A METRIZABLE TOPOLOGY ON THE CONTRACTING BOUNDARY OF A GROUP

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Abstract. The 'contracting boundary' of a proper geodesic metric space consists of equivalence classes of geodesic rays that behave like rays in a hyperbolic space. We introduce a geometrically relevant, quasi-isometry invariant topology on the contracting boundary. When the space is the Cayley graph of a finitely generated group we show that our new topology is metrizable.

1. Introduction

Boundary methods have enjoyed phenomenal success in the realm of (Gromov) hyperbolic groups and spaces. This has lead to development of boundary theories for other classes of spaces, most notably those that enjoy some measure of hyperbolic-like behavior, such as visual boundaries for CAT(0) spaces. However, unlike the hyperbolic case, quasi-isometric CAT(0) spaces do not necessarily have homeomorphic visual boundaries, and indeed the same group can act geometrically on CAT(0) spaces with non-homeomorphic visual boundaries [10]. Thus, it is not possible to unambiguously assign a visual boundary to a CAT(0) group.

Charney and Sultan [8] sought to rectify this problem by defining a 'contracting boundary' for CAT(0) spaces. Hyperbolic boundaries and visual boundaries of CAT(0) spaces can be constructed as equivalence classes of geodesic rays emanating from a fixed basepoint. These represent the metrically distinct ways of 'going to infinity'. Charney and Sultan's idea was to restrict attention to ways of going to infinity in hyperbolic directions: They consider equivalence classes of geodesic rays that are 'contracting', which is a way of quantifying how hyperbolic such rays are. They topologize the resulting set using a direct limit construction, and show that this topology is preserved by quasi-isometries. However, their construction has some drawbacks: basically, it has too many open sets. In general it is not even first countable. This makes it hard for sequences in the boundary to converge, even when geometric intuition suggests they ought to.

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In this paper we define a contracting boundary for proper geodesic metric spaces and endow it with a quasi-isometry invariant topology. When the space is a Cayley graph of a finitely generated group, we prove that the topology is metrizable, which is a significant improvement over the direct limit topology. Furthermore, our topology more closely resembles the topology of the boundary of a hyperbolic space, which we hope will make it easier to work with.

Our contracting boundary will consist of equivalence classes of 'contracting quasi-geodesics'. The definition of contraction we use follows that of Arzhantseva, Cashen, Gruber, and Hume [2]; this is weaker than that of Charney and Sultan, so our construction applies to more general spaces. For example, we get contracting quasi-geodesics from cyclic subgroups generated by non-peripheral elements of relatively hyperbolic groups [14, 20], pseudo-Anosov elements of mapping class groups [16, 13], fully irreducible free group automorphisms [1], and generalized loxodromic elements of acylindrically hyperbolic groups [19, 11, 4, 21]. On CAT(0) spaces the two definitions agree, so our boundary is the same as theirs *as a set*, but our topology is coarser. Cordes [9] has defined a 'Morse boundary' for proper geodesic metric spaces by applying Charney and Sultan's direct limit construction to the set of equivalence classes of Morse geodesic rays. It turns out that our notion of contracting geodesic is equivalent to the Morse condition, and our contracting boundary agrees with the Morse boundary as a set, but, again, our topology is coarser. If the underlying space is hyperbolic then all of these boundaries are homeomorphic to the Gromov boundary. At the other extreme, all of these boundaries are empty in spaces with no hyperbolic directions. In particular, it follows from work of Drutu and Sapir [12] that groups that are *wide*, that is, no asymptotic cone contains a cut point, will have empty contracting boundary. This includes groups satisfying a law: for instance, solvable groups or bounded torsion groups.

The boundary of a proper hyperbolic space can be topologized as follows. If ζ is a point in the boundary, an equivalence class of geodesic rays issuing from the chosen basepoint, we declare a small neighborhood of ζ to consist of boundary points η such that if $\alpha \in \zeta$ and $\beta \in \eta$ are representative geodesic rays then β closely fellow-travels α for a long time. In proving that this topology is invariant under quasi-isometries, hyperbolicity is used at two key points. The first is that quasi-isometries take geodesic rays uniformly close to geodesic rays. In general a quasi-isometry only takes a geodesic ray to a quasi-geodesic ray, but hyperbolicity implies that this is within bounded distance of a geodesic ray, with bound depending only on the quasi-isometry and hyperbolicity constants. The second use of hyperbolicity is to draw a clear distinction between fellow-travelling and not, which is used to show that the time for which two geodesics fellow-travel is roughly preserved by quasi-isometries. If α and β are non-asymptotic geodesic rays issuing from a common basepoint in a hyperbolic space, then closest point projection sends β to a bounded subset $\alpha([0, T_0])$ of α, and there is a transition in the behavior of β at time T_0. For $t < T_0$ the distance from $\beta(t)$ to α is bounded and the diameter of the projection of $\beta([0, t])$ to α grows like t. After this time β escapes *quickly* from α, that is, $d(\beta(t), \alpha)$ grows like $t - T_0$, and the diameter of the projection of $\beta([T_0, t])$ is bounded.

We recover the second point for non-hyperbolic spaces using the contraction property. Our definition of a *contracting* set Z, see Definition 3.2, is that the diameter of the projection of a ball tangent to Z is bounded by a sublinear function of the radius of the ball. Essentially this means that sets far from Z have large diameter compared to the diameter of their projection. In contrast to the hyperbolic case, it is not true, in general, that if α is a contracting geodesic ray and β is a geodesic ray not asymptotic to α then β has bounded projection to α. However, we *can* still characterize the escape of β from α by the relation between the growth of the projection of $\beta([0,t])$ to α and the distance from $\beta(t)$ to α. The main technical tool we introduce is a divagation estimate that says if α is contracting and β is a quasi-geodesic then β cannot wander slowly away from α; if it is to escape, it must do so quickly. More precisely, once β exceeds a threshold distance from α, depending on the quasi-geodesic constants of β and the contraction function for α, then the distance from $\beta(t)$ to α grows superlinearly compared to the growth of the projection of $\beta([0,t])$ to α. In fact, for the purpose of proving that fellow-travelling time is roughly preserved by quasi-isometries it will be enough to know that the this relationship is at least a fixed linear function.

The first point cannot be recovered, and, in fact, the topology as described above is not quasi-isometry invariant for non-hyperbolic spaces [7]. Instead, we introduce a finer topology that we call the *topology of fellow-travelling quasi-geodesics*. The idea is that η is close to ζ if all *quasi*-geodesics tending to η closely fellow-travel quasi-geodesics tending to ζ for a long time. See Definition 5.3 for a precise definition. Using our divagation estimates we show that this topology is quasi-isometry invariant.

The use of quasi-geodesic rays in our definition is quite natural in the setting of coarse geometry, since then the rays under consideration do not depend on the choice of a particular metric within a fixed quasi-isometry class. Geodesics, on the other hand, are highly sensitive to the choice of metric, and it is only the presence of a very strong hypothesis like global hyperbolicity that allows us to define a quasi-isometry invariant boundary topology using geodesics alone.

After some preliminaries in Section 2, we define the contraction property and recall/prove some basic technical results in Section 3 concerning the behavior of geodesics relative to contracting sets. In Section 4 we extend these results to quasi-geodesics, and derive the key divagation estimates, see Corollary 4.3 and Lemma 4.5.

In Section 5 introduce the topology of fellow-travelling quasi-geodesics and show that it is first countable, Hausdorff, and regular. In Section 6 we prove that it is also quasi-isometry invariant.

We compare other possible topologies in Section 7.

In Section 8 we consider the case of a finitely generated group. In this case we prove that the contracting boundary is second countable, hence metrizable.

We also prove a weak version of North-South dynamics for the action of a group on its contracting boundary in Section 9, in the spirit of Murray's work [17].

2. Preliminaries

Let X be a metric space with metric d. For $Z \subset X$, define:

- $N_r Z := \{x \in X \mid \exists z \in Z,\, d(z,x) < r\}$
- $N_r^c Z := \{x \in X \mid \forall z \in Z,\, d(z,x) \geq r\}$
- $\bar{N}_r Z := \{x \in X \mid \exists z \in Z,\, d(z,x) \leq r\}$
- $\bar{N}_r^c Z := \{x \in X \mid \forall z \in Z,\, d(z,x) > r\}$

For $L \geq 1$ and $A \geq 0$, a map $\phi\colon (X, d_X) \to (Y, d_Y)$ is an (L, A)–*quasi-isometric embedding* if for all x, $x' \in X$:

$$\frac{1}{L} d_X(x, x') - A \leq d_Y(\phi(x), \phi(x')) \leq L d_X(x, x') + A$$

If, in addition, $\bar{N}_A \phi(X) = Y$ then ϕ is an (L, A)–*quasi-isometry*. A *quasi-isometry inverse* $\bar{\phi}$ of a quasi-isometry $\phi\colon X \to Y$ is a quasi-isometry $\bar{\phi}\colon Y \to X$ such that the compositions $\phi \circ \bar{\phi}$ and $\bar{\phi} \circ \phi$ are both bounded distance from the identity map on the respective space.

