-1)/Q}$ where $Q = (m+1)\dim_{\mathbb{R}} \mathbb{K} - 2$.*

It is worth noting at this point that Q is the conformal dimension of the boundary of X. For large p this bound is far from optimal as we will see in the next section.

1.4. **Hyperbolic spaces.** We begin by considering the case of an infinite 3-regular tree.

Theorem 9. *Let T be the infinite 3-regular tree. Then $\Lambda^p_T(r) \simeq_p r^{\frac{p-1}{p}}$, for all $p \in [1, \infty)$.*

Note that when $p = \infty$, $\Lambda^p_T(r) \simeq r/\log(r)$ by Proposition 6.1. Using Theorem 9, together with results of Chou and Benjamini–Schramm on embeddings of trees into elementary amenable groups with exponential growth and non-amenable groups respectively [Cho80, BS97] we obtain the following corollary.

Corollary 10. *Let G be a finitely generated elementary amenable group with exponential growth or a finitely generated infinite non-amenable group. Then for all $p \in [1, \infty)$, $\Lambda^p_G(r) \gtrsim_p r^{\frac{p-1}{p}}$.*

We continue with a striking result which suggests a connection between conformal dimension of boundaries and a phase transition in the Poincaré profiles of hyperbolic groups.

Theorem 11. *Let G be a finitely generated hyperbolic group with equivariant conformal dimension Q (see Definition 12.4). For every $\varepsilon > 0$*

$$\Lambda^p_G(r) \lesssim \begin{cases} r^{\frac{Q-1}{Q}+\epsilon} & \text{if} \quad p \leq Q \\ r^{\frac{p-1}{p}} & \text{if} \quad p > Q. \end{cases}$$

If the equivariant conformal dimension is attained, we have:

$$\Lambda^p_G(r) \lesssim \begin{cases} r^{\frac{Q-1}{Q}} & \text{if} \quad 1 \leq p < Q \\ r^{\frac{Q-1}{Q}} \log^{\frac{1}{Q}}(r) & \text{if} \quad p = Q \\ r^{\frac{p-1}{p}} & \text{if} \quad p > Q. \end{cases}$$

In certain cases we are able to prove that these upper bounds are sharp. Our key examples are rank-one symmetric spaces, and a collection of groups $G_{m,n} = \langle s_1, \ldots, s_m \mid s_i^n, [s_1, s_2], \ldots, [s_{m-1}, s_m], [s_m, s_1] \rangle$, $m \geq 5, n \geq 3$ which occur as uniform lattices in the isometry groups of associated Fuchsian buildings $\Delta_{m,n}$, as studied by Bourdon and Bourdon–Pajot amongst others [Bou97, BP99]. Following the terminology of [Cap14] we call these *Bourdon buildings*. These groups are virtually torsion free [HW99], and commensurable to hyperbolic Coxeter groups when n is even [Hag06].

Theorem 12. *Let $X = \mathbb{H}^m_{\mathbb{K}}$ be a rank-one symmetric space for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $m \geq 2$ (with $m = 2$ when $\mathbb{K} = \mathbb{O}$), or let X be one of the groups $G_{m,n}$; in either case, let Q be the Ahlfors regular conformal dimension of the boundary of X. Then*

$$\Lambda^p_X(r) \simeq \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } p < Q \\ r^{\frac{Q-1}{Q}} \log(r)^{\frac{1}{Q}} & \text{if } p = Q \\ r^{\frac{p-1}{p}} & \text{if } p > Q \end{cases}$$

In the case of $\Lambda^1_{\mathbb{H}^m_{\mathbb{R}}}$, these sharp bounds for the separation profile appear in [BST12]. It is interesting to note that uniform lattices G in $PSL(2, \mathbb{R})$ satisfy $\Lambda^1_G(r) \simeq \log(r)$ and $\Lambda^p_G(r) \simeq r^{\frac{p-1}{p}}$ for all $p > 1$, while non-uniform lattices H satisfy $\Lambda^1_H(r) \simeq r^{\frac{p-1}{p}}$ for all $p \geq 1$. We have no other examples of this distinction between uniform and non-uniform lattices for any other p or for any groups of higher rank.

We do not define the conformal dimension here, but comment that for a rank-one symmetric space $\mathbb{H}^m_{\mathbb{K}}$ we have $Q = (m+1)\dim_{\mathbb{R}} \mathbb{K} - 2$ and for the groups $G_{m,n}$ we have $Q = 1 + \log(n-1)/\operatorname{arccosh}((m-2)/m)$, which takes a dense set of values in $(1, \infty)$ as m, n vary.

The upper bound in Theorem 12 is obtained by constructing specific functions on the boundary using an embedding of the space into a real hyperbolic space. The lower bound in Theorem 12 for rank-one symmetric spaces with $p < Q$ follows from Corollary 8. For the groups $G_{m,n}$, and for the sharp case $p = Q$, we require the following more general result.

Theorem 13. *Suppose that X is a visual Gromov hyperbolic graph with a visual metric ρ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a p-Poincaré inequality. Then for all $q \geq p$, $\Lambda^q_X(r) \gtrsim r^{(Q-1)/Q}$.*

Moreover, if $(\partial_\infty X, \rho)$ admits a Q-Poincaré inequality, then $\Lambda^Q_X(r) \gtrsim r^{1-1/Q} \log(r)^{1/Q}$.

Here a "p-Poincaré inequality" is in the sense of Heinonen–Koskela [HK98], namely an analytic property of the compact metric space $\partial_\infty X$. Such inequalities hold on boundaries of rank-one symmetric spaces, see e.g. [Jer86, HK98, MT10], so we can apply this lower bound to obtain an alternative proof of Corollary 8. For the groups $G_{m,n}$, the Poincaré inequalities are constructed in [BP99].

The sharp lower bounds on Λ^Q_X come from showing a suitable Poincaré inequality on annuli $B(o,2r)\setminus B(o,r)$ in X. It is interesting to observe that for $p<Q$, $p=Q$, and $p>Q$, the sharp lower bounds on $\Lambda^p X$ are realised by embedded spheres, annuli and trees respectively.

Finally, Theorems 9 and 12, together with the embedding theorem of Bonk–Schramm [BS00], imply that for every hyperbolic group G there is some p_0 such that for all $p>p_0$, we have $\Lambda^p_G(r)\simeq r^{\frac{p-1}{p}}$. The relationship between the infimal such p_0 and the conformal dimension of the boundary of G is one of the most intriguing aspects of these profiles.

1.5. **Consequences.** By applying Theorem 12 to the groups $G_{m,n}$, we find a new collection of functions which can be obtained as separation profiles of finitely generated groups:

Corollary 14. *There exists a dense subset A of $(0,1)$ such that for all $\alpha\in A$ there is a hyperbolic group G_α with $\mathrm{sep}_{G_\alpha}(r)\simeq r^\alpha$.*

The key purpose of a monotone coarse invariant is to be able to distinguish situations in which one space cannot be coarsely embedded into another. There are few general tools to do this; asymptotic dimension is one and growth (or equivalently, the L^∞-Poincaré profile) is another. Here we present and discuss some results of this form which cannot be obtained by studying growth and/or asymptotic dimension.

Corollary 15. *If there is a coarse embedding of $\mathbb{H}^k_{\mathbb{C}}$ into $\mathbb{H}^l_{\mathbb{R}}$, then $l>2k$. Likewise, if there is a coarse embedding of $\mathbb{H}^k_{\mathbb{H}}$ into $\mathbb{H}^l_{\mathbb{R}}$, then $l>4k+2$.*

To prove the analogous result for quasi-isometric embeddings, one can use the conformal dimension of the boundary, however, a coarse embedding does not necessarily induce a well-defined map between boundaries [BR13] so this approach cannot be expected to work. Using asymptotic dimension as an invariant one could only deduce that $l\geq 2k$ in the first case and $l\geq 4k$ in the second.

By [BS00], every hyperbolic group quasi-isometrically embeds into some $\mathbb{H}^k_{\mathbb{R}}$. A natural obstruction to a coarse embedding $G_k\to\mathbb{H}^k_{\mathbb{R}}$

is that the asymptotic dimension of G_k is greater than k. Poincaré profiles provide a different obstruction.

Corollary 16. *For every k there is a hyperbolic group G_k of asymptotic dimension 2 which does not coarsely embed into $\mathbb{H}^k_{\mathbb{R}}$.*

We can take G_k to be $G_{m(k),5}$ for some appropriately chosen $m(k)$ and apply Theorem 12.

It is in general very difficult to prove a statement of the form "a hyperbolic group H is not isomorphic to a subgroup of a hyperbolic group G". One may use torsion or asymptotic dimension in certain cases, here we show that the Poincaré profiles can exclude subgroups when the two methods listed above fail.

Corollary 17. *There exists a collection of (torsion-free) hyperbolic groups $(G_q)_{q\in\mathbb{Q}}$ with asymptotic dimension 2 such that whenever $i < j$ there is no coarse embedding from G_i to G_j. In particular, G_i is not virtually a subgroup of G_j.*

Indeed, the groups $G_{m,n}$ are virtually torsion-free so we may choose the G_q in Corollary 17 to be torsion-free. By results of Gersten, finitely presented subgroups of hyperbolic groups with cohomological dimension 2 (which equals the asymptotic dimension for torsion-free hyperbolic groups [BM91, BL07]) are hyperbolic but not necessarily quasi-convex. The fact that G_j is not a quasi-convex subgroup of G_i is immediate by considering the conformal dimension.

Remark 18. By a recent result of Pansu [Pan16], if a hyperbolic group H coarsely embeds into a hyperbolic group G, then the "L^p-cohomological dimension" of H is less than or equal to the conformal dimension of the boundary of G. In the cases of the Bourdon buildings and rank-one symmetric spaces, these two numbers turn out to coincide. This provides an alternative proof of Corollaries 15, 16 and 17.

1.6. **About the proofs.** The proof of the theorems described in the previous section employ a variety of techniques. In particular, the arguments needed for getting upper bounds are completely different than those for obtaining lower bounds.

1.6.1. *Upper bounds.* For groups with polynomial growth, our sharp upper bounds are obtained via an argument adapted from [Hum17] based on the fact that these groups have finite Assouad–Nagata dimension ([Hum17, Theorem 1.5] deals with the separation profile, corresponding to $p = 1$). Finite Assouad–Nagata dimension is a quantitative strengthening of finite asymptotic dimension. Contrary to the latter, finite Assouad–Nagata dimension is not monotone under coarse embedding (only quasi-isometric embedding) as it is sensitive to distortion of the metric. We come up with a new notion called *finite measured dimension* (see Definition 9.1), weaker than finite asymptotic dimension, which should be of independent interest. Our motivation here is that it is well adapted for providing upper bound on the Poincaré profiles.

Obtaining upper bounds for hyperbolic groups is trickier, and occupies all of §12. We need three different arguments, depending whether p lies below, above, or equals the conformal dimension. We show that hyperbolic groups admit in some sense "many hyperplanes", by using a theorem of Bonk–Schramm [BS00] to embed the group into a real hyperbolic space, which has an abundance of hyperplanes. We crucially use a version of Helly's theorem for CAT(0) spaces. Our argument for small p is largely inspired from [BST12] where the separation profile of the real hyperbolic plane is computed. It consists in a symmetrization argument. For large p, we construct for every finite set A, a p-Dirichlet function whose restriction to A provides a good test function.

1.6.2. *Lower bounds.* Obtaining lower bounds for groups with polynomial growth follows from well-known Poincaré inequalities in balls. It is interesting that the functional analytic interpretation of the separation profile gives us new estimates for nilpotent groups without effort.

The lower bounds for hyperbolic Lie groups and small p are obtained by considering parabolic closed nilpotent subgroups. The cases of Bourdon–Pajot buildings, and of the cases when $p = Q$, are more interesting and more subtle. For $p < Q$, we exploit the fact that their visual boundary satisfies (infinitesimal) Poincaré inequalities. We "pull down" these inequalities on a sphere of large radius in the space using a discretization argument developed in the first part of the paper. To get the sharp lower bound in the $p = Q$ case, we use the Poincaré inequalities on spheres and a curve counting argument to find a new Poincaré inequalities on annuli. A similar but simpler curve counting argument gives the lower bound for the 3-regular tree, and hence all spaces in which it embeds.

1.7. **Structure of the paper.** The paper splits roughly into three parts. The first part introduces Poincaré profiles as monotone regular (in particular coarse) invariants. After introducing our notations and fixing the class of metric measure spaces under consideration, we present the more general definition of Poincaré constants in Section 3 and explain some basic properties. We then introduce Poincaré profiles and prove Theorem 1 in Sections 4 and 5 respectively.

The second part deals with relationships between Poincaré profiles. The descriptions of extremal profiles (Propositions 2 and 3) and the connection with separation (Proposition 6) are proved in Section 6, and the dependence on p (Proposition 5) is discussed in Section 7.

The final part is dedicated to calculating profiles using the technology developed in the rest of the paper. Groups with polynomial growth (Theorem 7) are considered in Sections 8 and 9. For hyperbolic spaces, trees (Theorem 9), lower bounds (Theorem 13) and upper bounds are in Sections 10, 11 and 12 respectively, with applications (in particular, Theorem 12) discussed in Section 13.

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2. Notation and framework

We first introduce notation to be used throughout the paper.

Suppose $f, g : S \to [0, \infty)$ where $S = \mathbb{N}$ or $S = [0, \infty)$. We write $f \preceq_{u,v,\ldots} g$ if there exists a constant $C > 0$ depending only on $u, v, \ldots$ such that $f(x) \leq Cg(x)$ for all $x \in S$. If $f \preceq_{u,v,\ldots} g$ and $g \preceq_{u,v,\ldots} f$ then we write $f \asymp_{u,v,\ldots} g$. We drop the subscripts if the constants are understood.

We write $f \lesssim_{u,v,\ldots} g$ if there exists a constant $C > 0$ depending only on $u, v, \ldots$ such that $f(x) \leq Cg(Cx + C) + C$ for all $x \in S$; similarly, we write $f \simeq_{u,v,\ldots} g$ if $f \lesssim_{u,v,\ldots} g$ and $g \lesssim_{u,v,\ldots} f$.

Given a subset A of a metric space (X, d) and some $M \geq 0$ we define the closed M-neighbourhood of A to be

$$[A]_M = \{x \in X \,:\, d(x, A) \leq M\}.$$

Given a point $x \in X$ and $r \geq 0$ we denote by $B(x, r)$ the closed metric ball of radius r centred at x.

Let (Z, ν) be a measure space with positive finite measure. We denote the averaged integral by

$$\fint_Z f d\nu = \frac{1}{\nu(Z)} \int_Z f d\nu.$$

Given a function $f \in L^p(X, \mu)$, another measure μ' such that $f \in L^p(X, \mu')$ and a measurable subset $Z \subseteq X$ we write

$$||f||_{p,\mu'} = \left(\int_X |f(z)|^p \, d\mu'(z) \right)^{\frac{1}{p}} \quad \text{and}$$

$$||f||_{Z,p} = \left(\int_Z |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}}.$$

The L^∞ norms $||\cdot||_{\infty,\mu'}$ and $||\cdot||_{Z,\infty}$ are defined analogously.

Given a graph $\Gamma = (V\Gamma, E\Gamma)$ and a subset $A \subset V\Gamma$, the full (or induced) subgraph of Γ with vertex set A is the graph with vertex set A and edge set $\{xy \in E : x, y \in A\}$.

The purpose of the remainder of this section is to introduce the class of spaces we will consider in this paper.

Definition 2.1. A **metric measure space** is a triple (X, d, μ) where μ is a non-trivial, locally finite, Borel measure on a complete, separable metric space (X, d).

The key examples are: graphs of bounded degree, Riemannian manifolds with bounded geometry and compactly generated locally compact groups, so we will make the following standing assumptions.

We will assume throughout the paper that any metric measure space (X, d, μ) satisfies the following properties:

- X has **bounded packing on large scales**[2]: if there exists $r_0 \geq 0$ such that for all $r \geq r_0$, there exists $K_r > 0$ such that
$$\forall x \in X,\ \mu(B(x, 2r)) \leq K_r \mu(B(x, r)).$$
We then say that X has **bounded packing on scales** $\geq r_0$.
- X is k-**geodesic** for some $k > 0$: for every pair of points $x, y \in X$ there is a sequence $x = x_0, \ldots, x_n = y$ such that $d(x_{i-1}, x_i) \leq k$ for all i and $d(x, y) = \sum_{i=1}^n d(x_{i-1}, x_i)$.

Up to rescaling the metric and/or the measure we will assume that X is 1-geodesic and has bounded packing on scales $\geq r_0 = 1$.

A subspace $Z \subset X$ is always assumed to be 1-**thick** (a union of closed balls of radius 1), so in particular it has positive measure. We equip Z with the subspace measure and the induced 1-distance

$$d(z, z') = \inf \left\{ \sum_{i=1}^n d(z_{i-1}, z_i) \right\}$$

where the infimum is taken over all sequences $z = z_0, \ldots, z_n = z'$, such that each $z_i \in Z$ and $d(z_{i-1}, z_i) \leq 1$.

Note that (as in the case of a disconnected subgraph) the induced 1-distance will take values in $[0, \infty]$.

Remark 2.2. In the case of (the vertex set of) a bounded degree graph X, d is the shortest path metric and μ is the (vertex) counting

[2]If $r_0 = 0$, then we simply say that X has bounded packing.

measure. Subspaces Z are (vertex sets of) 1-thick subgraphs equipped with the vertex counting measure and their own shortest path metric (the induced 1-distance).

In a locally compact group G with compact generating set K, we equip G with a Haar measure (which is unique up to scaling) and the word metric $d = d_K$.

The reason for working with thick sets is justified by the following easy lemma (see [Tes08, Lemma 8.4]).

Lemma 2.3. *Assume X has bounded packing on scales $\geq r_0$, and let $A \subset X$ be r-thick for some $r \geq r_0$. Then for all $u > 0$,*

$$\mu([A]_u) \preceq_u \mu(A).$$

3. Poincaré constants

Let (X, d) be a metric space and let $a > 0$. Given a measurable function $f : X \to \mathbb{R}$, we define its **upper gradient at scale** a to be

$$|\nabla_a f|(x) = \sup_{y,y' \in B(x,a)} |f(y) - f(y')|.$$

Remark 3.1. We have slightly modified the notation from [Tes08], where the upper gradient was referred to as the "local norm of the gradient" and was denoted by $|\nabla f|_a$. The changes in this paper are for brevity; in what follows $\||\nabla_a f|\|_p$ will simply be denoted by $\|\nabla_a f\|_p$.

Definition 3.2. Let (Z, d, ν) be a metric measure space with finite measure and fix a scale $a > 0$. We define the **L^p-Poincaré constant at scale** a of Z to be

$$h_a^p(Z) = \inf_f \frac{\|\nabla_a f\|_p}{\|f\|_p},$$

where the infimum is taken over all $f \in L^p(Z, \nu)$ such that $f_Z := \fint_Z f d\nu = 0$ and $f \not\equiv 0$. We adopt the convention that $h_a^p(Z) = 0$ whenever $\nu(Z) = 0$.

Before continuing we list some basic properties of the Poincaré constant.

Lemma 3.3. *Let (Z, d, ν) be a metric measure space with finite measure.*

(i) *Let $\theta : (0, \infty) \to (0, \infty)$ be a non-decreasing function, and let (Z', d', ν') be a metric measure space such that $(Z, \nu) = (Z', \nu')$, and $d'(z_1, z_2) \leq \theta(d(z_1, z_2))$ for all z_1, z_2. Then for all $a > 0$,*

$$h_a^p(Z) \leq h_{\theta(a)}^p(Z').$$

(ii) *Let (Z', d', ν') be a metric measure space where $(Z, d) = (Z', d')$ and there exists some $M \geq 1$ such that $M^{-1}\nu(A) \leq \nu'(A) \leq M\nu(A)$ for every measurable $A \subseteq Z$. Then for all $a > 0$,*

$$h_a^p(Z') \leq 2M^{2/p} h_a^p(Z).$$

Proof. Part (i) is immediate. For part (ii), let $f : X \to \mathbb{R}$ be a measurable function such that $⨍ f d\mu = 0$ and let $m = ⨍ f d\nu'$. We see that

$$\|f\|_{p,\nu} \leq 2\|f - m\|_{p,\nu} \leq 2M^{1/p}\|f - m\|_{p,\nu'}$$

The first inequality above is the $C = -m$ case of inequality (3.5) proved in Lemma 3.4. On the other hand

$$\|\nabla f\|_{p,\nu} \leq M^{1/p}\|\nabla f\|_{p,\nu'},$$

so we are done. □

To obtain a sensible definition of the L^p-Poincaré constant it is necessary to only consider functions whose average is zero and to choose a notion of gradient. In both cases there are multiple ways to do this.

3.1. **Choice of average.** Given a measure space (Z, ν) with finite positive measure, there are multiple ways to define the "average" of a measurable function $f : (Z, \nu) \to \mathbb{R}$:

(1) the **average** $f_Z = ⨍_Z f d\nu$,
(2) a **median** m_f: any value such that $\nu(\{f < m_f\}) \leq \nu(Z)/2$ and $\nu(\{f > m_f\}) \leq \nu(Z)/2$,
(3) a p-**energy minimizer**: any value c_p such that $\inf_c ||f - c||_p$ is attained for $c = c_p$.

There is a simple comparison between the average and any energy minimizer, so choosing (1) or (3) gives comparable Poincaré constants.

Lemma 3.4. *Let (Z, ν) be a measure space with finite positive measure, and let $f : (Z, \nu) \to \mathbb{R}$ be a measurable function. For every $p \in [1, \infty)$ we have $||f - c_p||_p \leq ||f - f_Z||_p \leq 2\,||f - c_p||_p$.*

Proof. For any $C \in \mathbb{R}$ we have

$$\begin{aligned}
\|f - f_Z\|_p &\leq \|f + C\|_p + \|C + f_Z\|_p \\
&= \|f + C\|_p + \nu(Z)^{1/p}\left|C + \frac{1}{\nu(Z)}\int_Z f(z)d\nu(z)\right| \\
&\leq \|f + C\|_p + \nu(Z)^{-1+1/p}\int_Z |C + f(z)|d\nu(z) \qquad (3.5) \\
&\leq \|f + C\|_p + \nu(Z)^{-1+1/p}\|C + f\|_p\|1\|_{p/(p-1)} \\
&= 2\|f + C\|_p.
\end{aligned}$$

In addition, if $C = c_p$, then $\|f - c_p\|_p \leq \|f - f_Z\|_p$ by definition. $\square$

In the case of $p = 1$, this lemma combines with the following to show that taking either averages or medians will yield comparable Poincaré constants.

Lemma 3.6. *Let (Z, ν) be a measure space with finite positive measure ν and let $f : Z \to \mathbb{R}$ be a measurable function. Then a value c is a* 1-*energy minimizer c_1 of f if and only if it is a median m_f.*

Proof. For $c' > c$, a calculation gives:

$$\|f - c'\|_1 - \|f - c\|_1 = (c' - c)\big(\nu(\{f \leq c\}) - \nu(\{f \geq c'\})\big) + \int_{\{c<f<c'\}} (c + c' - 2f) d\nu. \tag{3.7}$$

If c minimizes $\|f - c\|_1$, (3.7) gives

$$0 \leq (c' - c)\big(\nu(\{f \leq c\}) - \nu(\{f \geq c'\})\big) + (c' - c)\nu(\{c < f < c'\}).$$

Letting $c' \to c$, we get $\nu(\{f > c\}) \leq \nu(\{f \leq c\})$. The same argument applied to $-f$ gives $\nu(\{f < c\}) \leq \nu(\{f \geq c\})$, so c is a median of f.

Conversely, if c is a median for f, (3.7) gives

$$\begin{aligned}
&\|f - c'\|_1 - \|f - c\|_1 \\
&\quad = (c' - c)\left(\nu(\{f \leq c\}) - \nu(\{f > c\})\right) \\
&\qquad\quad + (c' - c)\nu(\{c < f < c'\}) + \int_{\{c<f<c'\}} (c + c' - 2f) d\nu. \\
&\quad \geq (c' - c)(\tfrac{1}{2}\nu(Z) - \tfrac{1}{2}\nu(Z)) + \int_{\{c<f<c'\}} (2c' - 2f) \geq 0,
\end{aligned}$$

so increasing c cannot lower $\|f - c\|_1$. The same argument applied to $-f$ gives that the median c is also a minimizer for $\|f - c\|_1$. $\square$

Remark 3.8. For Γ a finite graph of constant degree d, $\lambda_{1,p}(\Gamma)$, the first eigenvalue of the p-Laplacian on Γ, may be calculated to be the infimum of $\left(\sum_{xy \in E\Gamma} |f(x) - f(y)|^p\right) / \left(\sum_{x \in V\Gamma} |f(x) - c_p(f)|^p d\right)$ over all non-constant f with $c_p(f)$ the energy minimizer of f (see [Bou12]). Thus by Lemma 3.4 we have $\lambda_{1,p}(\Gamma) \asymp h^p(\Gamma)^{1/p}$.

3.2. **Comparison with Lipschitz gradient.** Classical Poincaré inequalities on balls in $\mathbb{R}^n$ involve the L^p-norms of the usual gradient vector ∇f. For general metric spaces this makes no sense, but it is possible to define an analogue of the point-wise norm $|\nabla f|$. Given this, one can define what it means for a metric measure space to satisfy a Poincaré inequality in this infinitesimal sense (see Section 11).

Let (Z, d, ν) be a metric measure space with finite (positive) measure. We define the Lipschitz gradient to be

$$\mathrm{Lip}_x(f) = \limsup_{h\to 0} \sup_{y\in B(x,h)} \frac{|f(x)-f(y)|}{h}.$$

Given a metric space (Z, d) we can define

$$h^p_{\mathrm{Lip}}(Z) = \inf \frac{||\mathrm{Lip}_x(f)||_p}{||f||_p}$$

where the infimum is taken over all non-constant Lipschitz functions $f : Z \to \mathbb{R}$ with average 0.

Following §10.2 and §10.3 from [Tes08], one can show that—under suitable assumptions on a metric measure space—the Poincaré constant relative to the Lipschitz norm (for Lipschitz functions) is equivalent to the Poincaré constant with respect to the gradient at some fixed scale $\alpha > 0$.

Here, we will focus on one direction (the only one required in the paper, namely in the proof of Theorem 11.1) which relies solely on a bounded packing assumption:

Proposition 3.9. *Let (Z, d, ν) be a metric measure space with finite measure ≥ 1, let $a > 0$ and let $C \geq 1$. Assume that for all $x \in Z$, $\nu(B(x, 2a)) \leq C\nu(B(x, a/2))$. Then,*

$$h^p_{\mathrm{Lip}}(Z) \preceq_{C,a,p} h^p_a(Z).$$

Proof. We first need the following lemma:

Lemma 3.10. *Assume $h^p_a(Z) \leq 1/8$. Let $(P_x)_x$ be a family of probability measures on Z, such that P_x is supported in $B(x, a)$ for every $x \in Z$. Then there exists $f \in L^\infty$ such that*

$$\frac{\|\nabla_a f\|_p}{\|Pf - (Pf)_Z\|_p} \leq 4h^p_a(Z),$$

where $Pf(x) := \int f dP_x$.

Proof. We start with f with average 0, $f_Z = 0$, such that

$$\frac{\|\nabla_a f\|_p}{\|f\|_p} \leq 2h^p_a(Z) \leq \frac{1}{4}.$$

Observe that

$$\|f - Pf\|_p \leq \|\nabla_a f\|_p \leq \tfrac{1}{4}\|f\|_p,$$

from which we deduce that

$$|(Pf)_Z| = \left|\fint_Z Pf\right| = \left|\fint_Z Pf - f\right| \leq \frac{\|f - Pf\|_p}{\nu(Z)^{1/p}} \leq \frac{\|f\|_p}{4\nu(Z)^{1/p}}.$$

So $\|(Pf)_Z\|_p \leq \frac{1}{4}\|f\|_p$ and then we deduce by the triangle inequality that

$$\|Pf - (Pf)_Z\|_p \geq \frac{\|f\|_p}{2}. \qquad \square$$

The rest of the proof of the proposition is similar to that of [Tes08, Theorem 10.9]. For the convenience of the reader we sketch it. Define a 1-Lipschitz map $\theta : Z \times Z \to \mathbb{R}_+$ by $\theta(x, y) = d(y, B(x, a)^c)$. For $U \subset Z$ write

$$P_x(U) = \int_U \frac{\theta(x, y)}{K(x)} d\nu(y),$$

where $K(x) = \int_{B(x,a)} \theta(x, z) d\nu(z)$. Note that $K(x) \asymp_C \nu(B(x, a))$, and that by assumption, $\nu(B(x, a)) \asymp_C \nu(B(y, a))$ as soon as $d(x, y) < a$. Since θ is 1-Lipschitz with respect to x, we see that for all $f \in L^\infty(Z)$,

$$\mathrm{Lip}_x(Pf) \preceq_C |\nabla_a f|.$$

Note that if $h_a^p(Z) > \frac{1}{8}$, then the statement of the proposition follows trivially. Hence we can assume that $h_a^p(X) \leq \frac{1}{8}$. By Lemma 3.10 we deduce that there exists some function f such that

$$\frac{\|\mathrm{Lip}_x(Pf)\|_p}{\|Pf - (Pf)_Z\|_p} \preceq_C h_a^p(Z).$$

Hence the proposition follows. $\square$

4. Poincaré profiles for metric measure spaces

Our goal in this section is to generalise the Poincaré profile to the class of metric measure spaces defined in Section 2.

Definition 4.1. Let (X, d, μ) be a metric measure space satisfying our standing assumptions, and fix some number $a \geq 2$. We define the **L^p-Poincaré profile** $\Lambda^p_{X,a}(r)$ of X at scale a to be the supremum of $\mu(A)h_a^p(A)$ over all subspaces $A \subset X$ satisfying $\mu(A) \leq r$. If no such subspace exists, define $\Lambda^p_{X,a}(r) = 0$.

Recall that by assumption, we only consider 1-thick subsets of X to be subspaces.

Remark 4.2. As mentioned in the introduction, strictly speaking, we have defined the L^p-Neumann-Poincaré profile. We could alternatively define L^p-Dirichlet-Poincaré profiles using Dirichlet-Poincaré inequalities (considering the infimum over all functions which are 0 on the boundary of Γ in X, rather than those which have average 0). As defined above, the monotone coarse invariant we obtain detects only if the space has infinite diameter. A small modification to the definition (taking the infimum of $\mu(A)h_a^p(A)$ over all subspaces $A \subset X$ satisfying $\mu(A) \geq r$) yields a coarse invariant (it is not even monotone under quasi–isometric embeddings) which detects isoperimetry (and in particular, Følner amenability) in the case $p = 1$. Dirichlet-type Poincaré inequalities were introduced in [Cou00, Section 7.2] where they are called Sobolev inequalities (see also [Tes08] for a related notion of L^p-isoperimetric profile). They have been extensive studied in the cases $p = 1$, where they are equivalent to isoperimetric inequalities, and $p = 2$, where they govern the asymptotic behaviour of the probability of return of the simple symmetric random walk.

We first prove that the Poincaré profile does not actually depend on the choice of a.

Proposition 4.3. *Assume that (Z, d, ν) is a finite metric measure space. Then for all $a \geq 2$ and all $p \in [1, \infty)$ we have*

$$h_a^p(Z) \asymp_a h_2^p(Z).$$

Proof. We claim that for any $t \geq 0$,

$$\nu\left(\{|\nabla_a f| \geq t\}\right) \preceq_a \nu\left(\left\{|\nabla_2 f| \geq \frac{t}{5a}\right\}\right), \tag{4.4}$$

and $\nu(\{|\nabla_2 f| \geq t\}) \leq \nu(\{|\nabla_a f| \geq t\})$. Together these inequalities immediately imply the proposition. The second inequality is obvious. Let $z \in Z$, and let $x, y \in B(z, a)$. Then one can easily check that our standing assumption implies that there exists a 1-path $x = x_0, \dots, x_n = y$ within $B(z, a)$ such that $n \leq 5a$. By the triangle inequality, this means that for at least one $1 \leq i \leq n$, $|f(x_i) - f(x_{i-1})| \geq \frac{1}{5a}|f(x) - f(y)|$. Now for all $z' \in B(x_i, 1)$ this implies that $|\nabla_2 f|(z') \geq \frac{1}{5a}|f(x) - f(y)|$. Hence there is a 1-thick subset which is $2a$-dense in the set $\{|\nabla_a f| \geq t)\}$ on which $|\nabla_2 f|(z') \geq \frac{t}{5a}$. Thus, the left-hand inequality in (4.4) follows from Lemma 2.3. $\square$

Corollary 4.5. *Assume that (X, d, μ) satisfies our standing assumptions. Then for all $a, a' \geq 2$ and all $p \in [1, \infty)$ we have*

$$\Lambda_{X,a}^p \asymp_{a,a'} \Lambda_{X,a'}^p.$$

Moreover, by Lemma 3.3, choosing a bi-Lipschitz equivalent metric and/or measure does not affect the L^p-Poincaré profile $\Lambda_{X,a}^p$ for sufficiently large a (up to $\simeq$). In particular this means that for a compactly generated locally compact group, the L^p-Poincaré profile does not depend on the choice of Haar measure or on the choice of compact generating set.

In light of Corollary 4.5, we now refer to Λ^p_X as the L^p-Poincaré profile of X, without the need to specify a scale.

5. Regular maps and large scale equivalence

The goal of this section is to prove Theorem 1. Firstly, we formally introduce the notion of a coarse regular map and prove that the definition coincides with regular maps for bounded degree graphs.

5.1. **Regular maps.** We introduce a natural generalisation of a regular map between graphs suited to the context of metric measure spaces. In this section we show that Poincaré profiles are monotone non-decreasing under coarse regular maps.

Definition 5.1. A map $F : (X, d, \mu) \to (X', d', \mu')$ is said to be **coarse regular** if it satisfies the following properties:

(i) F is coarse Lipschitz: there exists an increasing function $\rho_+ : [0, \infty) \to [0, \infty)$ such that for all $x, y \in X$,

$$d(F(x), F(y)) \leq \rho_+(d(x, y));$$

(ii) F is coarsely measure preserving: there exists δ_0 such that for all $\delta \geq \delta_0$ and for all (1-thick) subspaces $A \subset X$,

$$\mu([A]_\delta) \asymp_\delta \mu'([F(A)]_\delta) \asymp_\delta \mu([F^{-1}(F(A))]_\delta).$$

The *parameters* of F are the constant δ_0 as well as the function ρ_+.

Remark 5.2. Coarse regular maps between spaces with bounded packing on large scales are stable under composition.

In applications, coarse regular maps often are embeddings of the following kind.

Definition 5.3. A coarse regular map $F : (X, d, \mu) \to (Y, d, \nu)$ is called a **large-scale embedding** if it is also a coarse embedding; there exists a function ρ_- such that $\lim_{t\to\infty} \rho_-(t) = \infty$ and for all $x, y \in X$,

$$\rho_-(d(x, y)) \leq d(F(x), F(y)).$$

If, in addition, $[F(X)]_C = Y$ for some $C \geq 0$ (in other words, if F is a coarse equivalence), then F is called a **large-scale equivalence**.

It is easy to see that the relation "there exists a large scale equivalence from X to Y" is an equivalence relation among metric measure spaces.

Lemma 5.4. *Let X, X' be simplicial graphs of bounded degree equipped with the shortest path metrics and vertex counting measures. A map $F : VX \to VX'$ is regular in the sense of* [BST12] *if and only if it is coarsely regular as a map between metric measure spaces.*

Proof. If F is regular, then by definition there exists a constant K such that F is K-Lipschitz, the image of every set of measure m has measure at most m and the pre-image of every set of measure m has measure at most Km. Since X and X' have bounded degree, F is coarsely regular.