A *geodesic* is an isometric embedding of an interval. A *quasi-geodesic* is a quasi-geodesic embedding of an interval. If $\alpha\colon I \to X$ is a quasi-geodesic, we often use α_t to denote $\alpha(t)$, and conflate α with its image in X. We use $\alpha + \beta$ and $\bar{\alpha}$ to denote concatenation and reversal, respectively.

A metric space is *geodesic* if every pair of points can be connected by a geodesic.

A metric space if *proper* if closed balls are compact.

It is often convenient to improve quasi-geodesics to be continuous, which can be accomplished by the following lemma.

Lemma 2.1 (Taming quasi-geodesics [6, Lemma III.H.1.11])**.** *If X is a geodesic metric space and $\gamma\colon [a, b] \to X$ is an (L, A)–quasi-geodesic then there exists a continuous $(L, 2(L + A))$–quasi-geodesic γ' such that $\gamma_a = \gamma'_a$, $\gamma_b = \gamma'_b$ and the Hausdorff distance between γ and γ' is at most $L + A$.*

Proof. Define γ' to agree with γ at the endpoints and at integer points of $[a, b]$, and then connect the dots by geodesic interpolation. □

A subspace Z of a geodesic metric space X is A–*quasi-convex* for some $A \geq 0$ if every geodesic connecting points in Z is contained in $\bar{N}_A Z$.

A subspace Z of a metric space X is μ–*Morse* for some $\mu : [1, \infty) \times [0, \infty) \to \mathbb{R}$ if for every $L \geq 1$ and every $A \geq 0$, every (L, A)–quasi-geodesic with endpoints in Z is contained in $\bar{N}_{\mu(L,A)} Z$.

If f and g are functions then we say $f \preceq g$ if there exists a constant $C > 0$ such that $f(x) \leq C g(Cx + c) + C$ for all x. If $f \preceq g$ and $g \preceq f$ then we write $f \asymp g$.

3. Contraction

Definition 3.1. We call a function ρ *sublinear* if it is non-decreasing, eventually non-negative, and $\lim_{r \to \infty} \rho(r)/r = 0$.

Definition 3.2. Let X be a proper geodesic metric space. Let Z be a closed subset of X, and let $\pi_Z\colon X \to 2^Z : x \mapsto \{z \in Z \mid d(x, z) = d(x, Z)\}$ be closest point projection to Z. Then, for a sublinear function ρ, we say that Z is ρ–*contracting* if for all x and y in X:

$$d(x, y) \leq d(x, Z) \implies \operatorname{diam} \pi_Z(x) \cup \pi_Z(y) \leq \rho(d(x, Z))$$

We say Z is *contracting* if there exists a sublinear function ρ such that Z is ρ–contracting.

We shorten π_Z to π when Z is clear from context.

Let us stress that the closest point projection map is set-valued, and there is no bound on the diameter of image sets other than that implied by the definition.

In a tree every convex subset is ρ–contracting where ρ is identically 0. More generally, in a hyperbolic space a set is contracting if and only if it is quasi-convex.

In fact, in this case more is true: the contraction function is bounded in terms of the hyperbolicity and quasi-convexity constants. We call a set *strongly contracting* if it is contracting with bounded contraction function.

The concept of strong contraction (sometimes simply called 'contraction' in the literature) has been studied before, notably by Minsky [16] to describe axes of pseudo-Anosov mapping classes in Teichmüller space, by Bestvina and Fujiwara [5] to describe axes of rank-one isometries of CAT(0) spaces (see also Sultan [22]), and by Algom-Kfir [1] to describe axes of fully irreducible free group automorphisms acting on Outer Space.

The more general Definition 3.2 was introduced by Arzhantseva, Cashen, Gruber, and Hume to characterize Morse geodesics in small cancellation groups [3]. It turns out that the contraction property is equivalent to the Morse property in arbitrary geodesic metric spaces [2].

We now recall some further results about contracting sets in a geodesic metric space X.

Lemma 3.3 ([2, Lemma 6.3])**.** *Given a sublinear function ρ and a constant $C \geq 0$ there exists a sublinear function $\rho' \asymp \rho$ such that if $Z \subset X$ and $Z' \subset X$ have Hausdorff distance at most C and Z is ρ–contracting then Z' is ρ'–contracting.*

Theorem 3.4 (Geodesic Image Theorem [2, Theorem 7.1])**.** *For $Z \subset X$, there exists a sublinear function ρ so that Z is ρ–contracting if and only if there exists a sublinear function ρ' and a constant κ_ρ so that for every geodesic segment γ, with endpoints denoted x and y, if $d(\gamma, Z) \geq \kappa_\rho$ then $\operatorname{diam} \pi(\gamma) \leq \rho'(\max\{d(x,Z),\, d(y,Z)\})$. Moreover ρ' and κ_ρ depend only on ρ and vice-versa, with $\rho' \asymp \rho$.*

An easy consequence is that there exists a κ'_ρ such that if γ is a geodesic segment with endpoints at distance at most κ_ρ from a ρ–contracting set Z then $\gamma \subset \bar{N}_{\kappa'_\rho}(Z)$.

The following is a special case of [2, Proposition 8.1].

Lemma 3.5. *Given a sublinear function ρ and a constant $C \geq 0$ there exists a constant B such that if α and β are ρ–contracting geodesics such that their initial points α_0 and β_0 satisfy $d(\alpha_0, \beta_0) = d(\alpha, \beta) \leq C$ then $\alpha \cup \beta$ is B–quasi-convex.*

The next two lemmas are easy-to-state generalizations of results that are known for strong contraction. The proofs are rather tedious, due to the weak hypotheses, so we postpone them until after Lemma 3.8.

Lemma 3.6. *Given a sublinear function ρ there is a sublinear function $\rho' \asymp \rho$ such that every subsegment of a ρ–contracting geodesic is ρ'–contracting.*

Lemma 3.7. *Given a sublinear function ρ there is a sublinear function $\rho' \asymp \rho$ such that if α and β are ρ–contracting geodesic rays or segments such that $\gamma := \bar{\alpha} + \beta$ is geodesic, then γ is ρ'–contracting.*

Given $C \geq 0$ a *geodesic C–almost triangle* is a trio of geodesic segments α^0, α^1 and α^2 such that the terminal endpoint of α^i is distance at most C from the initial endpoint of α^{i+1} for each $i \in \{0, 1, 2\}$, with superscripts taken modulo 3.

Lemma 3.8. *Given a sublinear function ρ and constant $C \geq 0$ there is a sublinear function $\rho' \asymp \rho$ such that if α, β, and γ are a geodesic C–almost triangle and α and β are ρ–contracting then γ is ρ'–contracting.*

Proof. By Lemma 3.5, there exists a B depending only on ρ and C such that $\alpha \cup \beta$ is B–quasi–convex. Thus, we can replace $\alpha \cup \beta$ by a single geodesic segment δ whose endpoints are C–close to the endpoints of γ. Furthermore, δ is a union of two subsegments, one of which has endpoints within distance B of α, and the other of which has endpoints within distance B of β. Consequently, by Theorem 3.4 there exists B' so that these two subsegments are B'–Hausdorff equivalent to subsegments of α and of β, respectively. Applying Lemma 3.6, Lemma 3.3, and Lemma 3.7, there is a $\rho'' \asymp \rho$ depending on ρ and B' such that δ is ρ''–contracting. Theorem 3.4 implies that since γ and δ are close at their endpoints, they stay close along their entire lengths, so their Hausdorff distance is determined by ρ'' and C, hence by ρ and C. Applying Lemma 3.3 again, we conclude γ is ρ'–contracting with $\rho' \asymp \rho'' \asymp \rho$ depending only on ρ and C. □

Definition 3.9. If Z is a subset of $\mathbb{R}$ define the *interval of* Z, $\mathrm{invl}(Z)$, to be the smallest closed interval containing Z. If $\gamma\colon I \to X$ is a geodesic and Z is a subset of γ let $\mathrm{invl}(Z) := \gamma(\mathrm{invl}(\gamma^{-1}(Z)))$.

Proof of Lemma 3.6. Let $\gamma\colon I \to X$ be a ρ–contracting geodesic. Let $J := [j_0, j_1]$ be a subinterval of I. Let $\rho'' \asymp \rho$ be the function given by Theorem 3.4, and let κ'_ρ be the constant defined there. We claim it suffices to take $\rho'(r) := 2(2\kappa'_\rho + \rho''(2r) + \rho(2r))$.

First we show that if $\pi_{\gamma_I}(x)$ misses γ_J then $\pi_{\gamma_J}(x)$ is relatively close to one of the endpoints of γ_J. This is automatic if $\operatorname{diam} \gamma_J \leq \rho(d(x, \gamma_J))$, so assume not. With this assumption, $\pi_{\gamma_I}(x)$ cannot contain points on both sides of γ_J, that is, if $\gamma^{-1}(\pi_{\gamma_I}(x))$ contains a point less than j_0 then it does not also contain one greater than j_1, and vice versa. Suppose that $\gamma^{-1}(\pi_{\gamma_I}(x))$ is contained in $(-\infty, j_0)$. Let β be a geodesic from x to a point y in $\pi_{\gamma_J}(x)$. There exists a first time s such that $d(\beta_s, \gamma_I) = \kappa_\rho$. By Theorem 3.4, $\operatorname{diam} \pi_{\gamma_I}(\beta|_{[0,s]}) \leq \rho''(d(x, \gamma_I))$. Suppose that $\gamma_{j_0} \in \mathrm{invl}(\pi_{\gamma_I}(\beta|_{[0,s]}))$. Then there is a first time $s' \in [0, s]$, such that $\pi_I(\beta_{s'})$ contains a point in $\gamma_{[j_0,\infty)}$. By the assumption on the diameter of γ_J, we actually have $\pi_{\gamma_I}(\beta_{s'}) \cap \gamma_J \neq \emptyset$, so $y \in \pi_{\gamma_J}(\beta_{s'}) \subset \pi_{\gamma_I}(\beta_{s'}) \subset \pi_{\gamma_I}(\beta|_{[0,s]})$ and $\operatorname{diam} \gamma_{j_0} \cup \pi_{\gamma_J}(x) \leq \rho''(d(x, \gamma_I))$. Otherwise, if $\gamma_{j_0} \notin \mathrm{invl}(\pi_{\gamma_I}(\beta|_{[0,s]}))$, then let $t > s$ be the first time such that $\gamma_{j_0} \in \mathrm{invl}\, \pi_{\gamma_I}(\beta|_{[s,t]})$. Again, $y \in \pi_{\gamma_I}(\beta_t)$. Since the points of β after β_s, are contained in $\bar{N}_{\kappa'_\rho}\gamma$, for all small $E > 0$ we have $\operatorname{diam} \pi_{\gamma_I}\beta_{t-E} \cup \pi_{\gamma_I}\beta_t \leq E + 2\kappa'_\rho$. Therefore, $d(\gamma_{j_0}, y) \leq d(y, \pi_{\gamma_I}(\beta_{t-E})) \leq E + 2\kappa'_\rho$, for all sufficiently small E. We conclude:

$$\operatorname{diam} \gamma_{j_0} \cup \pi_{\gamma_J}(x) \leq \max\{2\kappa'_\rho, \rho''(d(x, \gamma_I))\} \tag{1}$$

Now suppose x and y are points such that $d(x, y) \leq d(x, \gamma_J)$. Note that $d(y, \gamma_J) \leq 2d(x, \gamma_J)$. We must show $\operatorname{diam} \pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y)$ is bounded by a sublinear function of $d(x, \gamma_J)$. There are several cases, depending on whether $\pi_{\gamma_I}(x)$ and $\pi_{\gamma_I}(y)$ hit γ_J.

Case 1: $\gamma_J \cap \pi_{\gamma_I}(x) \neq \emptyset$ and $\gamma_J \cap \pi_{\gamma_I}(y) \neq \emptyset$. In this case $\pi_{\gamma_J}(x) \subset \pi_{\gamma_I}(x)$, and likewise for y, so:

$$\operatorname{diam} \pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y) \leq \operatorname{diam} \pi_{\gamma_I}(x) \cup \pi_{\gamma_I}(y) \leq \rho(d(x, \gamma_I)) = \rho(d(x, \gamma_J))$$

Case 2: $\gamma^{-1}(\pi_{\gamma_I}(x)) < j_0$ and $\gamma^{-1}(\pi_{\gamma_I}(y)) < j_0$. By (1) twice:

$$\begin{aligned} \operatorname{diam} \pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y) &\leq 2 \max\{2\kappa'_\rho, \rho''(2d(x, \gamma_I))\} \\ &\leq 2 \max\{2\kappa'_\rho, \rho''(2d(x, \gamma_J))\} \end{aligned}$$

Case 3: $\gamma^{-1}(\pi_{\gamma_I}(y)) < j_0$ and $\gamma_J \cap \pi_{\gamma_I}(x) \neq \emptyset$. In this case $\pi_{\gamma_J}(x) \subset \pi_{\gamma_I}(x)$ and $d(x,y) \leq d(x,\gamma_J) = d(x,\gamma_I)$, so $\operatorname{diam}\pi_{\gamma_I}(y) \cup \pi_{\gamma_J}(x) \leq \rho(d(x,\gamma_J))$. By hypothesis, $\gamma_{j_0} \in \operatorname{invl}(\pi_{\gamma_I}(y) \cup \pi_{\gamma_J}(x))$, and by (1): $\operatorname{diam}\gamma_{j_0} \cup \pi_{\gamma_J}(y) \leq \max\{2\kappa'_\rho, \rho''(2d(x,\gamma_J))\}$. Thus:

$$\operatorname{diam}\pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y) \leq \max\{\rho(d(x,\gamma_J)), 2\kappa'_\rho, \rho''(2d(x,\gamma_J))\}$$

Case 4: $\gamma^{-1}(\pi_{\gamma_I}(x)) < j_0$ and $\pi_{\gamma_I}(y) \cap \gamma_{[j_0,\infty)} \neq \emptyset$. If $j_1 - j_0 \leq 2\rho(2d(x,\gamma_J))$ then there is nothing more to prove, so assume not. Let β be a geodesic from x to y. For all $z \in \beta$:

$$d(x,z) + d(z,y) = d(x,y) \leq d(x,\gamma_J) \leq d(x,z) + d(z,\gamma_J)$$

This implies $d(z,y) \leq d(z,\gamma_J)$. Let z be the first point on β such that $\gamma^{-1}(\pi_{\gamma_I}(z))$ contains a point greater than or equal to j_0. By the hypothesis on $|J|$, $\gamma^{-1}(\pi_{\gamma_I}(z)) < j_1$. This means $\operatorname{diam}\pi_{\gamma_J}(z) \cup \pi_{\gamma_J}(y)$ is controlled by one of the previous cases, and it suffices to control $\operatorname{diam}\pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(z)$. Thus, we may assume that $\gamma^{-1}(\pi_{\gamma_I}(\beta \setminus \{y\})) < j_0$ and $\pi_{\gamma_J}(y) \subset \pi_{\gamma_I}(y)$.

We know from (1) that $\pi_{\gamma_J}(x)$ is $\max\{2\kappa'_\rho, \rho''(d(x,\gamma_J))\}$-close to γ_{j_0}, so it suffices to control $\operatorname{diam}\gamma_{j_0} \cup \pi_{\gamma_J}(y)$. Take a point $z \neq y$ on β such that $d(y,z) \leq d(y,\gamma_I)$. By hypothesis, $\gamma_{j_0} \in \operatorname{invl}\pi_{\gamma_I}(z) \cup \pi_{\gamma_I}(y)$, but $\operatorname{diam}\pi_{\gamma_I}(z) \cup \pi_{\gamma_I}(y) \leq \rho(d(y,\gamma_I)) = \rho(d(y,\gamma_J)) \leq \rho(2d(x,\gamma_J))$.

Up to symmetric arguments, this exhausts all the cases. □

Proof of Lemma 3.7. Let α and β be ρ–contracting geodesic segments or rays with $\alpha_0 = \beta_0$ such that $\gamma := \bar{\alpha} + \beta$ is geodesic.

First suppose that x is a point such that $\pi_\gamma(x) \cap \alpha \neq \emptyset$ and $\pi_\gamma(x) \cap \beta \neq \emptyset$. Let δ be a geodesic from x to $\alpha_0 = \beta_0$. Recall from Theorem 3.4 that once δ enters the κ_ρ–neighborhood of either α or β then it cannot leave the κ'_ρ–neighborhood. Thus, δ intersects at most one of $\bar{N}_{\kappa_\rho}\alpha \setminus N_{2\kappa'_\rho}\alpha_0$ or $\bar{N}_{\kappa_\rho}\beta \setminus N_{2\kappa'_\rho}\beta_0$. Without loss of generality, suppose δ does not intersect $\bar{N}_{\kappa_\rho}\beta \setminus N_{2\kappa'_\rho}\beta_0$. Let t be the first time such that $d(\delta_t, \beta) = \kappa_\rho$. Then $d(\delta_t, \beta_0) \leq 2\kappa'_\rho$ and, by Theorem 3.4, there is a sublinear $\rho'' \asymp \rho$ such that $\operatorname{diam}\pi_\beta(\delta|_{[0,t]}) \leq \rho''(d(x,\beta)) = \rho''(d(x,\gamma))$. In particular, this means $\operatorname{diam}\pi_\beta(x) \cup \beta_0 \leq \operatorname{diam}\pi_\beta(\delta) \leq 4\kappa'_\rho + \rho''(d(x,\gamma))$. Now let δ' be a geodesic from x to a point $x' \in \pi_\beta(x)$, and project δ' to α. Since $\bar{\alpha} + \beta$ is geodesic, $\operatorname{diam}\pi_\alpha(x) \cup \alpha_0 \leq \operatorname{diam}\pi_\alpha\delta' \leq \rho''(\max\{d(x,\alpha), d(x',\alpha)\})$ by Theorem 3.4. We have already established that $d(x',\alpha) \leq 4\kappa'_\rho + \rho''(d(x,\gamma))$. Since, ρ'' grows sublinearly, $d(x,\alpha) > 4\kappa'_\rho + \rho''(d(x,\gamma))$ except for $d(x,\alpha)$ less than some bound depending only on ρ and ρ''. We conclude that there is a sublinear function $\rho''' \asymp \rho$ depending only on ρ such that $\operatorname{diam}\pi_\alpha(x) \cup \{\alpha_0\} \leq \rho'''(d(x,\gamma))$ and $\operatorname{diam}\pi_\beta(x) \cup \{\beta_0\} \leq \rho'''(d(x,\gamma))$, hence $\operatorname{diam}\pi_\gamma(x) \leq 2\rho'''(d(x,\gamma))$.