Suppose F is coarsely regular, then it is $\rho_+(1)$-Lipschitz. Fix some suitable δ_0, let $x' = F(x) \in F(VX)$ and notice that the (1-thick) subspace $A = [x]_1$ satisfies

$$\left|F^{-1}(x')\right| \leq \left|[F^{-1}(F(A))]_\delta\right| \preceq_\delta |[A]_\delta| \leq |[x]_{\delta+1}| \preceq_\delta 1.$$

Thus, F is regular. □

The following proposition is the main goal of this section, and will be proved in §5.2.

Proposition 5.5. *Let $F : X \to X'$ be a coarsely regular map between metric measure spaces which satisfy our standing assumptions. Then for all $p \in [1, \infty)$,*

$$\Lambda^p_X \lesssim_p \Lambda^p_{X'}.$$

Theorem 1 follows immediately from Lemma 5.4 and this proposition. Note that by Proposition 4.3 it suffices to prove $\Lambda^p_{X,a} \lesssim_p \Lambda^p_{X',a'}$. for some $a, a' \geq 2$.

An important consequence of Proposition 5.5 is the following.

Proposition 5.6. *Let G and H be compactly generated locally compact groups, and let $\phi : H \to G$ be a proper continuous morphism (i.e. $\ker\phi$ is compact and $\phi(H)$ is a closed subgroup). We assume that both G and H are equipped with left-invariant Haar measures and word metrics with respect to some compact symmetric generating sets. Then, for all $p \in [1, \infty)$, $\Lambda^p_H \lesssim_p \Lambda^p_G$. If $\phi(H)$ is co-compact then $\Lambda^p_H \simeq_p \Lambda^p_G$.*

Proof. The morphism ϕ is a large-scale embedding hence it is coarsely regular. If $\phi(H)$ is co-compact then ϕ is a large-scale equivalence. The result then follows from Proposition 5.5. □

5.2. **Proof of Theorem** 1. The argument behind the proof is as follows: given a coarse regular map $F : X \to X'$ which is ρ_+-coarse Lipschitz and coarsely measure preserving for all $\delta \geq \delta_0$, and a subspace $Z \subseteq X$, we define $M = \max\{\rho_+(1), \delta_0\}$ and build metric measure space discretizations Y of Z and Y' of the 1-thick subspace $[[F(Z)]_M]_1$. By the definition of a coarse regular map and Lemma 2.3,

$$\mu_X(Z) \asymp_M \mu_X([Z]_M) \asymp_M \mu_{X'}([[F(Z)]_M]_1).$$

We then show that the process of taking a discretization yields spaces with equal measure and comparable Poincaré constants, and finally prove that Y and Y' have comparable Poincaré constants.

The first step of the proof consists in constructing discretizations of our spaces. We fix some $b \geq M$ (which we refer to as the *discretization parameter*). We let $Y \subset Z$ be a maximal $3b$-separated subset of Z. By maximality Z is covered by the union of balls $\bigcup_{y\in Y} B(y, 9b)$. We pick (measurably) a set A_y for each $y \in Y$ with the following properties:

- $B(y, b) \subset A_y \subset B(y, 9b)$;
- $(A_y)_{y\in Y}$ forms a measurable partition of Z.

We equip Y with the subspace distance and the measure $\nu_Y(y) = \nu(A_y)$. Let $\pi : Z \to Y$ be defined by "$\pi(z)$ is the only $y \in Y$ such that $z \in A_y$". Note that π is surjective, and a right-inverse of the inclusion $j : Y \to Z$. Moreover, $\pi^{-1}(y) = A_y$ for every $y \in Y$.

Remark 5.7. Observe that the choice of b ensures that Y has bounded packing at all scales ≥ 0, and that both π and j are large-scale equivalences. In particular, if Y' is a similar discretization of $[[F(Z)]_M]_1$, then $\Psi = \pi' \circ F \circ j$ is a coarse regular map. Moreover, if one chooses the discretization parameter b' large enough, then Ψ is surjective.

Our next goal is to compare the Poincaré constant of a subspace with that of its discretization.

Lemma 5.8. *Let (Z, d, ν) be a metric measure space with finite measure. Suppose (Y, d, ν_Y) is a discretization (with parameter $b \geq 2$) of Z as above. Then for all $a \geq b$,*

$$h^p_a(Y) \lesssim_a h^p_{20a}(Z), \quad \text{and} \quad h^p_a(Z) \leq h^p_{20a}(Y).$$

Proof. Let $f \in L^\infty(Z)$ be such that $⨍_Z f d\nu = 0$. We define $\phi \in \ell^\infty(Y)$ by $\phi(y) = ⨍_{A_y} f d\nu$. Clearly $⨍_Y \phi d\nu_Y = 0$ and $\|\phi \circ \pi\|_{Z,p} = \|\phi\|_{Y,p}$. Write

$f(z) = \phi(\pi(z)) + \fint_{A_{\pi(z)}}(f(z) - f(w))d\nu(w)$. Then

$$\begin{aligned}
\|f\|_{Z,p} &\le \|\phi\circ\pi\|_{Z,p} + \left(\int_Z \left|\fint_{A_{\pi(z)}} (f(z) - f(w))\, d\nu(w)\right|^p d\nu(z)\right)^{1/p} \\
&\le \|\phi\|_{Y,p} + \left(\int_Z \fint_{A_{\pi(z)}} |f(z) - f(w)|^p \, d\nu(w) d\nu(z)\right)^{1/p} \\
&\le \|\phi\|_{Y,p} + \left(\int_Z |\nabla_{10a} f|(z)^p\right)^{1/p} \\
&= \|\phi\|_{Y,p} + \|\nabla_{10a} f\|_p \ .
\end{aligned}$$

On the other hand, it is immediate from the definitions that $|\nabla_a\phi|(y) \le |\nabla_{20a} f|(z)$ for all $z \in A_y$.

We now prove the first inequality. If $h^p_{20a}(Z) \le \frac{1}{2}$, then for any $\epsilon \in (0, 1/6)$ we can find f as above so that

$$\frac{2}{3} \ge \frac{1}{2} + \epsilon \ge h^p_{20a}(Z) + \epsilon \ge \frac{\|\nabla_{20a} f\|_p}{\|f\|_p} \ge \frac{\|\nabla_{20a} f\|_p}{\|\phi\|_p + \|\nabla_{20a} f\|_p}.$$

Thus $\|\nabla_{20a} f\|_p \le 2\|\phi\|_p$ and

$$h^p_{20a}(Z) + \epsilon \ge \frac{\|\nabla_a \phi\|_p}{3\|\phi\|_p} \ge \frac{1}{3} h^p_a(Y).$$

Since ϵ was arbitrary, $h^p_a(Y) \le 3h^p_{20a}(Z)$. Moreover, it is easy to see that $h^p_a(Y) \lesssim_a 1$ (a much more general statement is proved in Proposition 7.1), so if $h^p_{20a}(Z) \ge \frac{1}{2}$, then $h^p_a(Y) \lesssim_a h^p_{20a}(Z)$.

The other direction is easier: given $\psi \in \ell^\infty(Y)$, such that $\fint_Y \psi d\nu_Y = 0$ we define $g = \sum_{y\in Y} \psi(y) 1_{A_y}$, where 1_{A_y} denotes the characteristic function of A_y. We clearly have $\fint g d\nu = 0$ and $\|g\|_p = \|\psi\|_p$. Hence we are left with comparing the gradients.

$$\begin{aligned}
\|\nabla_r g\|_p^p &= \sum_Y \nu(A_y) \fint_{A_y} \sup_{z',z''\in B(z,a)} |g(z') - g(z'')|^p d\nu(z) \\
&\le \sum_Y \nu(A_y) \sup_{z',z''\in B(y,10a)} |g(z') - g(z'')|^p \\
&\le \sum_Y \nu_Y(y) \sup_{y',y''\in B(y,20a)\cap Y} |\psi(y') - \psi(y'')|^p \\
&= \|\nabla_{20a}\psi\|_p^p.
\end{aligned}$$

□

Now we compare the Poincaré constants of discrete spaces related by a sufficiently nice surjective coarse regular map.

Lemma 5.9. *Let $\pi : (Y, d, \nu) \to (Y', d, \nu')$ be a map between two discrete metric measure spaces with finite (non-degenerate) measures, and assume that:*

- *π is surjective;*
- *$\nu(\pi^{-1}(y')) = \nu'(y')$.*

Then for all $a \geq 0$ and C such that $d(y, z) \leq a$ implies $d(\pi(y), \pi(z)) \leq Ca$, we have

$$h^p_a(Y) \leq h^p_{Ca}(Y').$$

Proof. Choose $f' \in \ell^\infty(Y')$ such that $\int f' d\mu' = 0$ and let $f = f' \circ \pi$. Clearly, we have $\int f d\mu = 0$, and

$$\|f\|_p = \|f'\|_p.$$

Moreover, for every $y \in Y$, if $y_1, y_2 \in B(y, a)$ then $\pi(y_1), \pi(y_2) \in B(\pi(y), Ca)$. So a straightforward computation shows that

$$\|\nabla_a f\|_p \leq \|\nabla_{Ca} f'\|_p. \qquad \square$$

As a result we obtain a version of Proposition 5.5 in the uniformly discrete case.

Corollary 5.10. *Let $\Psi : (Y, d, \nu) \to (Y', d, \nu')$ be a surjective coarse regular map between uniformly discrete spaces, which have bounded packing at any scale. Then, for all $a > 0$, there exists $C > 0$ such that*

$$h^p_a(Y) \preceq_a h^p_{Ca}(Y').$$

Proof. The assumptions imply that Ψ is coarse Lipschitz, surjective, and such that $\nu(\Psi^{-1}(y')) \asymp_a \nu'(y')$. Hence the corollary follows from Lemmas 3.3(ii) and 5.9: if we push the measure ν forward with Ψ to obtain a measure $\Psi_*\nu$ on Y' we have

$$h^p_a(Y, \nu) \preceq h^p_{Ca}(Y', \Psi_*\nu) \preceq h^p_{Ca}(Y', \nu'). \qquad \square$$

Combining these results we are in a position to prove Proposition 5.5.

Proof of Proposition 5.5. Let Z be a 1-thick subspace of X and define $Z' = [[F(Z)]_M]_1$ where $M = \max\{\delta_0, \rho_+(1)\}$. Then Z' is a 1-thick subspace of X' and $\mu(Z) \asymp_M \mu'(Z')$. Let b, b' be sufficiently large that the discretizations Y of Z and Y' of Z' satisfy the hypotheses of Lemma 5.8 for some suitable $a = a(b, b') \geq 2$ and so that $\Psi = \pi' \circ F \circ j$ is surjective. Note that b and b' may be chosen independently of the choice of subspace Z of X, hence a does not depend on Z.

Applying Corollary 5.10 we see that there exists a constant C depending only on a such that $h^p_a(Y) \preceq_a h^p_{Ca}(Y')$. Now, by Lemma 5.8 $h^p_a(Z) \preceq_{a,M} h^p_{C'a}(Z')$ where a, M, C' do not depend on Z.

Thus there is some M' depending only on M and Y' such that

$$\begin{array}{rcl} \Lambda^p_{X,a}(r) & = & \sup\{\mu(Z)h^p_a(Z) \,:\, \mu(Z) \leq r\} \\ & \lesssim_{a,M} & \sup\{\mu'(Z')h^p_{C'a}(Z') \,:\, Z' = [[F(Z)]_M]_1,\ \mu(Z) \leq r\} \\ & \leq & \Lambda^p_{X',C'a}(M'r). \end{array}$$

We conclude using Corollary 4.5. □

6. Extremal profiles: growth and separation

6.1. **Growth and the L^∞-Poincaré profile.** In this section we give the proof of Proposition 2. Recall our standing assumptions: a metric measure space (X, d, μ) is 1-geodesic and has bounded packing at scales $\geq r_0 = 1$. Recall also that the growth function $\gamma_X(k)$ is the supremum of the measures of balls of radius k in X, and the inverse growth function $\kappa_X(n)$ is the infimal s such that there exists a ball $B \subset X$ of radius s with measure $> n$. By assumption subspaces are 1-thick and equipped with a 1-geodesic metric.

Proposition 6.1. *Let (X, d, μ) be a metric measure space with unbounded growth function $\gamma_X : [1,\infty) \to (0,\infty)$, and let $a \geq 2$. Then*

$$\Lambda^\infty_{X,a}(r) \simeq_a \sup\left\{\frac{s}{\kappa_X(s)} \,:\, \gamma_X(1) \leq s \leq r\right\},$$

where we interpret $\sup\emptyset$ to be 0.

In all our applications, the function $\sup\left\{\frac{s}{\kappa_X(s)} \,:\, \gamma_X(1) \leq s \leq r\right\}$ will be equivalent to $\frac{r}{\kappa_X(r)}$ but in general this may not be the case. The proof requires a lemma.

Lemma 6.2. *Let Z be a subspace of X with diameter m and let $a \geq 2$. Then $h^\infty_a(Z) \leq \frac{4a}{m}$, and if every $y, z \in Z$ can be joined by a 1-path of length $\leq 2m$ then $h^\infty_a(Z) \geq \frac{1}{2m}$.*

Proof. Choose $x, y \in Z$ such that $d(x, y) \geq m - \delta$, and define $f(z) = d(x, z)$. It is clear that $f(x) = 0$ and $f(y) \geq m - \delta$, so $\|f - f_Z\|_\infty \geq \frac{m-\delta}{2}$, while $\|\nabla_a f\|_\infty \leq 2a$ by the triangle inequality. Thus $h^\infty_a(Z) \leq \frac{4a}{m-\delta}$ for all $\delta > 0$.

For the second inequality, fix $\delta > 0$ and let $f \in L^\infty(Z)$ satisfy $\inf_{z\in Z} f(z) = 0$. Choose y, z so that $(f(z) - f(y)) + \delta \geq \sup_{z\in Z}|f(z)| = \|f\|_\infty$.

By our hypothesis there exists a sequence of points $y = z_0, \dots, z_k = z$ such that $k \leq 2m$ and $d(z_i, z_{i+1}) \leq 1$ for all i. Therefore, $\nabla_a f(z_i) \geq \frac{1}{2m}(||f||_\infty - \delta)$ for some i, so $||\nabla_a f||_\infty \geq \frac{1}{2m}(||f||_\infty - \delta)$. Since we have $||f - f_Z||_\infty \leq ||f||_\infty$, letting $\delta \to 0$, we see that $h_a^\infty(Z) \geq \frac{1}{2m}$. □

Proof of Proposition 6.1. The upper bound on $\Lambda^\infty_{X,a}(r)$ follows immediately from Lemma 6.2. Indeed, if $\mu(Z) \leq r$ then

$$\mu(Z) h_a^\infty(Z) \lesssim_a \frac{\mu(Z)}{\operatorname{diam}(Z)} \leq \frac{\mu(Z)}{\kappa(\mu(Z))},$$

so if $\mu(Z) \geq \gamma_X(1)$ we are done. We can ignore Z with $\mu(Z)$ bounded by any fixed constant like $\gamma_X(1)$ since any $f \in L^\infty(Z)$ has a representative with $\|\nabla_a f\|_\infty \leq 2\|f\|_\infty$, and so $\mu(Z)h_a^\infty(Z) \leq 2\mu(Z)$ is bounded too.

We now prove the lower bound. Let $t \geq 2$ and choose $x_t \in X$ such that $\mu(B(x_t, t)) \geq \frac{1}{2}\gamma_X(t)$. Define Z_t to be the 1-thick subspace $[B(x_t, t-1)]_1$. By Lemma 2.3 there is a constant C (which does not depend on t) such that $\mu(Z_t) \leq \mu(B(x_t, t)) \leq \gamma_X(t) \leq C\mu(Z_t)$.

By Lemma 6.2 $h_a^\infty(Z_t) \in [\frac{1}{4t}, \frac{2a}{(t-1)}]$, so $\mu(Z_t)h_a^\infty(Z_t) \asymp_a \frac{\gamma_X(t)}{t}$.

There exists $s_0 \geq \gamma_X(1)$ so that for all $s \geq s_0$, $\kappa_X(s) \geq 3$. On any bounded interval in $[\gamma_X(1), \infty)$, κ_X is ≥ 1 and so $s/\kappa_X(s)$ is bounded, thus we may assume that s and r satisfy $s_0 \leq s \leq r$. Repeating the above argument, we see that $\gamma_X(t)/\gamma_X(t-1)$ has a uniform upper bound which is independent of t. Let $t = \kappa_X(s) - 1 \geq 2$, and so $\gamma_X(t) \leq s \leq \gamma_X(t+2) \preceq \gamma_X(t)$. Thus for $r \geq s_0$,

$$\Lambda^\infty_{X,a}(r) \succeq_a \frac{\gamma_X(t)}{t} \succeq \frac{s}{\kappa_X(s)}.$$ □

6.2. **Separation profiles of metric measure spaces.** We wish to extend the Cheeger constant definition of separation [Hum17] to the setting of metric measure spaces (X, d, μ) which are 1-geodesic and has bounded packing at scales ≥ 1.

Given a subspace $A \subset X$ (which as usual we assume is 1-thick and equipped with the induced measure and induced 1-geodesic metric) we define the **boundary at scale** $a \geq 2$ of A to be

$$\partial_a A = [A]_a \cap [A^c]_a$$

with the usual notation $A^c = X \setminus A$. For clarity, given a subspace Z of X and $A \subset Z$, we also define the boundary at scale a of A in Z to be $\partial_a^Z A = Z \cap \partial_a(A)$.

Definition 6.3. Let (Z, d, ν) be a metric measure space, where $\nu(Z)$ is finite and let $a \geq 2$. We define the **Cheeger constant at scale** a

of Z to be

$$h_a(Z) = \inf\left\{ \frac{\nu(\partial_a \Omega)}{\nu(\Omega)} \,:\, \nu(\Omega) \le \frac{\nu(Z)}{2} \right\}.$$

Let (X, d, μ) be a metric measure space. We define the function $\operatorname{sep}_{X,a}(r) = \sup\{\mu(Z)h_a(Z)\}$, where the supremum is taken over all (1-thick) subspaces $Z \subseteq X$ with $\mu(Z) \le r$, and is 0 if no such subspaces exist.