Now suppose x, $y \in X$ are points such that $d(x,y) \leq d(x,\gamma)$. There are several cases according to where $\pi_\gamma(x)$ and $\pi_\gamma(y)$ lie.

Case 1: $\pi_\gamma(x) \cap \alpha \neq \emptyset \neq \pi_\gamma(y) \cap \alpha$. Then $d(x,y) \leq d(x,\gamma) = d(x,\alpha)$, so contraction for α implies $\operatorname{diam}\pi_\alpha(x) \cup \pi_\alpha(y) \leq \rho(d(x,\alpha)) = \rho(d(x,\gamma))$. There are four sub-cases to check, according to whether $\pi_\gamma(x)$ and $\pi_\gamma(y)$ hit β. These are easy to check, with the worst bound being $\operatorname{diam}\pi_\gamma(x) \cup \pi_\gamma(y) \leq \rho(d(x,\gamma)) + 2\rho'''(2d(x,\gamma))$.

Case 2: $\pi_\gamma(x) \cap \beta = \emptyset = \pi_\gamma(y) \cap \alpha$. Let δ be a geodesic from x to y. Let w be the first point on δ such that $\pi_\gamma(w) \cap \beta \neq \emptyset$. Then $d(w,\alpha) = d(w,\beta) = d(w,\gamma) \leq 2d(x,\gamma)$ and $d(y,\beta) = d(y,\gamma) \leq 2d(x,\gamma)$. We can apply that α is ρ-contracting to the pair x, w since $d(x,w) \leq d(x,y) \leq d(x,\gamma) = d(x,\alpha)$. Likewise, we can apply that β is ρ-contracting to w, y since $d(x,w)+d(w,y) = d(x,y) \leq d(x,\gamma) \leq d(x,w)+d(w,\gamma)$ so $d(w,y) \leq d(w,\gamma)$. We conclude:

$$\begin{aligned}\operatorname{diam} \pi_\gamma(x) \cup \pi_\gamma(y) &\leq \operatorname{diam} \pi_\alpha(x) \cup \pi_\alpha(w) + \operatorname{diam} \pi_\gamma(w) \\ &\quad + \operatorname{diam} \pi_\beta(w) \cup \pi_\beta(y) \\ &\leq \rho(d(x,\alpha)) + 2\rho'''(d(w,\gamma)) + \rho(d(w,\beta)) \\ &\leq \rho(d(x,\gamma)) + 2\rho'''(2d(x,\gamma)) + \rho(2d(x,\gamma))\end{aligned}$$

By symmetry these two cases cover all possibilities, so it suffices to define $\rho'(r) := 2\rho(2r) + 2\rho'''(2r)$. □

4. Contraction and Quasi-geodesics

In this section we explore the behavior of a quasi-geodesic ray based at a point in a contracting set Z. The main conclusion is that such a ray can stay close to Z for an arbitrarily long time, but once it exceeds a certain threshold distance depending on the quasi-geodesic constants and the contraction function then the ray must escape Z at a definite linear rate.

Definition 4.1. Given a sublinear function ρ and constants $L \geq 1$ and $A \geq 0$, define:

$$\kappa(\rho, L, A) := \max\{3A, 3L^2, 1 + \inf\{R > 0 \mid \forall r \geq R,\, 3L^2\rho(r) \leq r\}\}$$

Define $\kappa'(\rho, L, A) := (L^2+2)(2\kappa(\rho, L, A) + A)$.

This definition implies that for $r \geq \kappa(\rho, L, A)$ we have:

$$r - L^2\rho(r) - A \geq \frac{1}{3}r \geq L^2\rho(r) \tag{2}$$

An inspection of the proof of [2, Theorem 7.1] gives that $\kappa(\rho, 1, 0) \geq \kappa_\rho$ and $\kappa'(\rho, 1, 0) \geq \kappa'_\rho$, so the results of the previous section still hold using $\kappa(\rho, 1, 0)$ and $\kappa'(\rho, 1, 0)$. Enlarging the constants lets us give unified proofs for geodesics and quasi-geodesics.

Theorem 4.2 (Quasi-geodesic Image Theorem)**.** *Let Z be ρ–contracting. Let $\beta\colon [0,T] \to X$ be a continuous (L, A)–quasi-geodesic segment. If $d(\beta, Z) \geq \kappa(\rho, L, A)$ then:*

$$\operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) \leq \frac{L^2+1}{L^2}\left(A + d(\beta_T, Z)\right) + \frac{L^2-1}{L^2} d(\beta_0, Z) + 2\rho(d(\beta_0, Z))$$

The proof generalizes the proof of the Geodesic Image Theorem to work for quasi-geodesics. We typically apply the result when $d(\beta_T, Z) = \kappa(\rho, L, A)$, in which case the theorem says that for fixed ρ, L, and A the projection diameter of β is bounded in terms of $d(\beta_0, Z)$. In particular, when β is geodesic, or, more generally, when $L = 1$, the bound is sublinear in $d(\beta_0, Z)$, and we recover a version of the Geodesic Image Theorem. With a little more work we can prove this stronger statement for quasi-geodesics as well. Although we do not need it in this paper, the stronger version may be of independent interest, so we include a proof at the end of this section (see Theorem 4.7).

Proof of Theorem 4.2. Let $t_0 := 0$. For each $i \in \mathbb{N}$ in turn, let t_{i+1} be the first time such that $d(\beta_{t_i}, \beta_{t_{i+1}}) = d(\beta_{t_i}, Z)$, or set $t_{i+1} = T$ if no such time exists. Let j be the first index such that $d(\beta_{t_j}, \beta_T) \leq d(\beta_{t_j}, Z)$.

$$\begin{aligned} T &= T - t_j + \sum_{i=0}^{j-1}(t_{i+1} - t_i) \\ &\geq \frac{1}{L}\left(d(\beta_{t_j}, \beta_T) - d(\beta_{t_j}, Z) + \sum_{i=0}^{j}(d(\beta_{t_i}, Z) - A)\right) \\ &\geq \frac{1}{L}\left(-d(\beta_T, Z) + \sum_{i=0}^{j}(d(\beta_{t_i}, Z) - A)\right) \end{aligned}$$

On the other hand:

$$\begin{aligned} \frac{T}{L} - A &\leq d(\beta_0, \beta_T) \\ &\leq d(\beta_0, Z) + \operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) + d(Z, \beta_T) \\ &\leq d(\beta_0, Z) + d(\beta_T, Z) + \sum_{i=0}^{j} \rho(d(\beta_{t_i}, Z)) \end{aligned}$$

Combining these gives:

$$\begin{aligned} &\sum_{i=1}^{j} \big(d(\beta_{t_i}, Z) - L^2\rho(d(\beta_{t_i}, Z)) - A\big) \\ &\qquad\leq d(\beta_T, Z) + L^2\left(A + d(\beta_0, Z) + d(\beta_T, Z)\right) \\ &\qquad\qquad - \big(d(\beta_0, Z) - L^2\rho(d(\beta_0, Z)) - A\big) \end{aligned}$$

By (2), the left-hand side is at least $L^2 \sum_{i=1}^{j} \rho(d(\beta_{t_i}, Z))$, so:

$$\begin{aligned} \operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) &\leq \sum_{i=0}^{j} \rho(d(\beta_{t_i}, Z)) \\ &\leq \frac{L^2+1}{L^2}\left(A + d(\beta_T, Z)\right) + \frac{L^2-1}{L^2} d(\beta_0, Z) + 2\rho(d(\beta_0, Z)) \end{aligned}$$

□

Corollary 4.3. *Let Z be ρ–contracting and let β be a continuous (L, A)–quasi-geodesic ray with $\beta_0 \in Z$. There are two possibilities:*

1. *The set $\{t \mid d(\beta_t, Z) \leq \kappa(\rho, L, A)\}$ is unbounded and β is contained in the $\kappa'(\rho, L, A)$–neighborhood of Z.*
2. *There exists a last time T_0 such that $d(\beta_{T_0}, Z) = \kappa(\rho, L, A)$ and:*

$$(\star) \qquad \forall t, \quad d(\beta_t, Z) \geq \frac{1}{2L}(t - T_0) - 2(A + \kappa(\rho, L, A))$$

Proof. Let $\kappa := \kappa(\rho, L, A)$. Let $[a, b]$ be a maximal interval such that $d(\beta_t, Z) \geq \kappa$ for $t \in [a, b]$ and $d(\beta_a, Z) = d(\beta_b, Z) = \kappa$.

For $t \in [a, b]$ we have $d(\beta_t, Z) \leq \kappa + L \cdot (b - a)/2 + A$. Since β is quasi-geodesic:

$$(b - a) \leq L(A + d(\beta_a, \beta_b)) \leq L(A + 2\kappa + \operatorname{diam} \pi(\beta_a) \cup \pi(\beta_b))$$

Theorem 4.2 implies:

$$\begin{aligned}\operatorname{diam}\pi(\beta_a)\cup\pi(\beta_b) &\leq \frac{L^2+1}{L^2}(A+\kappa)+\frac{L^2-1}{L^2}\kappa+\frac{2\kappa}{3L^2}\\ &= \frac{L^2+1}{L^2}A+\frac{6L^2+2}{3L^2}\kappa\end{aligned}$$

Putting these estimates together yields:

$$d(\beta_t, Z) < (L^2+2)(2\kappa+A) = \kappa'(\rho, L, A)$$

Thus, once β leaves the $\kappa'(\rho, L, A)$–neighborhood of Z it can never return to the $\kappa(\rho, L, A)$–neighborhood of Z. If $\{t \mid d(\beta_t, Z) \leq \kappa\}$ is unbounded then β never leaves the $\kappa'(\rho, L, A)$–neighborhood of Z.

Suppose now that there does exist some last time T_0 such that $d(\beta_{T_0}, Z) = \kappa$. Any segment $\beta_{[T_0,t]}$ stays outside $N_\kappa Z$, so apply Theorem 4.2 to see:

$$\begin{aligned}\frac{t-T_0}{L}-A &\leq d(\beta_t,\beta_{T_0})\\ &\leq d(\beta_t, Z)+\operatorname{diam}\pi(\beta_t)\cup\pi(\beta_{T_0})+\kappa\\ &\leq \frac{6L^2-1}{3L^2}d(\beta_t,Z)+\frac{L^2+1}{L^2}(A+\kappa)+\kappa\end{aligned}$$

Thus:

$$d(\beta_t, Z) \geq \frac{3L}{6L^2-1}(t-T_0)-\frac{6L^2+3}{6L^2-1}(A+\kappa)$$ □

Lemma 4.4. *If α is a ρ–contracting geodesic ray and β is a continuous (L, A)–quasi-geodesic ray asymptotic to α with $\alpha_0 = \beta_0$ then β is ρ'–contracting where $\rho' \asymp \rho$ depends only on ρ, L, and A.*

Proof. By Corollary 4.3, if β is asymptotic to α then β is contained in the $\kappa'(\rho, L, A)$–neighborhood of α. Then it is easy to see that the Hausdorff distance between α and β is bounded in terms of ρ, L, and A. The claim then follows from Lemma 3.3. □

Lemma 4.5. *Let α be a ρ-contracting geodesic ray, and let β be a continuous (L, A)–quasi-geodesic ray with $\alpha_0 = \beta_0 = o$. Given some R and J, suppose there exists a point $x \in \alpha$ with $d(x, o) \geq R$ and $d(x, \beta) \leq J$. Let y be the last point on the subsegment of α between o and x such that $d(y, \beta) = \kappa(\rho, L, A)$. There is a constant $\Lambda \leq 2$ and a function $\lambda(\phi, p, q)$ defined for sublinear ϕ, $p \geq 1$, and $q \geq 0$ such that λ is monotonically increasing in p and q and:*

$$d(x, y) \leq \Lambda J + \lambda(\rho, L, A)$$

Thus:

$$d(o, y) \geq R - \Lambda J - \lambda(\rho, L, A)$$

Proof. If $d(x, \beta) \leq \kappa(\rho, L, A)$ then $y = x$ and we are done. Otherwise, let a be the last time such that β_a is $\kappa(\rho, L, A)$–close to α between o and x, and let $y' \in \alpha$ be the last point of α with $d(\beta_a, y') = \kappa(\rho, L, A)$. Since $d(x, y) \leq d(x, y')$, it suffice to prove the lemma replacing y by y'.

Now let b be the first time such that $d(\beta_b, x) = J$. The subsegment $\beta_{[a,b]}$ stays outside $N_{\kappa(\rho,L,A)}\alpha$. Pick a geodesic from β_b to x and let w be the first point such that $d(w, \alpha) = \kappa(\rho, 1, 0)$. Pick $z \in \pi(\beta_b)$ and $v \in \pi(w)$, and let $W := d(\beta_b, w)$, $Y := d(y, z)$, $Z := d(z, v)$, and $X := d(v, x)$, see Figure 1.

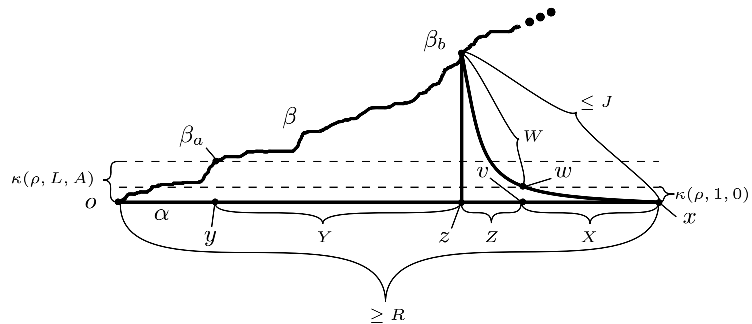

Figure 1. Setup for Lemma 4.5

We have $W \geq d(\beta_b, \alpha) - Z - \kappa(\rho, 1, 0)$, and $X \leq J - W + \kappa(\rho, 1, 0)$, so:

$$\begin{aligned} d(y, x) &\leq X + Y + Z \\ &\leq Y + Z + J - d(\beta_b, \alpha) + Z + 2\kappa(\rho, 1, 0) \end{aligned}$$

Apply Theorem 4.2 for L and A to bound Y and for 1 and 0 to bound Z. Simplifying the result yields:

$$d(y, x) \leq$$
$$J + 6\kappa(\rho, 1, 0) + \frac{L^2 + 1}{L^2}(A + \kappa(\rho, L, A)) - \frac{1}{L^2} d(\beta_b, \alpha) + 6\rho(d(\beta_b, \alpha))$$

Now use the facts that $\rho(d(\beta_b, \alpha)) \leq \frac{d(\beta_b, \alpha)}{3L^2}$ and $d(\beta_b, \alpha) \leq J$ to achieve:

$$\begin{aligned} d(y, x) &\leq \frac{L^2 + 1}{L^2} J + 6\kappa(\rho, 1, 0) + \frac{L^2 + 1}{L^2}(A + \kappa(\rho, L, A)) \\ &\leq 2J + 6\kappa(\rho, 1, 0) + 2(A + \kappa(\rho, L, A)) \end{aligned}$$

Set $\Lambda := 2$ and $\lambda(\phi, p, q) := 6\kappa(\phi, 1, 0) + 2(q + \kappa(\phi, p, q))$. □

Lemma 4.6. *Given a sublinear function ρ and constants $L \geq 1$, $A \geq 0$ there exist constants $L' \geq 1$ and $A' \geq 0$ such that if α is a ρ-contracting geodesic ray or segment and β is a continuous (L, A)–quasi-geodesic ray with $\alpha_0 = \beta_0 = o$, then we obtain a continuous (L', A')–quasi-geodesic by following α backward until α_{s_0}, then following a geodesic from α_{s_0} to β_{t_0}, then following β, where β_{t_0} is the last point of β at distance $\kappa(\rho, L, A)$ from α, and where α_{s_0} is the last point of α at distance $\kappa(\rho, L, A)$ from β_{t_0}.*

Proof. Define $\kappa := \kappa(\rho, L, A)$ and Λ and $\lambda := \lambda(\rho, L, A)$ from Lemma 4.5. Recall $\Lambda \leq 2 \leq 2L$. It suffices to take $A' := \left(\frac{(4L+1)\kappa+\lambda}{4L} + A\right)$ and $L' := 4L$. Since we have constructed a concatenation of three quasi-geodesic segments, it suffices to check that points on different segments are not too close together. Since $A' > A + \kappa$ we may ignore the short middle segment. Thus, we need to check for $s \geq s_0$ and $t \geq t_0$ that $d(\alpha_s, \beta_t) \geq \frac{s - s_0 + t - t_0 + \kappa}{L'} - A'$.