Remark 6.4. If Γ is a finite graph of bounded degree D then the boundary at scale a has comparable size to the vertex boundary, so the usual (vertex) Cheeger constant $h(\Gamma)$ satisfies $h(\Gamma) \asymp_{a,D} h_a(\Gamma)$. As a result, if X is an infinite graph of bounded degree D, then $\operatorname{sep}_{X,a} \simeq_{a,D} \operatorname{sep}_X$, where sep_X is the usual separation function for graphs. (See [Hum17, Propositions 2.2, 2.4].)

6.3. **Comparing Cheeger and L^1-Poincaré constants.** Our next goal is to prove Proposition 3. Along the way we will also prove Proposition 6.

Proposition 6.5. *Let (X, d, μ) be a metric measure space and let $a \ge 2$. Then*

$$\frac{1}{2}\operatorname{sep}_{X,a} \le \Lambda^1_{X,a} \le \operatorname{sep}_{X,a}.$$

We prove this by comparing the Cheeger constant and the L^1-Poincaré constant.

Proposition 6.6. *Let (X, d, μ) be a metric measure space. The following co-area formula holds for every non-negative measurable function $f : X \to \mathbb{R}$.*

$$\int_X |\nabla_a f|(x) d\mu(x) = \int_{\mathbb{R}_+} \mu\left(\partial_a\{f > t\}\right) dt \tag{6.7}$$

Proof. For every measurable subset $A \subset X$, we have

$$\mu(\partial_a A) = \int_X |\nabla_a 1_A|(x) d\mu(x). \tag{6.8}$$

Thus, (6.7) follows by integrating over X the following local equalities

$$|\nabla_a f|(x) = \int_{\mathbb{R}_+} |\nabla_a 1_{\{f>t\}}|(x) dt. \tag{6.9}$$

It remains to show that these equalities hold for all $x \in X$.

Notice that $|\nabla_a 1_{\{f>t\}}(x)| = 1$ if and only if there exists $y, y' \in B(x, a)$ with $f(y) > t$ and $f(y') \le t$. In particular, $|\nabla_a 1_{\{f>t\}}(x)|$

equals one for $t \in (\inf_{B(x,a)} f, \sup_{B(x,a)} f)$ and equals zero for $t \notin [\inf_{B(x,a)} f, \sup_{B(x,a)} f]$. Hence,

$$\int_{\mathbb{R}_+} |\nabla_a 1_{\{f>t\}}|(x)dt = \sup_{B(x,a)} f - \inf_{B(x,a)} f = |\nabla_a f|(x),$$

which proves (6.9). □

Using this co-area formula we can prove the required relation between $h_a(Z)$ and $h_a^1(Z)$.

Proposition 6.10. *Let (Z,d,ν) be a metric measure space with finite positive measure ν and let $a \geq 2$. Then*

$$h_a^1(Z) \leq h_a(Z) \leq 2h_a^1(Z).$$

Proof. Let $\Omega \subset Z$ such that $\nu(\Omega) \leq \nu(Z)/2$. We deduce from (6.8) that

$$\|\nabla_a f\|_1 = \nu(\partial_a \Omega),$$

where $f = 1_\Omega$. On the other hand,

$$\|f - f_Z\|_1 = \nu(\Omega)\left(1 - \frac{\nu(\Omega)}{\nu(Z)}\right) + (\nu(Z) - \nu(\Omega))\left(\frac{\nu(\Omega)}{\nu(Z)}\right) \geq \nu(\Omega).$$

Hence $h_a^1(Z) \leq h_a(Z)$.

By Lemmas 3.4 and 3.6, for each $\delta > 0$ we may choose $f \in L^1(Z,\nu)$ (with median 0) such that

$$\frac{\|\nabla_a f\|_1}{\|f\|_1} \leq 2h_a^1(Z) + \delta.$$

Let $f_+ = \max\{f, 0\}$ and $f_- = \min\{f, 0\}$. For any $s, s', t, t' > 0$ if $\frac{s+s'}{t+t'} \leq C$ then $\frac{s}{t} \leq C$ or $\frac{s'}{t'} \leq C$. Since $\|f\|_1 = \|f_-\|_1 + \|f_+\|_1$ and $\|\nabla_a f\|_1 = \|\nabla_a f_+\|_1 + \|\nabla_a f_-\|_1$, we deduce that up to replacing f by $-f$, we have

$$\frac{\|\nabla_a f_+\|_1}{\|f_+\|_1} \leq 2h_a^1(Z) + \delta.$$

Hence using (6.7) and the fact that

$$\|f_+\|_1 = \int_{\mathbb{R}_+} \nu(\{f > t\})dt,$$

we conclude that there exists some $t \geq 0$ such that the subset $\Omega_t = \{f > t\}$ satisfies

$$h_a(Z) \leq \frac{\nu(\partial_a \Omega_t)}{\nu(\Omega_t)} \leq 2h_a^1(Z) + \delta.$$

This proves the second inequality. □

Proof of Proposition 6: The first half of the above proof can easily be adapted to prove that $2^{1-p}h^p_a(Z)^p \leq h_a(Z)$. Hence, $h^p_a(Z)^p \leq 2^p h^1_a(Z)$. $\square$

7. Dependency on p

In this section we prove Proposition 5. One trivial upper bound can always be put on Poincaré constants.

Proposition 7.1. *Let (Z,d,ν) be a metric measure space with $\nu(Z)$ finite. Assume there is no $z \in Z$ with $\nu(\{z\}) > \frac{2}{3}\mu(Z)$. For all $p \in [1,\infty)$ and all $a \geq 2$, $h^p_a(Z) \leq 2\cdot 3^{\frac{1}{p}}$.*

Proof. By our standing assumptions (Definition 2.1), ν is measure isomorphic to a real interval and an at-most-countable collection of atoms. It is then easy to find a subset $Y \subset Z$ satisfying $\frac{1}{3}\nu(Z) \leq \nu(Y) \leq \frac{2}{3}\nu(Z)$. Let f be the characteristic function of Y.

Then $||f - f_Y||^p_p \geq \frac{\nu(Z)}{3\cdot 2^p}$ and $||\nabla_a f||^p_p \leq \nu(Z)$, thus $h^p_a(Z) \leq 2\cdot 3^{\frac{1}{p}}$. $\square$

Equipped with this we are now able to study the relationship between different Poincaré profiles of the same space and prove Proposition 5.

Proposition 7.2. *Let (Z,d,ν) be a metric measure space with $\nu(Z)$ finite. Assume there is no $z \in Z$ with $\nu(\{z\}) > \frac{2}{3}\nu(Z)$. Then for all $1 \leq p \leq q < \infty$ and all $a \geq 2$,*

$$h^q_a(Z) \succeq_{p,q} h^p_a(Z).$$

For all metric measure spaces (X,d,μ) (where μ is possibly infinite), and all $1 \leq p \leq q < \infty$,

$$\Lambda^q_X \succeq_{p,q} \Lambda^p_X.$$

Proof. Our goal is to prove that for any function $g : Z \to \mathbb{R}$, there is a function $f : Z \to \mathbb{R}$ such that

$$\frac{||\nabla_a g||_q}{||g - g_Z||_q} \succeq_{p,q} \frac{||\nabla_a f||_p}{||f - f_Z||_p} \geq h^p_a(Z).$$

Taking the infimum over all g would then yield the desired result. From this, we see that it suffices to consider all functions g which satisfy the upper bound $||\nabla_a g||_q \leq 6\,||g - g_Z||_q$ given by Proposition 7.1. By (3.5) we have that for all $C \in \mathbb{R}$, $6\,||g - g_Z||_q \leq 12\,||g - C||_q$.

For $a \in \mathbb{R}$ and $p \geq 1$, write $\{a\}^p = \text{sign}(a)|a|^p$. For each C, define $f^C : Z \to \mathbb{R}$ by $f^C(z) = \{g(z) + C\}^{q/p}$, for some $C \in \mathbb{R}$. Since f^C_Z is a continuous function of C, we fix C so that $f^C_Z = 0$. Set $f = f^C$.

For each $z \in Z$ let $\overline{(g+C)}_a(z) = \sup\{|g(z') + C| \,:\, d(z,z') \leq a\}$.

By the mean value theorem (see e.g. Matoušek [Mat97, Lemma 4]), for every $s,t\in\mathbb{R}$ and $\alpha\geq 1$,

$$|\{s\}^\alpha-\{t\}^\alpha|\leq\alpha(|s|^{\alpha-1}+|t|^{\alpha-1})|s-t|.$$

For each $z\in Z$ we apply this to $s=g(x)+C, t=g(y)+C, \alpha=\frac{q}{p}$ for all pairs of points $x,y\in B(z,a)$ and see that

$$|\nabla_a f|(z)\leq\frac{2q}{p}\overline{(g+C)}_a(z)^{\frac{q-p}{p}}|\nabla_a g|(z).$$

By the definition of ∇_a, $\overline{(g+C)}_a(z)\leq|g(z)+C|+|\nabla_a g|(z)$, so taking pth powers and integrating, we see that

$$\begin{aligned}
&||g+C||_q^q\,h_a^p(Z)^p=||f||_p^p\,h_a^p(Z)^p=||f-f_Z||_p^p\,h_a^p(Z)^p\\
&\leq\int_Z|\nabla_a f|(z)^p d\nu\\
&\leq\left(\frac{2q}{p}\right)^p\int_Z(|g(z)+C|+|\nabla_a g|(z))^{q-p}|\nabla_a g|(z)^p d\nu\\
&\overset{(\star)}{\leq}\left(\frac{2q}{p}\right)^p 2^{q-p}\left(\int_Z|g(z)+C|^{q-p}|\nabla_a g|(z)^p d\nu+||\nabla_a g||_q^q\right)\\
&\overset{(\dagger)}{\leq}\frac{2^q q^p}{p^p}\left(||g+C||_q^{q-p}||\nabla_a g||_q^p+12^{q-p}||g+C||_q^{q-p}||\nabla_a g||_q^p\right)\\
&\preceq_{p,q}||g+C||_q^{q-p}||\nabla_a g||_q^p,
\end{aligned}$$

where $(\star)$ follows from $(s+t)^\alpha\leq 2^\alpha(s^\alpha+t^\alpha)$ for any $s,t,\alpha>0$, and $(\dagger)$ follows from Hölder's inequality and $||\nabla_a g||_q\leq 12\,||g+C||_q$. Rearranging, taking pth roots, and applying (3.5) we have

$$||g-g_Z||_q\leq 2\,||g+C||_q\preceq_{p,q}\frac{||\nabla_a g||_q}{h_a^p(Z)}.\qquad\square$$

Remark 7.3. There are graphs X of bounded degree containing expanders, and by Propositions 7.2 and 3,

$$\Lambda_X^p(r_n)\succeq_p\Lambda_X^1(r_n)\asymp\mathrm{sep}_X(r_n)\succeq r_n$$

on some unbounded subsequence (r_n) [Hum17], but $\Lambda_X^\infty(r)\simeq r/\log(r)$ by Proposition 2, so one should not expect universal constants (independent of p,q) in the above proposition.

8. Poincaré profiles of groups with polynomial growth

The goal of this section is to prove the lower bound in Theorem 7.

Given a compactly generated locally compact group group G, with compact symmetric generating set K, let $d=d_K$ be the associated word metric and let μ be a left-invariant Haar measure. We refer to the triple (G, d, μ) as a **metric measure CGLC group**. By Lemma 3.3 and Corollary 4.5, the L^p-Poincaré profile of G is well-defined (up to $\simeq$).

Theorem 8.1. *Let (G, d, μ) be a metric measure CGLC group. If there exists some $m > 0$ such that $\gamma(r) \asymp r^m$, then for every $p \in [1, \infty]$, $\Lambda^p_G(r) \gtrsim_p r^{\frac{m-1}{m}}$.*

Note that the $p = \infty$ case follows immediately from Proposition 6.1. Moreover, by Proposition 7.2 $\Lambda^p_G \gtrsim_p \Lambda^1_G$ for all $p \in [1, \infty)$. Using Proposition 6.5 we see that Theorem 8.1 follows from

Theorem 8.2. *Let (G, d, μ) be a metric measure CGLC group. If $\mu(B(1, r)) \asymp r^m$ then for all a, r sufficiently large, there is a subset B_r of $B(1, r)$ with measure at least $\frac{1}{2}\mu(B(1, r))$ satisfying $h_a(B_r) \succeq_a r^{-1}$.*

This theorem will be our goal for the section. The proof is in three parts: the first part gives a general Poincaré inequality satisfied by any compactly generated locally compact group. Secondly we refine this inequality for groups with polynomial growth. In the third part we use this Poincaré inequality (specifically in the L^1 setting) to obtain lower bounds on the Cheeger constant at scale a of metric balls.

8.1. **A Poincaré inequality.** Poincaré inequalities are well known to hold for groups with polynomial growth, see for example [SC02]. In this subsection we present a generalisation of [Kle10, Theorem 2.2] (attributed to Saloff-Coste and explicitly appearing in the L^2 case in [DSC93]) to compactly generated locally compact groups in our framework. The proof below is also similar in nature to [HK00, Proposition 11.17] which is attributed to Varopoulos [Var87].

Theorem 8.3. *Let (G, d, μ) be a metric measure CGLC group. Let $\Delta : G \to \mathbb{R}$ be the modular function on G; i.e., for $U \subset G$ and $g \in G$, $\mu(Ug) = \mu(U)\Delta(g)$. Define $\Delta(K) = \sup_{g\in K} \Delta(g)$.*

For any $p \geq 1$, $a \geq 1$, for any metric ball $B = B(x_0, R)$ of radius R and any function $f \in L^1(G)$ we have the following:

$$\int_B |f(x) - f_B|^p \, d\mu(x) \leq \frac{(2R)^p \mu(2B)\Delta(K)^{2R}}{\mu(B)} \int_{3B} |\nabla_a f| \, (x)^p d\mu(x),$$

where for $\lambda > 0$, $\lambda B = B(x_0, \lambda R)$.

Proof. We may assume $x_0 = e$. Recall that

$$|\nabla_a f| \, (x) = \sup \{ |f(y) - f(z)| \, : \, y, z \in B(x, a) \} .$$

If $a \leq a'$ then $|\nabla_a f|(x) \leq |\nabla_{a'} f|(x)$ so it suffices to prove the result above for $a = 1$.

For every $z \in 2B$, we choose a geodesic $\gamma_z : \{0, 1, \ldots, k\} \to G$ with $\gamma_z(0) = e$ and $\gamma_z(k) = z$.

For $x, y \in B(R)$, let $z = x^{-1}y$, and let $|\gamma_z| = k$ be the length of the corresponding path. Then by the triangle and Hölder's inequality,

$$|f(x) - f(y)|^p \leq \left| \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i)) \right|^p \leq |\gamma_z|^{p-1} \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i))^p.$$

For fixed $z \in 2B$, consider the map $F : (x, i) \mapsto (x\gamma_z(i), i)$. This is clearly injective, so $(x, i) \mapsto x\gamma_z(i)$ is at most $2R$-to-1, and

$$\begin{aligned}
\int_B \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i))^p d\mu(x) &= \sum_{i=1}^{|\gamma_z|} \int_B |\nabla_1 f|(x\gamma_z(i))^p d\mu(x) \\
&= \sum_{i=1}^{|\gamma_z|} \int_{B\cdot\gamma_z(i)} |\nabla_1 f|(x)^p \Delta(\gamma_z(i)^{-1}) d\mu(x) \\
&\leq 2R \sup_{g \in 2B} \Delta(g) \int_{3B} |\nabla_1 f|(x)^p d\mu(x).
\end{aligned}$$

Since $2B = K^{2R}$ we have $\sum_{g \in 2B} \Delta(g) = \Delta(K^{2R}) \leq \Delta(K)^{2R}$, so

$$\begin{aligned}
\int_B |f - f_B|^p \, d\mu &\leq \int_B \left| \int_B |f(x_1) - f(x_2)| \frac{d\mu(x_2)}{\mu(B)} \right|^p d\mu(x_1) \\
&\leq \frac{1}{\mu(B)} \int_{B \times B} |f(x_1) - f(x_2)|^p \, d\mu(x_1) d\mu(x_2) \\
&\leq \frac{(2R)^{p-1}}{\mu(B)} \int_{x \in B} \int_{z \in 2B} \sum_{i=1}^{|\gamma_z|} |\nabla_1 f|(x\gamma_z(i))^p d\mu(z) d\mu(x) \\
&\leq \frac{(2R)^p \Delta(K)^{2R}}{\mu(B)} \int_{z \in 2B} \int_{x \in 3B} |\nabla_1 f|(x)^p d\mu(x) d\mu(z) \\
&\leq \frac{(2R)^p \Delta(K)^{2R} \mu(2B)}{\mu(B)} \int_{3B} |\nabla_1 f|(x)^p d\mu(x).
\end{aligned}$$

□

8.2. **CGLC groups with polynomial growth.** We begin by refining the above Poincaré inequality.

Lemma 8.4. *If $\liminf_{r \to \infty} \frac{1}{r} \log(\mu(B(1, r))) = 0$, then G is unimodular.*

Proof. Suppose G is not unimodular, then there exists some $g \in G$ such that $\Delta(g) > 1$. Since Δ is multiplicative, there is some $k \in K$ with $\Delta(k) > 1$.

Now, for each n, $Kk^n \subseteq B(1, n+1)$, so $\mu(B(1, n+1)) > \Delta(k)^n \mu(K)$, and therefore $\liminf_{r\to\infty} \frac{1}{r}\log(\mu(B(1,r))) > 0$. □

From this we obtain the following refinement of a special case of Theorem 8.3.

Corollary 8.5. *If G has polynomial growth then there exists a constant C such that, for any $p \geq 1$ and $a \geq 1$, for any metric ball $B = B(x_0, R)$ of radius R and any function $f \in L^p(G)$ we have the following:*

$$\int_B |f(x) - f_B|^p \, d\mu(x) \leq CR^p \int_{3B} |\nabla_a f|\,(x)^p d\mu(x). \tag{8.6}$$

Using this refined Poincaré inequality (specifically the case $p = 1$) we will now present a proof of Theorem 8.2 via a series of lemmas. The goal is to prove that any subset A of B such that both $A \cap B$ and $A^c \cap B$ have measure proportional to B must have large boundary inside B. It is not sufficient to apply the Poincaré inequality (8.6) to the characteristic function of A inside B as we cannot distinguish the contribution coming from the boundary of A in B with that coming from the boundary of B in X. The solution is to apply the Poincaré inequality (8.6) "deep inside" B.

From this we will show that there is a large subset of B with sufficiently large Cheeger constant. This step is modelled on ideas from [Hum17] relating the cut size and Cheeger constant definitions of separation.

Definition 8.7. Let X be a metric space, let $x \in X$, and let $r, s \in \mathbb{R}_+$ with $s > r$. The (r, s)**-corona** around x is the set $C_{r,s}(x) = B(x, s) \setminus B(x, r)$.

Lemma 8.8. *Let (G, d, μ) be a metric measure CGLC group with $\gamma(r) \asymp r^m$. For each $\delta \in (0, 1)$ there exists some $\epsilon > 0$ such that for every $x \in G$ and r sufficiently large, we have $\mu(C_{r,(1+\epsilon)r}(x)) \leq \delta\mu(B(x, r))$.*

Proof. By [Tes07, Lemma 24], there exist constants $\alpha, \beta > 0$ independent of r such that $\mu(C_{r-s,r}(x)) \geq \alpha\mu(C_{r,r+s}(x))$ for every $x \in G$ whenever $4\beta < s \leq r$.

Let $\epsilon' \in (0, 1)$, let $r \geq 8\beta$ and for each $1 \leq i \leq k = \lfloor -\log_2 \epsilon' \rfloor$, let $b_i = \mu(C_{(1-2^i\epsilon')r,r})$.

By construction $b_{i+1} \geq (1 + \alpha)b_i$ for all $i \geq 1$, so $b_k \geq (1 + \alpha)^{k-1} b_1$.

Fix $\delta \in (0,1)$. If $\mu(C_{r,(1+\epsilon')r}) > \delta\mu(B(x,r))$, then $\mu(C_{r,(1+\epsilon')r}) \geq \delta b_k \geq \delta(1+\alpha)^{k-1}b_1$. But, by [Tes07, Lemma 24], $b_1 \geq \alpha\mu(C_{r,(1+\epsilon')r})$, so $\alpha\delta(1+\alpha)^{k-1} \leq 1$.

Thus $k \leq \log_{1+\alpha}(\frac{1}{\alpha\delta}) + 1$, which implies that

$$\epsilon' \geq \epsilon_{\alpha,\delta} := \frac{\alpha\delta}{4\log_{1+\alpha}(2)}.$$

The conclusion of the lemma holds for all $\epsilon < \epsilon_{\alpha,\delta}$. □

Lemma 8.9. *Let (G,d,μ) be a metric measure CGLC group with $\gamma(r) \asymp r^m$. There exist constants $r_0, \epsilon, k > 0$ such that the following holds for all $r \geq r_0$.*

For any $A \subset B(x,r)$ with $\frac{1}{4}\gamma(r) \leq \mu(A) \leq \frac{1}{2}\gamma(r)$, there exists a point $w \in B(x,r)$ such that $B(w,3\epsilon r) \subset B(x,r)$, and such that $\mu(B(w,\epsilon r) \cap A) \geq k\gamma(r)$ and $\mu(B(w,\epsilon r) \cap A^c) \geq k\gamma(r)$.

Proof. By Lemma 8.8, for all r sufficiently large, and ϵ sufficiently small the corona $C_{(1-3\epsilon)r,r}(x)$ has size $< \frac{1}{10}\gamma(r)$ for every $x \in X$. Now fix $k > 0$ such that $\gamma(\epsilon r) \geq \frac{80}{3}k\gamma(r)$ for all $r \geq r_0$. Applying Lemma 8.8 with $\delta = \frac{k}{2}$ we deduce that

$$\gamma(\epsilon r + 1) - \gamma(\epsilon r) \leq \frac{k}{2}\gamma(\epsilon r) \leq \frac{k}{2}\gamma(r), \tag{8.10}$$

holds whenever ϵ is sufficiently small and r sufficiently large.

Since $\mu(A \cap B(x,(1-3\epsilon)r)) \geq \frac{3}{20}\gamma(r)$, there exists a point $y \in B(x,(1-3\epsilon)r))$ such that $\mu(A \cap B(y,\epsilon r)) \geq \frac{3}{20}\gamma(\epsilon r) \geq 2k\gamma(r)$. Similarly, there is some $z \in B(x,(1-3\epsilon)r))$ such that $\mu(A^c \cap B(z,\epsilon r)) \geq 2k\gamma(r)$. Now, by our choice of k, for every $v \in B(x,(1-3\epsilon)r)$,

$$\max\{\mu(A \cap B(v,\epsilon r)), \mu(A^c \cap B(v,\epsilon r))\} \geq \tfrac{1}{2}\gamma(\epsilon r) \geq 2k\gamma(r).$$

Since $y, z \in B(x,(1-\epsilon)r))$ there is a sequence $y = v_0, v_1, \ldots, v_l = z$ such that $d(v_{i-1},v_i) = 1$, $l \leq 2r$ and $\{v_i\} \subset B(x,(1-\epsilon)r))$. By (8.10), we see that the measure of the symmetric difference of $B(v_i,\epsilon r)$ and $B(v_{i+1},\epsilon r)$ is at most $k\gamma(r)$ for all i.

Choose i maximal such that $\mu(A \cap B(v_i,\epsilon r)) \geq 2k\gamma(r)$. If $i = l$ then we choose $w = v_l$ and the proof is complete. If $i < l$ then $\mu(A^c \cap B(v_{i+1},\epsilon r)) \geq 2k\gamma(r)$, but since the symmetric difference of $B(v_i,\epsilon r)$ and $B(v_{i+1},\epsilon r)$ has measure at most $k\gamma(r)$, we see that $\mu(A^c \cap B(v_i,\epsilon r)) \geq k\gamma(r)$ and we set $w = v_i$. □

With this lemma we can show that large subsets of balls have large boundaries inside the ball.

Proposition 8.11. *Let (G,d,μ) be a metric measure CGLC group with $\gamma(r) \asymp r^m$. For every a sufficiently large, there exists a constant $k = k(a)$ such that for every ball B of radius r, and any subspace $A \subset B(x,r)$ with $\frac{1}{4}\gamma(r) \leq \mu(A) \leq \frac{1}{2}\gamma(r)$, we have $\mu(\partial_a^B A) \geq kr^{m-1}$.*

Proof. Let $A \subset B(x,r) = B$ be such that $\frac{1}{4}\mu(B) \leq \mu(A) \leq \frac{1}{2}\mu(B)$. By Lemma 8.9 there exists some $w \in B(x,(1-3\epsilon)r)$ such that $\mu(B(w,\epsilon r) \cap A),\ \mu(B(w,\epsilon r) \cap A^c) \geq k\mu(A)$. Applying the Poincaré inequality (8.6) with $p = 1$ to the characteristic function $\mathbf{1}_A$ on the ball $B(w,\epsilon r)$ we see that

$$\tfrac{1}{2}k\mu(A) \leq C\epsilon r\mu(\partial_a^{B(w,3\epsilon r)}A).$$

Since $B(w,3\epsilon r) \subseteq B$ we deduce that there exists a constant $k' > 0$ (independent of r) such that

$$\mu(\partial_a^B A) \geq \frac{k'}{r}\mu(B). \qquad \square$$

The last step in this argument ensures that there is a large subset of the ball with suitable Cheeger constant at scale a. This is a generalisation of a similar result for graphs presented in [Hum17].

Proposition 8.12. *Let (X,d,μ) be a metric measure space such that $\inf_{x\in X}\mu(B(x,1)) = c > 0$, and let $a,r \geq 2$. If there exists a constant $\lambda = \lambda(a,r) \leq \frac{1}{4}$ and a ball $B = B(x,r)$ of radius r, such that for any subspace $A \subset B$ with $\frac{1}{4}\mu(B) \leq \mu(A) \leq \frac{1}{2}\mu(B)$, we have $\mu(\partial_a^B A) \geq \lambda\mu(A)$, then there exists some 1-thick subspace B' of B such that $\mu(B') \geq \frac{1}{2}\mu(B)$ and $h_a(B') \geq \frac{\lambda}{2}$.*

Proof. Fix B as above. Given any subset A_0 of B such that $\mu(A_0) \leq \frac{1}{2}\mu(B)$ and $\mu(\partial_a A_0) < \frac{\lambda}{2}\mu(B)$, we have $\mu(A_0) < \frac{1}{4}\mu(B)$ by assumption.

Let m be the supremum of the measures of all subsets A_0 satisfying the above, and let A_1 be such a subset with measure at least $m - \frac{2c}{\lambda}$. Define $A' = [B \setminus [A_1]_a]_1 \subseteq B \setminus A_1$. Note that A' is 1-thick and $\mu(A') \geq \mu(B) - \mu(A_1) - \mu(\partial_a A_1) \geq \frac{5}{8}\mu(B)$.

We wish to show that $h_a(A') \geq \lambda/2$. Suppose for a contradiction that there exists some subset $E \subset A'$ with $\mu(E) \leq \frac{1}{2}\mu(A')$ and $\mu(\partial_a^{A'}E) < \frac{\lambda}{2}\mu(E)$. Since $\partial_a^B E \subseteq \partial_a^{A'}E \cup \partial_a^B A_1$, we have $\mu(\partial_a^B(E \cup A_1)) < \frac{\lambda}{2}(\mu(E) + \mu(A_1))$. From this we deduce that $\mu(E) + \mu(A_1) > \frac{1}{2}\mu(B)$, or, if this is not the case, then $\mu(E) \leq \frac{2c}{\lambda}$ by the choice of A_1.

In the second case we are done: $\partial_a^{A'}E$ contains a ball of radius 1, so $c \leq \mu(\partial_a^{A'}E) < \frac{\lambda}{2}\mu(E) \leq c$ which is a contradiction.

Otherwise $\mu(E \cup A_1) \in (\frac{1}{2}\mu(B), \frac{3}{4}\mu(B))$ so $E' = B \setminus (E \cup A_1)$ satisfies $\mu(E') \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B))$ and $\mu(\partial_a^B E') = \mu(\partial_a^B(E \cup A_1)) < \frac{\lambda}{2}\mu(B)$, which is also a contradiction. $\square$

Proof of Theorem 8.2. This follows immediately from Propositions 8.11 and 8.12 with $\lambda = k/r$. □

9. Upper bounds and large-scale dimension

The goal of this section is to obtain upper bounds on the Poincaré profiles of a metric measure space which is finite dimensional in the sense of the definition below. In doing so, we will prove that the lower bound for groups of polynomial growth in section 8 is sharp to complete the proof of Theorem 7.

Definition 9.1. Let (X, d, μ) be a metric measure space. We say X has **measurable dimension at most** n ($\mathrm{mdim}(X) \leq n$) if, for all $r \geq 0$ we can write $X = X_0 \cup \cdots \cup X_n$ and decompose each $X_i = \bigcup X_{ij}$ so that each X_{ij} is 1-thick, $\sup(\mu(X_{ij})) < \infty$ and $d(X_{ij}, X_{ij'}) \geq r$ whenever $j \neq j'$.

If $\mathrm{mdim}(X) \leq n$ we define the function $\gamma_n(r)$ to be the infimal value of $\sup(\mu(X_{ij})) + 1$ taken over all decompositions of X satisfying the above hypotheses.

Notice that $\gamma_n(r)$ is non-decreasing as a function of r.

A simple comparison can be made with asymptotic dimension when the metric measure space has **bounded geometry**: for all $r \geq 0$ there exists some C_r such that $\mu(B(x, r)) \leq C_r$ for all $x \in X$.

Lemma 9.2. *Let (X, d, μ) be a metric measure space with bounded geometry. Then the asymptotic dimension of X is at least* $\mathrm{mdim}(X)$*.*

Proof. Suppose $asdim(X) \leq n$. This implies that for all $r \geq 0$ one can decompose $X = X'_0 \cup \ldots \cup X'_n$ and further decompose each $X'_i = \bigcup X'_{ij}$ so that $\sup\left\{\mathrm{diam}(X'_{ij})\right\} = K_r < \infty$ and $d(X_{ij}, X_{ij'}) \geq r + 2$ whenever $j \neq j'$.

Define $X_{ij} = \bigcup_{y \in X'_{ij}} B(y, 1)$. Each X_{ij} is 1-thick, it has diameter at most $L = K_r + 2$ and $d(X_{ij}, X_{ij'}) \geq r$ whenever $j \neq j'$. Since X has bounded geometry, $\mu(X_{ij}) \leq C_L$ for all i, j. □

Lemma 9.3. *Let (X, d, μ) and (Y, d', μ') be metric measure spaces and suppose Y has bounded packing at scales ≥ 1. If there exists a coarsely regular map $F : X \to Y$, then* $\mathrm{mdim}(X) \leq \mathrm{mdim}(Y)$*. Moreover, for all suitable n we have $\gamma_n^X \lesssim_n \gamma_n^Y$.*

Proof. Suppose $\mathrm{mdim}(Y) \leq n$. Then for all $r \geq 0$ one can write $Y = \bigcup_{i=0}^n \bigcup_j Y_{ij}^r$ where each Y_{ij}^r is 1-thick, $\mu'(Y_{ij}^r) \leq C$ for some C and all i, j, and $d'(Y_{ij}^r, Y_{ij'}^r) > \rho_+(r + 2)$ whenever $j \neq j'$.

Let $X_{ij}^r = [F^{-1}(Y_{ij}^r)]_1$. By Definition 5.1(i), $d(X_{ij}^r, X_{ij'}^r) > r$ whenever $j \neq j'$, and by (ii) $\mu(X_{ij}^r) \asymp \mu'([Y_{ij}^r]_1) \preceq \mu'(Y_{ij}^r)$ by Lemma 2.3. □

Remark 9.4. One can remove the assumption that Y has bounded packing at scales ≥ 1 by removing the assumption that each X_{ij} is 1-thick in the definition of measurable dimension.

Proposition 9.5. *Let (X, d, μ) be a metric measure space with $\mu(X) = \infty$ and measurable dimension at most n. For all $\delta > 0$,*

$$\Lambda_X^p(r) \lesssim_n \sup\left\{\gamma_n(t+\delta)/t \,:\, \gamma_n(t) \leq r/(4n+4)\right\}.$$

Proof. If γ_n is bounded then μ is bounded, which is a contradiction.

Choose $s > 4(n+1)\gamma_n(0)$ and assume $\mu(A) = s \leq r$. Fix $\delta > 0$ and find t so that $4(n+1)\gamma_n(t) \leq \mu(A) \leq 4(n+1)\gamma_n(t+\delta)$. Select a decomposition of X into sets X_{ij}^t as above where $\mu(X_{ij}^t) \leq \gamma_n(t)$ for all i, j.

Then there exists some i such that $\mu(A \cap X_i) \geq \frac{1}{n+1}\mu(A) \geq 4\gamma_n(t)$. Without loss of generality, assume $i = 0$. Choose J so that

$$X_0' := \bigcup_{j \in J} X_{0j}^t \quad \text{satisfies} \quad \frac{\mu(A)}{4(n+1)} \leq \mu(A \cap X_0') \leq \frac{\mu(A)}{2(n+1)}.$$

Set $X_0'' = X_0^t \setminus X_0'$ and let $f_t : A \to \mathbb{R}$ be the function $f(x) = \frac{1}{t}\min\{t, d_X(x, X_0')\}$.

Now f_t is $\frac{1}{t}$-Lipschitz, so $\int_A |\nabla_2 f|^p \leq \frac{2^p}{t^p}\mu(A)$. Since f takes values in $[0,1]$ and has value 0 on X_0' and value 1 on X_0'' each of measure $\geq \mu(A)/4(n+1)$, we see that $\int_A |f - f_R|^p \, d\mu(x) \geq (\frac{1}{2})^p \frac{1}{4(n+1)}\mu(A)$.

Thus, $h_a^p(A) \leq \frac{4}{t}(n+1)^2 \preceq_n \frac{1}{t}$. As this holds for every measurable $A \subset X$ of finite measure the result follows. □

Remark 9.6. Under nice circumstances, for instance when a space X has a cobounded isometry group, and finite asymptotic dimension where the K_r can be bounded by an affine function of r (sometimes called linearly controlled or asymptotic Assouad–Nagata dimension), the function $\gamma_n(s_r + \delta)/s_r$ is equivalent (up to $\simeq$) to $r/\kappa(r)$ where κ is the inverse growth function. This is easily deduced from the argument in the proof of Proposition 6.1.

Proof of Theorem 7. Let (G, d, μ) be a CGLC metric measure group with $\mu(B(1,r)) \asymp r^m$. Such groups have finite asymptotic Assouad–Nagata dimension ([Bre14, Theorem 1.2] and [HP13, Theorem 5.5]), so by Proposition 9.5, $\Lambda_G^p(r) \lesssim r^{\frac{m-1}{m}}$ for all $p \geq 1$. The lower bound is proved in Theorem 8.1. □

Example 9.7. As another example, for X equal to the product of two 3-regular trees we have $\Lambda_X^p(r) \simeq r/\log(r)$ for all $p \in [1, \infty]$: The case $p = \infty$ follows immediately from Proposition 2. By [BST12, Theorem

3.1] and Proposition 6.5, the lower bound holds when $p = 1$, so the lower bound for general p follows from Proposition 7.2. For the upper bound, X has exponential growth, a cobounded isometry group, and asymptotic Assouad–Nagata dimension 2, so by Proposition 9.5, $\Lambda^p_X(r) \lesssim r/\log(r)$ for all $p \geq 1$.

10. Trees

In this section, we calculate the Poincaré profile for regular trees.

Theorem 10.1 (Theorem 9)**.** *Let T be the infinite 3-regular tree. Then for every $p \in [1, \infty)$, $\Lambda^p_T(r) \asymp_p r^{(p-1)/p}$.*

For $p = 1$ this is immediate from [BST12]. This theorem immediately implies the following corollary for groups admitting quasi-isometric embeddings of such trees.