For such s and t, let $x := \alpha_s$, $y := \alpha_{s_0}$, and $z := \beta_t$. By Lemma 4.5, $s - s_0 = d(x, y) \leq \Lambda d(x, z) + \lambda < 2Ld(x, z) + \lambda$. Choose some point $z' \in \pi_\alpha(z)$. By Corollary 4.3 ($\star$) we have $d(z, x) \geq d(z, z') \geq \frac{t - t_0}{2L} - 2(A + \kappa)$. Now average these two lower bounds for $d(x,z)$:

$$\begin{aligned} d(\alpha_s,\beta_t) = d(x,z) &\geq \frac{1}{2}\left(\frac{s-s_0}{2L} - \frac{\lambda}{2L} + \frac{t-t_0}{2L} - 2(A+\kappa)\right) \\ &\geq \frac{s-s_0+t-t_0+\kappa}{4L} - \left(\frac{\lambda}{4L} + \frac{4L+1}{4L}\kappa + A\right) \end{aligned}$$ □

To close this section we give the stronger formulation of the Quasi-geodesic Image Theorem:

Theorem 4.7. *Given a sublinear function ρ and constants $L \geq 1$ and $A \geq 0$ there is a sublinear function ρ' such that if Z is ρ–contracting and $\beta\colon [0,T] \to X$ is a continuous (L,A)–quasi-geodesic segment with $d(\beta,Z) = d(\beta_T,Z) = \kappa(\rho,L,A)$ then* $\operatorname{diam}\pi(\beta_0)\cup\pi(\beta_T) \leq \rho'(d(\beta_0,Z))$.

Proof. Define $\rho'(r) := \sup_\beta \operatorname{diam}\pi(\beta_0)\cup\pi(\beta_T)$ where the supremum is taken over all continuous (L,A)–quasi-geodesic segments β such that $d(\beta,Z) = \kappa(\rho,L,A)$ is realized at one endpoint of β and the other endpoint is at distance at most r from Z. Suppose that ρ' is not sublinear, so suppose $\limsup_{r\to\infty}\rho'(r)/r = 2\epsilon > 0$. Then there exists a sequence $(r_i) \to \infty$ such that for each i there exists a continuous (L,A)–quasi-geodesic segment $\beta^{(i)}\colon [0,T_i] \to X$ with $d(\beta^{(i)}_{T_i},Z) = \kappa(\rho,L,A)$ and $d(\beta^{(i)}_0,Z) \leq r_i$ and $\operatorname{diam}\pi(\beta^{(i)}_0)\cup\pi(\beta^{(i)}_{T_i}) \geq \epsilon r_i$, so that $\operatorname{diam}\pi(\beta^{(i)}_0)\cup\pi(\beta^{(i)}_{T_i}) \geq \epsilon d(\beta^{(i)}_0,Z)$.

For $n\in\mathbb{N}$ define κ_n large enough so that for all $r \geq \kappa_n$ we have $r - L^2\rho(r) - A \geq nL^2\rho(r)$ (recalling (2), $\kappa_1 = \kappa(\rho,L,A)$). The proof of Theorem 4.2 shows that if a continuous (L,A)–quasi-geodesic segment stays outside the κ_n–neighborhood of Z then:

$$\operatorname{diam}\pi(\beta_0)\cup\pi(\beta_T) \leq \frac{1}{n}\left(2A + \frac{L^2+1}{L^2}d(\beta_T,Z) + d(\beta_0,Z)\right) \tag{3}$$

For $\epsilon > 0$ as above, choose $n\in\mathbb{N}$ large enough that $n\epsilon > 2$. For all sufficiently large i we have that $\operatorname{diam}\pi(\beta^{(i)}_0)\cup\pi(\beta^{(i)}_{T_i}) > 2A + \frac{L^2+1}{L^2}\kappa_1 + \kappa_n$. By (3) for $n=1$, we have $d(\beta^{(i)}_0,Z) > \kappa_n$. Let $s_i > 0$ be the first time such that $d(\beta^{(i)}_{s_i},Z) = \kappa_n$.

$$\begin{aligned} \epsilon d(\beta^{(i)}_0,Z) &\leq \operatorname{diam}\pi(\beta^{(i)}_{T_i})\cup\pi(\beta^{(i)}_0) \\ &\leq \operatorname{diam}\pi(\beta^{(i)}_{T_i})\cup\pi(\beta^{(i)}_{s_i}) + \operatorname{diam}\pi(\beta^{(i)}_{s_i})\cup\pi(\beta^{(i)}_0) \\ &\leq \left(2A + \frac{L^2+1}{L^2}\kappa_1 + \kappa_n\right) + \frac{1}{n}\left(2A + \frac{L^2+1}{L^2}\kappa_n + d(\beta^{(i)}_0,Z)\right) \\ &\leq \left(2A + \frac{L^2+1}{L^2}\kappa_1 + \kappa_n\right) + \frac{\epsilon}{2}\left(2A + \frac{L^2+1}{L^2}\kappa_n + d(\beta^{(i)}_0,Z)\right) \end{aligned}$$

Solving for $d(\beta^{(i)}_0,Z)$, we find that it is bounded, independent of i. By (3) for $n=1$, this would bound $\operatorname{diam}\pi(\beta^{(i)}_0)\cup\pi(\beta^{(i)}_{T_i})$, independent of i, whereas we have assumed $\operatorname{diam}\pi(\beta^{(i)}_0)\cup\pi(\beta^{(i)}_{T_i}) \geq \epsilon r_i \to \infty$. This is a contradiction, so we conclude $\lim_{r\to\infty}\rho'(r)/r = 0$. □

5. The contracting boundary and the topology of fellow-travelling quasi-geodesics

Definition 5.1. Let X be a proper geodesic metric space with basepoint o. Define $\partial_c X$ to be the set of contracting quasi-geodesic rays based at o modulo Hausdorff equivalence.

Lemma 5.2. *For each $\zeta \in \partial_c X$ the set of contracting geodesic rays in ζ is non-empty and the function*

$$\rho_\zeta(r) := \sup_{\alpha,x,y} \operatorname{diam} \pi_\alpha(x) \cup \pi_\alpha(y)$$

where the supremum is taken over geodesics $\alpha \in \zeta$ and points x and y such that $d(x,y) \leq d(x,\alpha) \leq r$, is sublinear and every geodesic in ζ is ρ_ζ–contracting.

Proof. By definition, ζ is an equivalence class of contracting quasi-geodesic rays, so there exists some ρ'–contracting (L,A)–quasi-geodesic ray $\beta \in \zeta$ based at o. Since X is proper, a sequence of geodesic segments connecting o to β_i for $i \in \mathbb{N}$ has a subsequence that converges to a geodesic α'. By Theorem 3.4, all of these geodesic segments, hence α' as well, are contained in a bounded neighborhood of β, with bound depending only on ρ', so there do exist geodesics asymptotic to β. Furthermore, Corollary 4.3 implies that geodesic rays asymptotic to β have uniformly bounded Hausdorff distance from β, with bound depending on ρ', L, and A. By Lemma 3.3, all such geodesics are ρ''–contracting for some $\rho'' \asymp \rho'$ depending on ρ', L, and A.

The function ρ_ζ is non-decreasing and bounds projection diameters by definition. The fact that there exists a sublinear function ρ'' such that all geodesics in ζ are ρ''–contracting implies $\rho_\zeta \leq \rho''$, so ρ_ζ is also sublinear. □

Definition 5.3. Let X be a proper geodesic metric space. Take $\zeta \in \partial_c X$. Fix a geodesic ray $\alpha^\zeta \in \zeta$. For each $r \geq 1$ define $U(\zeta, r)$ to be the set of points $\eta \in \partial_c X$ such that for all $L \geq 1$ and $A \geq 0$ and every continuous (L,A)–quasi-geodesic ray $\beta \in \eta$ we have $d(\beta, \alpha^\zeta \cap N_r^c o) \leq \kappa(\rho_\zeta, L, A)$.

Informally, $\eta \in U(\zeta, r)$ means that inside the ball of radius r about the basepoint quasi-geodesics in η fellow-travel α^ζ just as closely as quasi-geodesics in ζ do. Alternatively, quasi-geodesics in η do not escape from α^ζ until after they leave the ball of radius r about the basepoint.

Definition 5.4. Define the *topology of fellow-travelling quasi-geodesics* on $\partial_c X$ by:

$$\mathcal{FQ} := \{U \subset \partial_c X \mid \forall \zeta \in U, \exists r \geq 1, U(\zeta, r) \subset U\}$$

The contracting boundary equipped with this topology is denoted $\partial_c^{\mathcal{FQ}} X$.

Note that the sets $U(\zeta, r)$ are not necessarily open.

Observation 5.5. Suppose $\eta \notin U(\zeta, r)$. By definition, for some L and A there exists a continuous (L,A)–quasi-geodesic $\beta \in \eta$ such that $d(\beta, \alpha^\zeta \cap N_r^c o) > \kappa(\rho_\zeta, L, A)$. Since $o \in \beta$, this is not possible if $\kappa(\rho_\zeta, L, A) \geq r$. Thus, in light of Definition 4.1, the quasi-geodesic β witnessing $\eta \notin U(\zeta, r)$ must be an (L,A)–quasi-geodesic with $L^2 < r/3$ and $A < r/3$.