Corollary 10.2. *If (G, d, μ) is a CGLC measure group which is non-amenable, non-unimodular, or is compact-by-elementary amenable and has exponential growth, then for any $p \geq 1$, $\Lambda^p_G(r) \gtrsim_{G,p} r^{(p-1)/p}$.*

Proof. In the first two cases this follows from [BS97], and in the third from [Cho80]. □

In this section, for a graph X, and a function $f : VX \to \mathbb{R}$, we define $|\nabla f| : EX \to \mathbb{R}$ as $|\nabla f|(e) = |f(x) - f(y)|$ where $e \in EX$ has endpoints $x, y \in VX$. If X has maximum vertex degree d then for each $p \geq 1$,

$$\|\nabla_2 f\|_p \asymp_d \|\nabla f\|_p = \left(\sum_{e \in EX} |\nabla f|(e)^p \right)^{1/p}.$$

A key step in proving Theorem 10.1 is to reduce to an estimate on complete graphs in the spirit of, for example, Spielman [Spi15, Section 4.7].

Proposition 10.3. *For any $r \in \mathbb{N}, r \geq 2$ and $p \in [1, \infty)$, letting K_r denote the complete graph on r vertices, we have*

$$r^{1/p} \leq \inf \left\{ \frac{\|\nabla f\|_p}{\|f - f_{K_r}\|_p} \ : \ f : VK_r \to \mathbb{R}, f \not\equiv f_{K_r} \right\} \preceq_p r^{1/p}.$$

Proof. Let $f : VK_r \to \mathbb{R}$ be any non-constant function on K_r. Then

$$\begin{aligned}
\|f - f_{K_r}\|_p^p &= \sum_x \left| f(x) - \frac{1}{r}\sum_y f(y) \right|^p \\
&\leq \frac{1}{r^p}\sum_x \left(\sum_y |f(x) - f(y)| \right)^p \\
&\leq \frac{1}{r^p}\sum_x \left(\sum_y |f(x) - f(y)|^p \right) r^{p-1} \\
&= r^{-1}\|\nabla f\|_p^p.
\end{aligned}$$

This proves the first inequality; the second can be seen by considering a function which is 1 and -1 on one vertex each, and zero everywhere else. ☐

Proof of Theorem 10.1. First we show the upper bound, which is relatively simple.

Suppose $A \subset T$ is a graph of size $|A| = r$; we can find a vertex x so that on deleting this vertex, all remaining connected components have size $\leq r/2$. Group these components into sets U, V of size $\in [r/4, 3r/4]$. Let $f : A \to [-1, 1]$ be identically -1 on U, 1 on V and 0 on x.

Clearly $\|f - f_A\|_p^p \geq \frac{1}{4}r$, and since ∇f is only non-zero on edges adjacent to x, $\|\nabla f\|_p^p \leq 3$. Thus $h^p(A) \leq (12/r)^{1/p}$ and

$$\Lambda^p(r) = \sup_{|A|\leq r} |A| h^p(A) \leq 12 r^{(p-1)/p}.$$

Second, we show the lower bound.

For any $r > 0$ there exists a ball $B = B(x_0, t) \subset T$ of size $\asymp 2^t \asymp r$, so we can assume $r = |B|$ and it then suffices to show that $h^p(B) \succeq |B|^{-1/p}$, with constant independent of B.

Let K_r be the complete graph on r vertices. Suppose that a non-constant function $f : B \to \mathbb{R}$ is given. Consider f as a function on the complete graph $K_r = K_{|B|}$. In light of Proposition 10.3, to show that $h^p(B) \succeq |B|^{-1/p}$, it suffices to show that

$$\sum_{e \in EB} |\nabla f(e)|^p \geq \frac{1}{2|B|^2} \sum_{x,y \in B} |f(x) - f(y)|^p,$$

for then $h^p(B) \succeq |B|^{-2/p} r^{1/p} \succeq |B|^{-1/p}$.

Now for each $x, y \in B$, let γ_{xy} be the simple path in T joining x to y. Observe that $|f(x) - f(y)| \leq \sum_{e \in \gamma_{xy}} |\nabla f(e)|$.

For each $e \in EB$, let N_e be the number of such simple paths that pass through e. Observe that $N_e \asymp 2^t \cdot 2^{t-d(x_0,e)}$, where $d(x_0, e)$ is the distance from the centre of the ball to the edge e.

Using Hölder's inequality, we have

$$\begin{aligned}
\sum_{x,y\in B} |f(x)-f(y)|^p &\leq \sum_{x,y\in B} \left(\sum_{e\subset\gamma_{xy}} |\nabla f(e)| \right)^p \\
&= \sum_{x,y\in B} \left(\sum_{e\subset\gamma_{xy}} |\nabla f(e)| N_e^{-1/p} N_e^{1/p} \right)^p \\
&\leq \sum_{x,y\in B} \left(\sum_{e\subset\gamma_{xy}} |\nabla f(e)|^p N_e^{-1} \right) \left(\sum_{e\subset\gamma_{xy}} N_e^{1/(p-1)} \right)^{p-1}
\end{aligned}$$

For each simple path, $N_e^{1/(p-1)}$ takes values in (two) geometric series, with ratio depending only on p and maximum value $\preceq (2^{2t})^{1/(p-1)} \asymp |B|^{2/(p-1)}$, and so the sum inside the second parentheses above is also $\preceq |B|^{2/(p-1)}$. Thus,

$$\begin{aligned}
\sum_{x,y\in B} |f(x)-f(y)|^p &\preceq \sum_{x,y\in B} \sum_{e\in\gamma_{xy}} |\nabla f(e)|^p N_e^{-1} |B|^2 \\
&\leq 2|B|^2 \sum_{e\in B} |\nabla f(e)|^p,
\end{aligned}$$

and so we are done. □

11. Lower bounds for hyperbolic spaces with boundary Poincaré inequalities

In this section we find lower bounds on Poincaré profiles for hyperbolic groups whose boundaries admit Poincaré inequalities in the sense of Heinonen and Koskela (Theorem 13). In section 13 we will apply these results to rank 1 symmetric spaces, and a family of hyperbolic buildings studied by Bourdon and Pajot.

Suppose a metric space (Z, ρ) is **Ahlfors Q-regular**, i.e. there is a measure μ on Z so that for every ball $B(z, r)$ in Z with $r \leq \operatorname{diam}(Z)$, we have $\mu(B(z,r)) \asymp r^Q$. (We may take μ to be the Hausdorff Q-measure on Z.) For $p, q \geq 1$, we say (Z, ρ) admits a **(q,p)-Poincaré inequality** (with constant $L \geq 1$) if for every Lipschitz function $f : Z \to \mathbb{R}$ and every ball $B(z,r) \subset Z$,

$$\left(\fint_{B(z,r)} |f - f_{B(z,r)}|^q \, d\mu \right)^{1/q} \leq Lr \left(\fint_{B(z,Lr)} (\mathrm{Lip}_x f)^p \, d\mu(x) \right)^{1/p},$$

where for $U \subset Z$, $f_U = \fint_U f \, d\mu = \frac{1}{\mu(U)} \int_U f \, d\mu$, and

$$\mathrm{Lip}_x f = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}.$$

If $q = 1$, we say Z admits a p-**Poincaré inequality**. By Hölder's inequality, if Z admits a p-Poincaré inequality, it admits a q-Poincaré inequality for all $q \geq p$. Moreover, since Z is doubling, it will admit (q,q)-Poincaré inequalities for all $q \geq p$ by [HK00, Theorem 5.1].

A geodesic metric measure space (X,d,μ) is **Gromov hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$: for every geodesic triangle $T = (\gamma_1, \gamma_2, \gamma_3)$, we have $\gamma_1 \subseteq [\gamma_2 \cup \gamma_3]_\delta$. It is **visual** if there exists $x_0 \in X$ and $C \geq 0$ so that every $x \in X$ belongs to a C-quasi-geodesic ray $\gamma : [0,\infty) \to X$ with $\gamma(0) = x_0$. Gromov hyperbolic metric spaces have a boundary at infinity $\partial_\infty X$ which comes with a family of metrics: if X is visual with respect to x_0, a **visual metric** ρ on $\partial_\infty X$ based at $x_0 \in X$ with visibility parameter $\epsilon > 0$ is a metric satisfying $\rho(\cdot,\cdot) \asymp \exp(-\epsilon(\cdot|\cdot)_{x_0})$, where $(\cdot|\cdot)_{x_0}$ denotes the Gromov product with respect to x_0. For more background and discussion, see [BS00, BP03].

We can now state the first main result of this section (cf. Theorem 13).

Theorem 11.1. *Suppose that X is a visual Gromov hyperbolic graph with a visual metric ρ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a p-Poincaré inequality. Then for all $q \geq p$, $\Lambda^q_X(r) \gtrsim r^{1-1/Q}$.*

By taking discretizations, one can apply this result to rank-1 symmetric spaces, amongst other examples.

Proof. Consider $\partial_\infty X$ with the metric ρ, which admits a p-Poincaré inequality with some constant $L \geq 1$. As a consequence, $(\partial_\infty X, \rho)$ is quasi-convex, so ρ is bi-Lipschitz equivalent to a geodesic metric. Therefore we may assume that ρ is geodesic, and so our standing assumptions hold.

Following Bourdon–Pajot [BP03, Section 2.1], we ensure that $Z = (\partial_\infty X, \rho)$ has diameter $1/2$ by rescaling, and define a graph Γ which approximates Z: Γ has vertex set $\{z^i_t : t \in \mathbb{N}, 1 \leq i \leq k(t)\}$ where for each $t \in \mathbb{N}$, $\Gamma_t = \{z^1_t, \ldots, z^{k(t)}_t\}$ is a maximal e^{-t}-separated net in Z. To each z^i_t we associate a ball $B(z^i_t, e^{-t}) \subset Z$, and we join z^i_t and z^j_u by an edge if and only if $|t-u| \leq 1$ and $B(z^i_t, e^{-t}) \cap B(z^j_u, e^{-u}) \neq \emptyset$.

By Bourdon–Pajot [BP03, Proposition 2.1, Corollary 2.4], Γ, with the path metric d, is a bounded degree hyperbolic graph which is quasi-isometric to X, and so it suffices to show the Poincaré profile bound for Γ.

We now consider the sequence $Z_t = (Z, \rho_t, \mu_t)$ of metric measure spaces, where $\rho_t = 6e^t\rho$, and $\mu_t = e^{Qt}\mu$. Note that $\mu_t(Z_t) \asymp e^{Qt}$. We deduce from the Poincaré inequality satisfied by Z that Z_t satisfies for any Lipschitz function f on Z_t, for all $q \geq p$

$$\left(\fint |f - f_{Z_t}|^q \, d\mu_t\right)^{1/q} \preceq e^t \left(\fint (\mathrm{Lip}_x f)^q \, d\mu_t(x)\right)^{1/q},$$

and therefore that

$$h^q_{\mathrm{Lip}}(Z_t) \succeq e^{-t}$$

with constant independent of t. By Proposition 3.9, this implies that

$$h^q_2(Z_t) \succeq e^{-t}.$$

Now equip Γ_t with the counting measure and the distance induced from its inclusion in Z_t. Since Γ_t is a maximal 6-separated subset of Z_t, we can find a partition

$$Z_t = \bigsqcup_{\gamma\in\Gamma_t} A_\gamma,$$

where

$$B_{\rho_t}(\gamma, 2) \subset A_\gamma \subset B_{\rho_t}(\gamma, 18).$$

By the Ahlfors regularity of Z_t, $\mu(A_\gamma) \asymp 1$. Hence by Lemmas 5.8 and 3.3(ii), we deduce that

$$h^q_{40}(\Gamma_t, \rho_t) \succeq e^{-t}.$$

In order to conclude, we need to show that there exists a constant C such that two vertices $x, y \in \Gamma_t$ such that $\rho_t(x, y) \leq 40$ satisfy $d(x, y) \leq C$ (where $d(x, y)$ is their distance in Γ). Indeed, that will show that

$$h^q_C(\Gamma_t, d) \succeq e^{-t}, \tag{11.2}$$

and since $|\Gamma_t| \asymp e^{Qt}$,

$$\Lambda^q_\Gamma(r) \succeq r^{1-1/Q}.$$

By [BP03, Lemma 2.2], for $x, y \in \Gamma$ corresponding to balls $B_x, B_y \subset Z$, $e^{-(x|y)} \asymp \mathrm{diam}(B_x \cup B_y)$, where $(x|y)$ denotes the Gromov product with respect to the base point z^1_1. For $x, y \in \Gamma_t$, we have $(x|y)$ equal to $t - \frac{1}{2}d(x, y)$ up to a uniform additive error, and $\mathrm{diam}(B_x \cup B_y) \asymp e^{-t} + \rho(x, y)$, so

$$e^{-t}e^{\frac{1}{2}d(x,y)} \asymp \mathrm{diam}(B_x \cup B_y) \asymp e^{-t} + \rho(x, y).$$

Thus, $\rho(x,y) \leq \frac{40}{6}e^{-t}$ implies that $d(x,y) \preceq 1$, which completes the proof of Theorem 11.1. $\square$

We will see in section 13 that for many spaces, Theorem 11.1 gives sharp lower bounds for Λ_X^q when $q \in [1,Q)$. For $q = Q$, however, one can do better.

Theorem 11.3. *Suppose that X is a visual Gromov hyperbolic graph with a visual metric ρ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a Q-Poincaré inequality. Then $\Lambda_X^Q(r) \gtrsim r^{1-1/Q}\log(r)^{1/Q}$.*

Proof. We continue with the notation of the proof of Theorem 11.1. Given $s < t \in \mathbb{N}$, let $B_{s,t}$ be the full subgraph of Γ containing the layers $\Gamma_{s+1}, \Gamma_{s+2}, \ldots, \Gamma_t$. (Later we will take $s = \lfloor t/2 \rfloor$.) The strategy of the proof is to use the Poincaré inequality in each layer to get a stronger constant for all of $B_{s,t}$.

Let us be given a function $f : B_{s,t} \to \mathbb{R}$, i.e. a function on $VB_{s,t}$. For $x \in \Gamma$, define $i_x \in \mathbb{N}$ to satisfy $x \in \Gamma_{i_x}$. Given $x \in \Gamma$ and $i \leq i_x$, let $\pi_i(x) \in \Gamma_i$ be (one of) the points in Γ_i so that the point in Z corresponding to x lies in the ball of radius e^{-i} corresponding to $\pi_i(x)$; the allowed choices of $\pi_i(x)$ are all at distance 1 from each other.

For $i = s+1, \ldots, t$, there are $i - s$ layers in $B_{s,t}$ with labels $\leq i$.

Lemma 11.4. *There is an assignment $B_{s,t} \to \mathbb{N}$ that maps each $x \in B_{s,t}$ to a layer $c_x \in \{s+1, \ldots, i_x\}$, so that for any $z \in B_{s,t}$ and any c, i with $c \leq i_z \leq i \leq t$ we have*

$$|\{x \in \Gamma_i : \pi_{i_z}(x) = z \text{ and } c_x = c\}| \preceq \frac{e^{Q(i-i_z)}}{i-s} \leq \frac{e^{Q(t-s)}}{t-s}, \tag{11.5}$$

where the constant of '$\preceq$' is independent of s, t, z, c and i.

This follows from a colouring argument that we defer until later.

Similarly to the proofs in Section 10, we bound

$$\begin{aligned} \|f - f_{B_{s,t}}\|_p^p &= \sum_x \left| f(x) - \frac{1}{|B_{s,t}|} \sum_y f(y) \right|^p \\ &\leq \frac{1}{|B_{s,t}|} \sum_x \sum_y |f(x) - f(y)|^p. \end{aligned} \tag{11.6}$$

(We refrain from setting $p = Q$ at present to clarify the role this plays in the proof.)

At a cost of multiplying by 2, we can restrict to sum only over x, y where $i_x \leq i_y$. In particular, $c_x \leq i_y$. Given such x, y, we consider the path α_x that follows $x, \pi_{i_x-1}(x), \ldots, \pi_{c_x}(x)$, and also the path $\beta_{x,y}$ that follows along $\pi_{c_x}(y), \pi_{c_x+1}(y), \ldots, \pi_{i_y-1}(y), y$.