Proposition 5.6. *$\mathcal{FQ}$ is a topology on $\partial_c X$ and for each $\zeta \in \partial_c X$ the collection $\{U(\zeta, n) \mid n \in \mathbb{N}\}$ is a neighborhood basis at ζ.*

Corollary 5.7. *$\partial_c^{\mathcal{FQ}}X$ is first countable.*

Proof of Proposition 5.6. For every $\zeta \in \partial_c X$ and $1 \leq r < r'$ we have $\zeta \in U(\zeta, r') \subset U(\zeta, r)$. The nesting is immediate from Definition 5.3, and $\zeta \in U(\zeta, r)$ by Corollary 4.3. Now it is easy to see that $\mathcal{FQ}$ is a topology. That a set of the form $U(\zeta, r)$ is a neighborhood of ζ in this topology follows from showing that the set

$$U := \{\eta \in U(\zeta, r) \mid \exists R_\eta,\, U(\eta, R_\eta) \subset U(\zeta, r)\}$$

is open, since then $\zeta \in U \subset U(\zeta, r)$. Now if $\eta \in U$ then there exists R_η so that $U(\eta, R_\eta) \subset U(\zeta, r)$. Lemma 5.8 below gives that there exists R so that for all $\xi \in U(\eta, R)$ there exists R' with $U(\xi, R') \subset U(\eta, R_\eta) \subset U(\zeta, r)$. Therefore $U(\eta, R) \subset U$ and so U is open. □

Lemma 5.8. *For every sublinear function ρ and $r \geq 1$ there exists a number $\psi(\rho, r) > r$ such that for every $R \geq \psi(\rho, r)$ and every $\zeta \in \partial_c X$ such that $\rho_\zeta \leq \rho$ we have that for every $\eta \in U(\zeta, R)$ there exists an R' such that $U(\eta, R') \subset U(\zeta, r)$.*

Proof. Let $r \geq 1$ be arbitrary. Suppose that $\eta \in U(\zeta, R)$ for some R. Let $\alpha := \alpha^\zeta \in \zeta$ and $\beta := \alpha^\eta \in \eta$ be the geodesics of Definition 5.3. By hypothesis, α is ρ–contracting. Suppose β is ρ'–contracting. Suppose there is no R' such that $U(\eta, R') \subset U(\zeta, r)$. Then for all $i \in \mathbb{N}$ there exists $\xi_i \in U(\eta, i) \setminus U(\zeta, r)$. The fact that $\xi_i \notin U(\zeta, r)$ means that there exist L_i, A_i, and a continuous (L_i, A_i)–quasi-geodesic ray $\gamma^{(i)} \in \xi_i$ such that $d(\gamma^{(i)}, \alpha \cap N_r^c o) > \kappa(\rho, L_i, A_i)$. On the other hand, $\xi_i \in U(\eta, i)$ implies $d(\gamma^{(i)}, \beta \cap N_i^c o) \leq \kappa(\rho', L_i, A_i)$.

By Observation 5.5, we have L_i^2, $A_i < r/3$ for all i. By properness and the fact that the $\gamma^{(i)}$ are uniformly $(\sqrt{r/3}, r/3)$–quasi-geodesic, we can pass to a subsequence so that $(\gamma^{(i)}|_{\mathbb{N}})$ converges to an (L, A)–quasi-geodesic γ for some L^2, $A \leq r/3$. Tame it to get a continuous $(L, 2(L+A))$–quasi-geodesic $\hat{\gamma}$ that agrees with γ on $\mathbb{N}$. The fact that $d(\gamma^{(i)}, \beta \cap N_i^c o) \leq \kappa(\rho', L_i, A_i)$ implies that the initial segment of $\gamma^{(i)}$ contained in the ball of radius $i - \kappa(\rho', \sqrt{r/3}, r/3)$ is contained in the $\kappa'(\rho', \sqrt{r/3}, r/3)$–neighborhood of β. Thus, $\hat{\gamma} \in \eta$.

Let x be the last point of α such that $d(x, \hat{\gamma}) = \kappa(\rho, L, 2(L+A))$. Let z_i be the last point of α such that $d(z_i, \gamma^{(i)}) = \kappa(\rho, L_i, A_i)$.

By definition of $\gamma^{(i)}$ we have $d(o, z_i) < r$.

Since $\hat{\gamma} \in \eta \in U(\zeta, R)$ we have $R \leq d(o, x)$.

There is an integer s such that $d(\hat{\gamma}_s, x) \leq \kappa(\rho, L, 2(L+A)) + 2(L+A) + L/2$. Thus, for all sufficiently large i we have $d(\gamma_s^{(i)}, x) < \kappa(\rho, L, 2(L+A)) + 2(L+A) + L/2 + 1$. An application of Lemma 4.5 shows there is an R_1 depending on ρ and r such that $d(o, z_i) \geq d(o, x) - R_1$.

Putting these estimates together and defining $\psi(\rho, r) := R_1 - r$ yields:

$$r > d(o, z_i) \geq R - R_1 = R - (\psi(\rho, r) - r) \implies R < \psi(\rho, r)$$

Therefore, given ζ and r, for every $R \geq \psi(\rho, r)$ and every $\eta \in U(\zeta, R)$ there exists R_η such that $U(\eta, R_\eta) \subset U(\zeta, r)$. □

Corollary 5.9. *For every $\zeta \in \partial_c X$ and $r \geq 1$ there exists an open set U such that $U(\zeta, \psi(\rho_\zeta, r)) \subset U \subset U(\zeta, r)$.*

Proposition 5.10. *The topology $\mathcal{FQ}$ does not depend on the choice of basepoint or on the choices of the representative geodesic rays for each point in $\partial_c X$.*

Proof. Let $\mathcal{C}$ be the set of contracting quasi-geodesic rays based at o and let $\mathcal{C}'$ be the set of contracting quasi-geodesic rays based at o'. There is a map $\phi\colon \mathcal{C} \to \mathcal{C}'$ by prefixing $\gamma \in \mathcal{C}$ with a chosen geodesic segment from o' to o. The map ϕ clearly induces a bijection $\partial_c\phi$ between contracting boundaries of X with respect to different basepoints, and the inverse map can be achieved by simply prefixing quasi-geodesic rays by a geodesic from o to o'. We check that $\partial_c\phi$ is an open map. For $\zeta \in \partial_c^{\mathcal{FQ}}X$ and $r \geq 1$ we show for sufficiently large R that $U'(\partial_c\phi(\zeta), R) \subset \partial_c\phi(U(\zeta, r))$, where $U'(\partial_c\phi(\zeta), R)$ denotes the appropriate neighborhood of $\partial_c\phi(\zeta)$ defined with o' as basepoint.

Let $\alpha := \alpha^\zeta$ be the reference geodesic for ζ based at o, and let α' be the reference geodesic for $\partial_c\phi(\zeta)$ based at o'. Then α' is bounded Hausdorff distance from α. Suppose α is ρ–contracting and α' is ρ'–contracting. Theorem 3.4 implies that α' eventually comes within distance $\kappa(\rho, 1, 0)$ of α, and Theorem 4.2 implies that this first happens at some time no later than $d(o', \alpha) + 3\kappa(\rho, 1, 0) + 2\rho(d(o', \alpha))$. After that time α' remains in the $\kappa'(\rho, 1, 0)$–neighborhood of α. Assume $R > d(o', \alpha) + 3\kappa(\rho, 1, 0) + 2\rho(d(o', \alpha))$.

Assume further that $R > r + 2d(o, o')$ and suppose $\eta \in U'(\partial_c\phi(\zeta), R)$. Let $\gamma \in \partial_c\phi^{-1}(\eta)$ be an arbitrary continuous (L, A)–quasi-geodesic. Our goal is to show that if R is chosen sufficiently large with respect to ρ, ρ', and r, then such a γ must come within distance $\kappa(\rho, L, A)$ of α outside $N_r o$. We then conclude $\partial_c\phi^{-1}(U'(\partial_c\phi(\zeta), R)) \subset U(\zeta, r)$. By Observation 5.5, it suffices to consider the case L^2, $A < r/3$.

Now, $\gamma' := \phi(\gamma) \in \eta$ is a continuous $(L, A + 2d(o, o'))$–quasi-geodesic. Since $\eta \in U'(\partial_c\phi(\zeta), R)$ there exists a point $x' \in \alpha'$ such that $d(\gamma', x') \leq \kappa(\rho', L, A + 2d(o, o'))$ and $d(x', o') \geq R$. The first restriction on R implies there is a point $x \in \alpha$ such that $d(x, x') \leq \kappa'(\rho, 1, 0)$, so $d(\gamma', x) \leq \kappa'(\rho, 1, 0) + \kappa(\rho', L, A + 2d(o, o'))$. We also have $d(x, o) \geq R - \kappa'(\rho, 1, 0) - d(o, o')$. Assuming further that $R > 2d(o, o') + 2\kappa'(\rho, 1, 0) + \kappa(\rho', L, A + 2d(o, o'))$, we have that the point of γ' close to x is actually a point of γ. Let y be the last point of α at distance $\kappa(\rho, L, A)$ from γ, and apply Lemma 4.5 to find:

$$\begin{aligned} d(o, y) \geq R &- \kappa'(\rho, 1, 0) - d(o, o') \\ &- \Lambda(\kappa'(\rho, 1, 0) + \kappa(\rho', \sqrt{r/3}, r/3 + 2d(o, o'))) - \lambda(\rho, \sqrt{r/3}, r/3) \end{aligned}$$