Continuing from (11.6), since

$$|f(x)-f(y)| \leq \left(\sum_{z\in\alpha_x} |\nabla_1 f|(z)\right) + |f(\pi_{c_x}(x)) - f(\pi_{c_x}(y))| + \left(\sum_{z\in\beta_{x,y}} |\nabla_1 f|(z)\right),$$

we use the inequality $(a+b+c)^p \leq 3^p(a^p+b^p+c^p)$ to find the following:

$$\|f - f_{B_{s,t}}\|_p^p \preceq \frac{1}{|B_{s,t}|}\sum_{\substack{x,y\\ i_x\leq i_y}} \left(\sum_{z\in\alpha_x} |\nabla_1 f|(z)\right)^p + \frac{1}{|B_{s,t}|}\sum_{\substack{x,y\\ i_x\leq i_y}} |f(\pi_{c_x}(x)) - f(\pi_{c_x}(y))|^p + \frac{1}{|B_{s,t}|}\sum_{\substack{x,y\\ i_x\leq i_y}} \left(\sum_{z\in\beta_{x,y}} |\nabla_1 f|(z)\right)^p. \tag{11.7}$$

We denote the resulting three terms of the sum by S_1, S_2, and S_3. For each $z \in B_{s,t}$, let M_z be the number of pairs (x, y) so that α_x passes through z, and likewise N_z for $\beta_{x,y}$. Let us bound the first term of (11.7), S_1.

$$\begin{aligned} S_1 &= \frac{1}{|B_{s,t}|}\sum_{\substack{x,y\\ i_x\leq i_y}} \left(\sum_{z\in\alpha_x} |\nabla_1 f|(z) M_z^{-1/p} M_z^{1/p}\right)^p \\ &\leq \frac{1}{|B_{s,t}|}\sum_{\substack{x,y\\ i_x\leq i_y}} \left(\sum_{z\in\alpha_x} |\nabla_1 f|(z)^p M_z^{-1}\right)\left(\sum_{z\in\alpha_x} M_z^{1/(p-1)}\right)^{p-1} \end{aligned}$$

when $p > 1$. If $z \in \Gamma_{s+j}$ for some $j \in \{1, \dots, t-s\}$, then by (11.5) the number of possible choices of x is

$$\preceq \sum_{i=s+j}^{t} j\frac{e^{Q(i-s-j)}}{i-s} \preceq \frac{j}{t-s} e^{Q(t-s-j)}$$

and there are $\leq |B_{s,t}|$ possible choices of y so that $z \in \alpha_x$. Thus

$$M_z \preceq \frac{j}{t-s} e^{Q(t-s-j)} |B_{s,t}| = \frac{j}{e^{Qj}} \cdot \frac{e^{Q(t-s)}|B_{s,t}|}{t-s}.$$

For any $p > 1$, $\sum_{j\geq 1}(je^{-Qj})^{1/(p-1)}$ is bounded by some constant depending only on Q and p. Whether $p > 1$ or $p = 1$, we get that

$$\begin{aligned}
S_1 &\preceq \frac{e^{Q(t-s)}}{t-s}\sum_{\substack{x,y\\ i_x\leq i_y}}\left(\sum_{z\in\alpha_x}|\nabla_1 f|(z)^p M_z^{-1}\right)\\
&= \frac{e^{Q(t-s)}}{t-s}\sum_{z}|\nabla_1 f|(z)^p\left(\sum_{x,y:i_x\leq i_y, z\in\alpha_x} M_z^{-1}\right) = \frac{e^{Q(t-s)}}{t-s}\left\|\nabla_1 f\right\|_p^p.
\end{aligned}$$

A very similar calculation lets us bound S_3: if $z \in \Gamma_{s+j}$ for some $j \in \{1, \ldots, t-s\}$, then by (11.5) there are $\preceq \frac{j}{t-s}|B_{s,t}|$ possible choices of x and $\preceq e^{Q(t-s-j)}$ possible choices of y so that $z \in \beta_{x,y}$. Thus

$$N_z \preceq \frac{j}{t-s}|B_{s,t}| \cdot e^{Q(t-s-j)} = \frac{j}{e^{Q_j}} \cdot \frac{e^{Q(t-s)}|B_{s,t}|}{t-s},$$

and the rest of the calculation goes through as before to give $S_3 \preceq \frac{1}{t-s}e^{Q(t-s)}\|\nabla_1 f\|_p^p$.

It remains to bound S_2. Suppose we have $x', y' \in \Gamma_{s+j}$ for some $j \in \{1, \ldots, t-s\}$. Let $P_{x',y'}$ be the number of pairs $x, y \in B_{s,t}$ so that $i_x \leq i_y$ and $\pi_{c_x}(x) = x'$ and $\pi_{c_x}(y) = y'$. Using again (11.5), we can bound $P_{x',y'}$ by the product of the number of choices of x, which is $\preceq \frac{1}{t-s}e^{Q(t-s-j)}$, and the number of choices of y, which is $\preceq e^{Q(t-s-j)}$. Thus

$$\begin{aligned}
S_2 &= \frac{1}{|B_{s,t}|}\sum_{\substack{x,y\\ i_x\leq i_y}}|f(\pi_{c_x}(x)) - f(\pi_{c_x}(y))|^p\\
&= \frac{1}{|B_{s,t}|}\sum_{j=1}^{t-s}\sum_{x',y'\in\Gamma_{s+j}} P_{x',y'}|f(x') - f(y')|^p\\
&\preceq \frac{e^{2Q(t-s)}}{(t-s)|B_{s,t}|}\sum_{j=1}^{t-s} e^{-2Qj}\sum_{x',y'\in\Gamma_{s+j}}|f(x') - f(y')|^p. \qquad (11.8)
\end{aligned}$$

Fixing for a moment our choice of j, let f_j be the average value of f restricted to Γ_{s+j}. Assuming $Z = (\partial_\infty X, \rho)$ satisfies a p-Poincaré inequality, we apply (11.2) to Γ_{s+j} to obtain:

$$\sum_{x'\in\Gamma_{s+j}}|f(x') - f_j|^p \preceq e^{p(s+j)}\sum_{x'\in\Gamma_{s+j}}|\nabla_C f|(x')^p.$$

Applying this twice, we have that

$$\begin{aligned}\sum_{x',y'\in\Gamma_{s+j}} |f(x')-f(y')|^p &\leq 2^p \sum_{x',y'\in\Gamma_{s+j}} \left(|f(x')-f_j|^p+|f(y')-f_j|^p\right)\\ &\preceq e^{p(s+j)}|\Gamma_{s+j}| \sum_{x'\in\Gamma_{s+j}} |\nabla_C f|(x')^p.\end{aligned}$$

Since $|\Gamma_{s+j}| \asymp e^{Q(s+j)}$, and $|B_{s,t}| \asymp e^{Qt}$, on substituting this back in to (11.8), we get

$$\begin{aligned}S_2 &\preceq \frac{e^{2Q(t-s)}}{(t-s)|B_{s,t}|}\sum_{j=1}^{t-s} e^{-2Qj}e^{p(s+j)}e^{Q(s+j)} \sum_{x'\in\Gamma_{s+j}} |\nabla_C f|(x')^p\\ &\asymp \frac{e^{Qt+(p-Q)s}}{t-s}\sum_{j=1}^{t-s} e^{(p-Q)j} \sum_{x'\in\Gamma_{s+j}} |\nabla_C f|(x')^p.\end{aligned}$$

Provided $p=Q$, this simplifies to

$$S_2 \preceq \frac{e^{Qt}}{t-s}\left\|\nabla_C f\right\|_Q^Q.$$

Our bounds for S_1 and S_3 are dominated by our bounds for S_2, so we set $s=\lfloor t/2\rfloor$ and conclude by (11.7) that

$$\|f-f_{B_{s,t}}\|_Q^Q \preceq \frac{e^{Qt}}{t}\left\|\nabla_C f\right\|_Q^Q \asymp \frac{|B_{s,t}|}{\log|B_{s,t}|}\left\|\nabla_C f\right\|_Q^Q. \qquad \square$$

It remains to show the colouring argument giving (11.5).

Proof of Lemma 11.4. Recall that we are defining a colouring map $B_{s,t} \to \{s+1,\dots,t\}, x\mapsto c_x$.

For each $i\in\{s+1,\dots,t\}$, the vertices of Γ_i correspond to a maximal e^{-i}-separated net in Z. By Ahlfors Q-regularity, there exists C so that the number of e^{-i} separated points in any r-ball in Z is $\leq C(r/e^{-i})^Q = Cr^Qe^{iQ}$. So if we let $r_i=\frac{1}{2}(i-s)^{1/Q}C^{-1/Q}e^{-i}$, we guarantee that any r_i-ball in Z meets at most $(i-s)$ points corresponding to vertices of Γ_i.

Define $\Gamma_i \to \{s+1,\dots,i\}, x\mapsto c_x$ to be any mapping so that no two points at distance $\leq r_i$ in Z are mapped to the same value. The existence of such a mapping follows from Zorn's lemma applied to the collection of all such partially defined functions.

Doing this for each i, we obtain our mapping $B_{s,t}\to\{s+1,\dots,t\}$. To verify that (11.5) holds, observe that for any $z\in B_{s,t}$ and c,i satisfying $c\leq i_z\leq i\leq t$ the set $\{x\in\Gamma_i : \pi_{i_x}(x)=z \text{ and } c_x=i_z\}$ is an r_i-separated set in $B(z,e^{-i_z})\subset Z$, therefore by Ahlfors regularity it has cardinality

$$\preceq \left(\frac{e^{-i_z}}{r_i}\right)^Q \preceq \left(\frac{e^{-i_z}}{(i-s)^{1/Q}e^{-i}}\right)^Q = \frac{e^{Q(i-i_z)}}{(i-s)} \leq \frac{e^{Q(t-s)}}{t-s}. \qquad \square$$

12. Upper bounds for hyperbolic spaces with hyperplanes

In this section we present an approach to finding upper bounds on the L^p-Poincaré profiles of hyperbolic spaces. Our hypotheses are as follows:

(1) (X, d, μ) is a δ-hyperbolic geodesic metric measure space, and it is visual with respect to a given point $x_0 \in X$: there exists $C \geq 0$ so that every $x \in X$ belongs to a C-quasi-geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x_0$.
(2) There exists a constant $h(X) > 0$ (called the volume entropy) and a constant $C \geq 0$ such that for every $R > 0$, $h(X)R - C \leq \log_e(\mu(B_R(x_0))) \leq h(X)R + C$.
(3) There is a visual metric ρ on $\partial_\infty X$ based at $x_0 \in X$ with visibility parameter $\epsilon > 0$; i.e., $\rho(\cdot, \cdot) \asymp \exp(-\epsilon(\cdot|\cdot)_{x_0})$, where $(\cdot|\cdot)_{x_0}$ denotes the Gromov product with respect to x_0.

For our last hypothesis, we require the following notion.

Definition 12.1. Let (X, d) be a metric space and $x_0 \in X$. For $C \geq 1$, a subset $A \subseteq X$ is said to be a **C-asymptotic shadow of** x_0 if, for every $x \in A$ there is a C-quasi-geodesic ray $\gamma_x : [0, \infty) \to X$ with $\gamma_x(0) = x_0$ and $d(\gamma_x(r_x)) = x$ for some r_x, and $\gamma_x[r_x, \infty) \subseteq A$. (Recall that a C-quasi-geodesic ray is a (C, C)-quasi-isometric embedding of $[0, \infty)$.)

The final hypothesis only needs to hold for large a, where $a \geq 2$ is the constant of thickness in Definition 4.1. Let $\mathrm{Isom}_\mu(X)$ be the group of μ-preserving isometries of X.

(4) There exist constants $\kappa, N, C > 0$ such that for any a-thick subspace Z of X with measure at least N, there is some $\psi \in \mathrm{Isom}_\mu(X)$, and there exist two measurable subsets $H^\pm$ of X which are C-asymptotic shadows of x_0, and satisfy the inequalities $\rho(\partial_\infty H^+, \partial_\infty H^-) \geq \kappa$, $\mu(\psi(Z) \cap H^+) \geq \kappa\mu(Z)$ and $\mu(\psi(Z) \cap H^-) \geq \kappa\mu(Z)$.

These properties are satisfied for suitable geometric actions of a hyperbolic group, as we will see in subsection 12.2.

Proposition 12.2. *If G is a non-elementary hyperbolic group which acts geometrically and on a space (X, d, μ) and preserving μ, then for* *any $x_0 \in X$ and visual metric ρ on $\partial_\infty X$ based at x_0 with visual parameter ϵ, (X,d,μ) satisfies properties* (1)–(4) *for suitable δ, C and $h(X)$. Moreover, $(\partial_\infty X, \rho)$ is Ahlfors $h(X)/\epsilon$-regular.*

Properties (1)–(3) are already known to hold in this generality, so our efforts will be focused on property (4). Given these properties, we find the following bounds on the Poincaré profile of X. Note that

Theorem 12.3. *Suppose X satisfies conditions (1)–(4) above for some fixed $\delta, C, \epsilon, \kappa, N$ and set $Q = h(X)/\epsilon$. Then we have the following bounds on Λ^p_X:*

$$\Lambda^p_{X,a}(r) \lesssim_{\delta,C,\kappa,N} \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } p < Q, \\ r^{\frac{p-1}{p}} \log(r)^{\frac{1}{p}} & \text{if } p = Q, \\ r^{\frac{p-1}{p}} & \text{if } p > Q. \end{cases}$$

To find the best bound possible for the Poincaré profiles Λ^p_G of a hyperbolic group G, it is natural to consider the following concept.

Definition 12.4. The **equivariant conformal dimension** of a hyperbolic group G is defined to be the infimum of the Hausdorff dimension of $(\partial_\infty X, \rho)$ where $\partial_\infty X$ is the boundary of a space X on which G acts geometrically and ρ is a visual metric on $\partial_\infty X$. We say the equivariant conformal dimension is attained if the infimum is realised.

Equivalently, we minimise $h(X)/\epsilon$ over all such actions, metrics and permissible visibility parameters.

Using Proposition 12.2 and Theorem 12.3 we are now ready to prove Theorem 11.

Corollary 12.5. *Let G be a hyperbolic group and let Q be its equivariant conformal dimension. Then, for any $\epsilon > 0$,*

$$\Lambda^p_G(r) \lesssim \begin{cases} r^{\frac{Q-1}{Q}+\epsilon} & \text{if} \quad p \leq Q \\ r^{\frac{p-1}{p}} & \text{if} \quad p > Q. \end{cases}$$

If the equivariant conformal dimension is attained, we have:

$$\Lambda^p_G(r) \lesssim \begin{cases} r^{\frac{Q-1}{Q}} & \text{if} \quad 1 \leq p < Q \\ r^{\frac{Q-1}{Q}} \log^{\frac{1}{Q}}(r) & \text{if} \quad p = Q \\ r^{\frac{p-1}{p}} & \text{if} \quad p > Q. \end{cases}$$

12.1. **Helly's theorem and centrepoints.** Inspired by the arguments presented in [BST12, Section 4], we show that finite measure thick subsets of real hyperbolic spaces have "medians". To find a suitable centrepoint of a subset, we use Helly's theorem (cf. [MTTV97]).

The version suitable for our needs is the following variation on a result of Ivanov [Iva14].

Theorem 12.6 (Ivanov)**.** *Let X be a uniquely geodesic space of compact topological dimension $k < \infty$ (for example, a CAT(0) space of geometric dimension k). Let $\mathcal{H}$ be a (possibly infinite) collection of closed convex subsets of X, with the property that there exists a compact convex set $Y \subset X$ so that for any $H_1, \dots, H_{k+1} \in \mathcal{H}$ we have $Y \cap H_1 \cap \cdots \cap H_{k+1} \neq \emptyset$. Then $\bigcap_{H \in \mathcal{H}} H \supset \bigcap_{H \in \mathcal{H}} H \cap Y \neq \emptyset$.*

Proof. If not, then for any $y \in Y$ there exists $H_y \in \mathcal{H}$ with $y \notin H_y$. Since Y is compact, for some $y_1, \dots, y_m$ we have that $\{X \setminus H_{y_i}\}_{i=1,\dots,m}$ is a finite subcover of the open cover $\{X \setminus H_y\}_{y \in Y}$ of Y. By assumption, any $k+1$ of the finite collection of convex sets $\{Y, H_{y_1}, \dots, H_{y_m}\}$ have non-empty intersection (in Y), and so Helly's Theorem [Iva14, Theorem 1.1] implies that there exists $y \in Y \cap H_{y_1} \cap \cdots \cap H_{y_m} \neq \emptyset$. This is a contradiction, since y is not covered by $\{X \setminus H_{y_i}\}_{i=1,\dots,m}$. □

Lemma 12.7. *(Centrepoint theorem) Let $a > 0$. There exists a constant $c = c(k, a) > 0$ such that for any a-thick subset Z of $\mathbb{H}^k_{\mathbb{R}}$ with finite measure, there is a point $x \in \mathbb{H}^k_{\mathbb{R}}$ such that for any half-space H of $\mathbb{H}^k_{\mathbb{R}}$ containing x, we have $\mu(H \cap Z) \geq c\mu(Z)$.*

Proof. By assumption $Z = \bigcup_{i \in I} B(z_i, a)$ for some $\{z_i\}_{i \in I} \subset Z$. Let Z' be an $2a$-separated $4a$-net in $\{z_i : i \in I\}$. It follows that $|Z'| \asymp_a \mu(Z)$ since $|Z'|\, \mu(B(z_i, a)) \leq \mu(Z) \leq |Z'|\, \mu(B(z_i, 5a))$ for some (any) z_i.

Let Y be a large closed (convex) ball containing Z'. Let $\mathcal{Z}$ be the set of all closed half-spaces of $\mathbb{H}^k_{\mathbb{R}}$ containing more than $\frac{k}{k+1}|Z'|$ of the points in Z'. Thus the intersection of any $k+1$ of the sets in $\mathcal{Z}$ has non-empty intersection with Y.

Applying Theorem 12.6, and the fact that $\mathbb{H}^k_{\mathbb{R}}$ has geometric dimension k, there exists some $x \in \bigcap_{H \in \mathcal{Z}} H$. Thus for any half-space $H \subset \mathbb{H}^k_{\mathbb{R}}$ with $|H \cap Z'| > \frac{k}{k+1}|Z'|$ we have $x \in H$. It is a short exercise to see that x is contained in every half-space H such that $|Z' \cap H| > \frac{k}{k+1}|Z'|$ if and only if every half-space H containing x satisfies $|Z' \cap H| > \frac{1}{k+1}|Z'|$.

Let H be a half-space containing x and let $Z'_H = Z' \cap H$. It is clear that $\mu(B(z, r) \cap H) \geq \frac{1}{2}\mu(B(z, r))$ for any $z \in Z'_H$ and any $r \geq 0$, so

$$\mu(Z \cap H) \geq \frac{\mu(B(z, a))}{2(k+1)}|Z'| \asymp_{k,a} \mu(Z). \qquad \square$$

We can use a measure-preserving isometry to move such a centrepoint x to the origin $o \in \mathbb{H}^k_{\mathbb{R}}$ in the Poincaré ball model, and now show that hypothesis (4) of Theorem 12.3 is satisfied for $\mathbb{H}^k_{\mathbb{R}}$.

Lemma 12.8. *There exist constants $\kappa, C > 0$ so that for any a-thick subset $Z \subset \mathbb{H}^k_{\mathbb{R}}$, and $o \in \mathbb{H}^k_{\mathbb{R}}$ a centrepoint of Z, there exist C-asymptotic shadows of o denoted by $H^-, H^+ \subset \mathbb{H}^k_{\mathbb{R}}$ so that we have $\rho(\partial_\infty H^-, \partial_\infty H^+) \geq \kappa$ and that $\mu(Z \cap H^-), \mu(Z \cap H^+) \geq \kappa\mu(Z)$.*

Proof. Fix $a > 0$ and $c = c(k, a) > 0$ the constants from Lemma 12.7.

Let $H \subset \mathbb{H}^k_{\mathbb{R}}$ be a hyperplane containing o, and let $\alpha > 0$. We denote by H^α the union of all two-sided geodesics passing through o and with end points in the α-neighbourhood of the boundary $\partial_\infty H \subset \partial_\infty \mathbb{H}^k_{\mathbb{R}} = \mathbb{S}^{k-1}$.

We start with an argument inspired by the proof of [BST12, Proposition 4.1]. Consider for every $r > 0$ the sphere $S_r = \{x \in \mathbb{H}^k_{\mathbb{R}},\ d(x, o) = r\}$ equipped with its Riemannian measure ν_r. Note that

$$\nu_r(S_r \cap H^\alpha) = \eta(\alpha)\nu_r(S_r)$$

for some increasing function η satisfying $\lim_{\alpha \to 0} \eta(\alpha) = 0$. We now fix $\alpha > 0$ so that $\eta(\alpha) \leq \frac{c}{2}$.

Recall that hyperplanes passing through o are characterized by their normal vector at o, and therefore are parametrized by the projective space P^{k-1}. We consider the Lebesgue probability measure ν on P^{k-1}. Given $\theta \in P^{k-1}$ we define H_θ to be the hyperplane through o with normal vector θ. Recall that $Z \subset \mathbb{H}^k_{\mathbb{R}}$ is a measurable subset of finite measure, so for each r

$$\int_{P^{k-1}} \nu_r(Z \cap H^\alpha_\theta \cap S_r) d\nu(\theta) = \nu_r(Z \cap S_r) \frac{\nu_r(S_r \cap H^\alpha)}{\nu_r(S_r)} = \nu_r(Z \cap S_r)\eta(\alpha).$$

Integrating over r, we deduce that

$$\int_{P^{k-1}} \mu(Z \cap H^\alpha_\theta) d\nu(\theta) = \mu(Z)\eta(\alpha),$$

and so for some hyperplane H_Z we have $\mu(Z \cap H^\alpha_Z) \leq \mu(Z)\eta(\alpha) \leq \frac{c}{2}\mu(Z)$.

Let H^-, H^+ be the two connected components of the complement of H^α_Z; these are convex and asymptotic shadows of o, and satisfy $\mu(H^- \cap Z), \mu(H^+ \cap Z) \geq \frac{c}{2}\mu(Z)$. Moreover, $\rho(\partial_\infty H^-, \partial_\infty H^+) \geq 2\alpha$. ☐

12.2. **Hyperbolic groups and centrepoints.** In this subsection, we prove Proposition 12.2.

Proof of Proposition 12.2. Property (1) follows from a standard argument with the Arzela–Ascoli theorem, see e.g. [BS00]. Property (2) follows from [Coo93, Theorem 7.2], and $(\partial_\infty X, \rho)$ is Ahlfors Q-regular with $Q = \frac{1}{\epsilon}h(X)$. Property (3) is the definition of a visual metric, so it remains only to show that property (4) is satisfied.

We require a probably well-known basic fact about convex hulls of quasi-convex subsets of real hyperbolic spaces. Recall that a subset Y of a geodesic metric space is K-quasi-convex if every geodesic that connect a pair of points of Y lies within the K-neighbourhood of Y. It turns out that in real hyperbolic spaces, quasi-convex subsets are "nearly" convex in a stronger sense:

Lemma 12.9. *Given $K \geq 0$, there exists $N = N(K, k)$ such that for every K-quasi-convex subset $Z \subset \mathbb{H}^k_{\mathbb{R}}$, the convex hull of Z is contained in the N-neighbourhood of Z.*

Proof. Note that in Klein model of $\mathbb{H}^k_{\mathbb{R}}$, the hyperbolic convex hull coincides with the Euclidean one. By Carathéodory's theorem, we deduce that any point of the convex hull of Z is a convex combination of some points $z_1, \ldots, z_m \in Z$, with $m \leq k+1$. Using the quasi-convexity of Z, the lemma follows by induction on m. □

We now show that (4) holds for X. Let X be a δ_X-hyperbolic Cayley graph of the hyperbolic group G. By a result of Bonk–Schramm [BS00], there exists constants $k \in \mathbb{N}, \lambda_\psi \geq 1, C_\psi \geq 0$ and a (λ_ψ, C_ψ)-quasi-isometric embedding $\psi : X \to \mathbb{H}^k_{\mathbb{R}}$. By post-composing ψ with an appropriate element of $\mathrm{Isom}_\mu(\mathbb{H}^k_{\mathbb{R}})$ if necessary, we may assume $\psi(1) = o$, the origin in the Poincaré ball model of $\mathbb{H}^k_{\mathbb{R}}$.

Given a finite subset Y of VX, define $Y' \subset \mathbb{H}^k_{\mathbb{R}}$ to be the closed 2-neighbourhood of $\psi(Y)$. By Lemma 12.7, there is a constant $c = c(k) > 0$ and a point $x' \in \mathbb{H}^k_{\mathbb{R}}$ such that for any half-space H of $\mathbb{H}^k_{\mathbb{R}}$ containing X we have $\mu(H \cap Y') \geq c\mu(Y')$. Such x' is contained in the convex hull of $\psi(Y)$, so by Lemma 12.9, $d_{\mathbb{H}^k_{\mathbb{R}}}(x', \psi(x)) \leq N(k)$ for some $x \in X$. By applying a left-translation in G (by an element g) we may assume $x = 1$, while by applying an isometry $\phi \in \mathrm{Isom}(\mathbb{H}^k_{\mathbb{R}})$, we may assume $x' = o$. Define $f = \phi \circ \psi \circ g^{-1} : X \to \mathbb{H}^k_{\mathbb{R}}$ and let $\partial_\infty f$ be the induced map $\partial_\infty f : \partial_\infty X \to \mathbb{S}^{k-1}$, where $\partial_\infty X$ is endowed with a visual metric ρ based at 1 and $\mathbb{S}^{k-1} = \partial_\infty \mathbb{H}^k_{\mathbb{R}}$ is endowed with the Euclidean (visual) metric ρ_{Euc}.

By Lemma 12.8, there exist constants κ, C and C-asymptotic shadows of o denoted $H^\pm$ so that $\rho_{Euc}(\partial_\infty H^-, \partial_\infty H^+) \geq 4\kappa$ and that $\mu(Y' \cap H^\pm) \geq 4\kappa\mu(Y')$. Since $f(1) = o$, it follows that

$$\rho(\partial_\infty f^{-1}[\partial_\infty H^-]_\kappa, \partial_\infty f^{-1}[\partial_\infty H^+]_\kappa) \geq \kappa'$$

for some $\kappa' > 0$ which does not depend on the choices of ϕ and g used to construct f. (It is not *a priori* obvious that either of $\partial_\infty f^{-1}[\partial_\infty H^\pm]_\kappa$ is non-empty.)

Define $H^{\pm}_X$ to be the set of all points $y \in X \setminus B(1,R)$ contained in the A-neighbourhood of the set of all geodesic rays in X from 1 to a point in $\partial_\infty f^{-1}([\partial_\infty H^{\pm}]_\kappa)$, where A and R are determined below.

We claim that there exist A, R so that if $y \in Y$ satisfies $d_X(1,y) \geq R$ and $B(f(y),2) \cap H^{\pm} \neq \emptyset$, then $y \in H^{\pm}_X$. Let $z \in H^{\pm}$ satisfy $d_{\mathbb{H}^k_{\mathbb{R}}}(z, f(y)) \leq 2$, and let γ be the unique geodesic ray in $\mathbb{H}^k_{\mathbb{R}}$ starting at o and containing z (we assume y is sufficiently far from 1 that $z \neq o$); denote the boundary point of γ by ζ. Since X is C_X-visual for some C_X, there exists a C_X-quasi-geodesic ray β in X from 1 that contains y; denote by η the boundary point of β in $\partial_\infty X$. The Gromov product of ζ and $\partial_\infty f(\eta)$ (relative to o) is bounded from below by $d_{\mathbb{H}^k_{\mathbb{R}}}(o, f(y))$ up to a uniform additive error, so by insisting that $d_X(1,y) \geq R$ is sufficiently large, we may assume that $\rho_{Euc}(\zeta, \partial_\infty f(\eta)) \leq \kappa$, hence $\partial_\infty f(\eta) \in [\partial_\infty H^{\pm}]_\kappa$. By the Morse Lemma, β is contained in a uniform neighbourhood of a geodesic ray from 1 to η, and hence for a suitable choice of A will be contained in $H^{\pm}_X$ outside $B(1,R)$. For these choices of R, A we have that $y \in H^{\pm}_X$ as desired.

From this, and the fact that f is a quasi-isometry with fixed constants, it follows that there exist $\eta, \eta' > 0$ so that $|Y \cap H^{\pm}_X| \geq \eta\mu(Y' \cap H^{\pm}) \geq \eta\kappa\mu(Y') \geq \eta\kappa\eta'|Y|$.

The proof of Proposition 12.2 is complete. □

12.3. **Upper bounds for the Poincaré profile.**

Proof of Theorem 12.3. Let $x_0 \in X$ and $a \geq 2$ be fixed so that (4) holds. Let Z be an a-thick subspace of X of sufficiently large finite measure (to be determined later). Apply (4) to move Z; without loss of generality we may assume that $\psi = id$. Let $H^{\pm}$ be the corresponding C-asymptotic shadows of x_0.

Define $\partial_\infty \phi : (\partial_\infty X, \rho) \to [0,1]$ by

$$\partial_\infty \phi(z) = \min\{1, \max\{0, \tfrac{3}{\kappa}\rho(z, \partial_\infty H^-) - 1\}\};$$

this is a $\frac{3}{\kappa}$-Lipschitz function so that $\partial_\infty \phi$ is zero on $[\partial_\infty H^-]_{\kappa/3}$ and one on $[\partial_\infty H^+]_{\kappa/3}$.

We choose a function $\phi : X \to [0,1]$ by setting $\phi(x) = \partial_\infty \phi(\eta)$ where $\eta \in \partial_\infty X$ is the endpoint of some C-quasi-geodesic $\gamma_x : [0,\infty) \to X$ with $\gamma_x(0) = x_0$ and $\gamma_x(t) = x$ for some t. Regardless of the choices made in defining this function we have the following control: for any $x, y \in X$ with $d(x,y) \leq C'$ there exists $K = K(\delta, C, C', \rho, \kappa)$ so that

$$|\phi(x) - \phi(y)| \leq K \exp(-\epsilon d(x, x_0)). \tag{12.10}$$

By a similar argument, there exists $L > 0$ so that if $d(x, x_0) \geq L$ and $x \in H^-$ then the endpoint η of γ_x used to define $\phi(x)$ satisfies $\rho(\eta, \partial_\infty H^-) \leq \kappa/3$, and so $\phi(x) = 0$. Likewise, if $x \in H^+$ and $d(x, x_0) \geq L$ then $\phi(x) = 1$.

By assuming that $\mu(Z)$ is greater than $\frac{2}{\kappa}\mu(B(x_0, L))$, we know—by assumption (4)—that $\mu(Z \cap H^- \setminus B(x_0, L))$ and $\mu(Z \cap H^+ \setminus B(x_0, L))$ are both $\geq \frac{\kappa}{2}\mu(Z)$. Switching the roles of $H^\pm$ if necessary, we assume $\phi_Z \geq 1/2$ and so

$$||\phi - \phi_Z||^p_{Z,p} \geq |\phi_Z|^p \mu(Z \cap H^- \setminus B(x_0, L)) \geq 2^{-p-1}\kappa\mu(Z). \tag{12.11}$$

We now bound $||\nabla_a \phi||_{B,p}$ on the ball $B = B(x_0, r)$. Since we have $\mu(B(x_0, R)) \asymp \exp(h(X)R)$, (12.10) gives

$$||\nabla_a \phi||^p_{B,p} \preceq_{K,\kappa,p} \int_{t=0}^{r} \exp(h(X)t)\exp(-p\epsilon t)dt. \tag{12.12}$$

We now consider the three cases for p separately. (Recall that $h(X) = \epsilon Q$.)

Case 1, $p > Q$: Equation (12.12) gives that $||\nabla_a \phi||^p_{X,p}$ is bounded by some constant D only depending on K, κ and p, so (12.11) gives $h^p_a(Z) \preceq_{K,\kappa,p} \mu(Z)^{-1/p}$ for any subspace Z and the case $p > Q$ follows.

Case 2, $p < Q$: The function ϕ is no longer a p-Dirichlet, but its gradient is well-behaved. Indeed, (12.10) gives $|\nabla_a \phi|(x) \preceq \exp(-\epsilon d(x, x_0))$, so

$$\|\nabla_a \phi\|^p_{B,p} \preceq \| \exp(-\epsilon d(\cdot, x_0))\|^p_{Z,p}. \tag{12.13}$$

Now we wish to put an upper bound on $||\nabla_a \phi||^p_{Z,p} \,/\, ||\phi - \phi_Z||^p_{Z,p}$, which by (12.11) is bounded by $||\nabla_a \phi||^p_{Z,p} / \mu(Z)$ up to a uniform multiplicative error. Thus by (12.13) it suffices to maximize $||\exp(-\epsilon d(\cdot, x_0))||^p_{Z,p}$ among all sets Z with the same measure.

But, up to a uniform multiplicative error, $||\exp(-\epsilon d(\cdot, x_0))||^p_{Z,p}$ is maximised when $d(\cdot, x_0)$ is minimised as a function from Z to $\mathbb{R}$. Clearly this occurs when Z is a metric ball centred at x_0.

By (12.12), for $Z = B(x_0, r)$,

$$||\nabla_a \phi||^p_{Z,p} \preceq \exp(h(X)r) \cdot \exp(-p\epsilon r) \asymp \mu(Z) \cdot \mu(Z)^{-p/Q}, \tag{12.14}$$

thus $h^p_a(Z) \preceq \mu(Z)^{-1/Q}$ and the bound on $\Lambda^p_{X,a}(\mu(Z))$ follows.

Case 3, $p = Q$: If $p = Q$ then the same argument as in Case 2 shows that inequality (12.14) is maximised for a metric ball, so

$$||\nabla_a \phi||^p_{Z,p} \preceq \int_{t=0}^{r} \exp(h(X)t)\exp(-p\epsilon t)dt = r \asymp \log(\mu(Z)),$$

so $h^p_a(Z) \preceq \log(\mu(Z))^{1/p} \cdot \mu(Z)^{-1/p}$ and thus the bound on Λ^p_X for $p = Q$ follows. $\square$

13. Applications to buildings and symmetric spaces

We use results from Sections 11 and 12 to calculate Poincaré profiles of buildings and rank-one symmetric spaces (Theorem 12).

Bourdon and Pajot [BP99] showed that a family of Fuchsian buildings earlier studied by Bourdon [Bou97] have boundaries that admit 1-Poincaré inequalities.

Definition 13.1. Let $m \geq 5, n \geq 3$ be given. Let R be the regular, right-angled hyperbolic polygon with m sides. Let $I = I_{m,n}$ be the Fuchsian building where the chambers are isometric to R, each edge is adjacent to n copies of R, and the vertex links are copies of the complete bipartite graph with n, n vertices.

The group

$$G_{m,n} = \langle s_1, \ldots, s_m \mid s_i^n, [s_i, s_{i+1}] \, \forall i \rangle,$$

where indices are modulo m, acts cellularly and geometrically on $I_{m,n}$. By [BP99, Theorem 1.1], $\partial_\infty G_{m,n} = \partial_\infty I_{m,n}$ carries an Ahlfors $Q_{m,n}$-regular metric, where $Q_{m,n} = 1 + \log(n-1)/\mathrm{arccosh}((m-2)/m) \in (1, \infty)$, and which admits a 1-Poincaré inequality in the sense of Heinonen–Koskela (Section 11).

The apartments in $I_{m,n}$ are each copies of the hyperbolic plane tiled by right-angled regular m-gons. As such, they have separation at least $\log(r)$; the boundary geometry lets us find much larger lower bounds.

Theorem 13.2. *Given $m \geq 5, n \geq 3$, and $p \in [1, \infty)$,*

$$\Lambda^p_{I_{m,n}}(r) \simeq_p \begin{cases} r^{1-1/Q_{m,n}} & \text{if } p < Q_{m,n} \\ r^{1-1/Q_{m,n}} \log(r)^{1/Q_{m,n}} & \text{if } p = Q_{m,n} \\ r^{1-1/p} & \text{if } p > Q_{m,n}. \end{cases}$$

Proof. The lower bounds follow from Theorem 11.1 for $p < Q_{m,n}$, Theorem 11.3 for $p = Q_{m,n}$ and Corollary 10.2 for $p \geq Q_{m,n}$. The upper bounds follow from Corollary 12.5. $\square$

Finally, we calculate the Poincaré profiles of rank-one symmetric spaces. The case of $p = 1$ for $\mathbb{H}^k_{\mathbb{R}}$ is dealt with by [BST12, Proposition 4.1] and Proposition 6.5, but all other cases are new.

Theorem 13.3. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be a real division algebra, and let $X = \mathbb{H}^m_{\mathbb{K}}$ be a rank-one symmetric space for $m \geq 2$ (and $m = 2$*

when $\mathbb{K} = \mathbb{O}$*). Let* $Q = (m+1)\dim_{\mathbb{R}} \mathbb{K} - 2$*, then*

$$\Lambda^p_{\mathbb{H}^m_{\mathbb{K}}}(r) \simeq \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } p < Q \\ r^{\frac{Q-1}{Q}} \log(r)^{\frac{1}{Q}} & \text{if } p = Q \\ r^{\frac{p-1}{p}} & \text{if } p > Q \end{cases}$$

Proof. The boundary of a rank-one symmetric space carries a visual metric that is Ahlfors Q-regular for the given exponent, and satisfies a 1-Poincaré inequality. The result then follows from Theorem 11.1, Theorem 11.3, Corollary 10.2, and Theorem 12.3. □

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Mathematical Institute, University of Oxford, Oxford, OX2 6GG.
E-mail address: `david.hume@maths.ox.ac.uk`

School of Mathematics, University of Bristol, Bristol, BS8 1TX.
E-mail address: `john.mackay@bristol.ac.uk`

Université Paris-Sud, Orsay, France.
E-mail address: `romain.tessera@math.u-psud.fr`