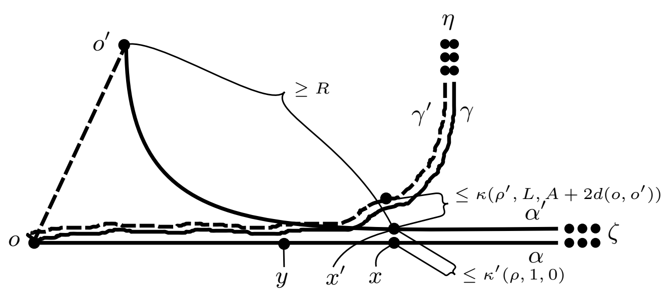

Figure 2. Change of basepoint

Assuming that R was chosen large enough to guarantee the right-hand side is at least r, we have that γ comes within distance $\kappa(\rho, L, A)$ of α outside $N_r o$. $\square$

Proposition 5.11. *$\partial_c^{\mathcal{FQ}}X$ is Hausdorff.*

Proof. Let ζ and η be distinct points in $\partial_c X$. Let $\alpha := \alpha^\zeta$ and $\beta := \alpha^\eta$ be representative geodesic rays. Let R be large enough that the $\kappa'(\rho_\zeta, 1, 0)$–neighborhood of $\alpha_{[R,\infty)}$ is disjoint from the $\kappa'(\rho_\eta, 1, 0)$–neighborhood of $\beta_{[R,\infty)}$. Such an R exists by Corollary 4.3.

Choose $\xi \in U(\zeta, R)$. Let $\gamma \in \xi$ be a geodesic ray. Since $\xi \in U(\zeta, R)$ there exists a point $x \in \alpha$ and $y \in \gamma$ with $d(x, o) \geq R$ and $d(x, y) \leq \kappa(\rho_\zeta, 1, 0)$. By construction $d(y, \beta) > \kappa'(\rho_\eta, 1, 0)$, so, by Corollary 4.3, the final visit of γ to the $\kappa(\rho_\eta, 1, 0)$–neighborhood of β must have occurred inside the ball of radius R about o. Thus, $\xi \notin U(\eta, R)$. □

Proposition 5.12. *$\partial_c^{\mathcal{FQ}} X$ is regular.*

Proof. Suppose $C \subset \partial_c^{\mathcal{FQ}} X$ is closed and $\zeta \in C^c$. Then C^c is a neighborhood of ζ, so there exists r' such that for all $r \geq r'$ we have $U(\zeta, r) \subset C^c$. Suppose:

$$\forall \zeta \in \partial_c^{\mathcal{FQ}} X,\ \exists r' \geq 1,\ \forall r \geq r',\ \exists R > r,\ \overline{U(\zeta, R)} \subset U(\zeta, r) \tag{4}$$

Then there exists an $R > r$ such that $\overline{U(\zeta, R)} \subset U(\zeta, r) \subset C^c$, so C is contained in an open set $\overline{U(\zeta, R)}^c$ that is disjoint from $U(\zeta, R)$. By Proposition 5.6, $U(\zeta, R)$ is a neighborhood of ζ, so it contains an open set that contains ζ and is disjoint from $\overline{U(\zeta, R)}^c$. Thus, (4) implies regularity.

Now we show $\partial_c^{\mathcal{FQ}} X$ satisfies (4). Given $\zeta \in \partial_c^{\mathcal{FQ}} X$ and $r \geq 1$, suppose $R > r$ and suppose there exists $\eta \in \overline{U(\zeta, R)} \cap U(\zeta, r)^c$. Since $\eta \notin U(\zeta, r)$, there exist, by Observation 5.5, constants L^2, $A < r/3$ and a continuous (L, A)–quasi-geodesic $\gamma \in \eta$ such that γ does not come within distance $\kappa(\rho_\zeta, L, A)$ of α^ζ outside $N_r o$. Since $\eta \in \overline{U(\zeta, R)}$, for every $s \in \mathbb{N}$ there exists $\xi_s \in U(\zeta, R) \cap U(\eta, s)$. The idea of the proof is to construct a quasi-geodesic δ that has a long initial segment that agrees with γ, a tail that agrees with that of α^{ξ_s} for some sufficiently large s, and a geodesic segment β connecting them. Since the tail of δ agrees with α^{ξ_s} we have $\delta \in \xi_s \in U(\zeta, R)$. Provide we control the quasi-geodesic constants of δ, this will imply that the initial segment of δ, which agrees with γ, comes close to α^ζ, contradicting the fact that γ witnesses $\eta \notin U(\zeta, r)$. For this to work we must arrange that the quasi-geodesic constants of δ depend only on ζ and r, whereas it would be easier to compute an additive constant also depending on η. We overcome this technical challenge by choosing β to be long enough to overwhelm that additive constant.

Take T sufficiently large so that no $(2L+1, A)$-quasi-geodesic ray starting at o that intersects $N_{12\kappa'(\rho_\eta,1,0)+4\kappa'(\rho_\eta,L,A)}\alpha^\eta_{[T,\infty)}$ can ever return to $N_{\kappa'(\rho_\zeta,2L+1,A)}\alpha^\zeta$. Let $T'' := T - 4\kappa'(\rho_\eta, 1, 0) - \kappa'(\rho_\eta, L, A)$, and let T' be the first time such that $d(\gamma_{T'}, \alpha^\eta_{[T'',\infty)}) \leq \kappa'(\rho_\eta, L, A)$. Let $S \gg T$ be a number depending on η, L, and A, to be determined in the course of the proof. Fix some $s > S$, and let $r_0 \geq S$ and $t_0 \leq T'$ be times such that $d(\gamma_{t_0}, \alpha^{\xi_s}_{r_0}) = d(\gamma_{[0,T']}, \alpha^{\xi_s}_{[S,\infty)})$. Let β be a geodesic between γ_{t_0} and $\alpha^{\xi_s}_{r_0}$. Define $\delta := \gamma_{[0,t_0]} + \beta + \alpha^{\xi_s}_{[r_0,\infty)}$.

First, let us estimate the length of β. The point γ_{t_0} is $\kappa'(\rho_\eta, L, A)$–close to some point α^η_u, and, by definition of T', $u \leq T''$. Some point $\alpha^{\xi_s}_v$ passes within

$2\kappa'(\rho_\eta,1,0)$ of α^η_u, so that:

$$\begin{aligned}|\beta| &= d(\gamma_{t_0},\alpha^{\xi_s}_{r_0})\\ &\geq r_0 - v - 2\kappa'(\rho_\eta,1,0) - \kappa'(\rho_\eta,L,A)\\ &\geq S - u - 4\kappa'(\rho_\eta,1,0) - \kappa'(\rho_\eta,L,A)\\ &\geq S - T\end{aligned} \tag{5}$$

It is convenient at this point to explain the choice of T. We have that $\gamma_{T'}$ is $\kappa'(\rho_\eta,L,A)$–close to some point $\alpha^\eta_{u'}$ with $u'\geq T''$, and some point $\alpha^{\xi_s}_{v'}$ passes within $2\kappa'(\rho_\eta,1,0)$ of $\alpha^\eta_{u'}$. Observe that $v'\geq T''-2\kappa'(\rho_\eta,1,0)$. So:

$$|\beta| = d(\gamma_{t_0},\alpha^{\xi_s}_{r_0}) \leq d(\gamma_{T'},\alpha^{\xi_s}_{r_0}) \leq r_0 - v' + 2\kappa'(\rho_\eta,1,0) + \kappa'(\rho_\eta,L,A)$$

Therefore (5) implies that $v \geq v' - 4\kappa'(\rho_\eta,1,0) - 2\kappa'(\rho_\eta,L,A) \geq T'' - 6\kappa'(\rho_\eta,1,0) - 2\kappa'(\rho_\eta,L,A)$, and so:

$$\begin{aligned}d(\gamma_{t_0},\alpha^\eta_{[T,\infty)}) &\leq \max\{T-u,0\} + \kappa'(\rho_\eta,L,A)\\ &\leq \max\{T-v+2\kappa'(\rho_\eta,1,0),0\} + \kappa'(\rho_\eta,L,A)\\ &\leq 12\kappa'(\rho_\eta,1,0) + 4\kappa'(\rho_\eta,L,A)\end{aligned}$$

This tells us that if we show δ is a $(2L+1,A)$–quasi-geodesic then the definition of T implies that δ can only come $\kappa'(\rho_\zeta,2L+1,A)$–close to α^ζ on the subsegment of δ that agrees with γ.

We now prove that δ is a $(2L+1,A)$–quasi-geodesic provided that S was chosen to be sufficiently large. Since δ is a concatenation of quasi-geodesics, we only need to check that points on different pieces of δ are not closer than they ought to be with respect to the parameterization.

First we claim $\gamma_{[0,t_0]}+\beta$ is an (L',A)–quasi-geodesic for $L' := 2L+1$. This is true for $\gamma_{[0,t_0]}$ and β. Suppose, for $A' := \frac{LA}{L'} \leq A$, there are $t\leq t_0$ and $u \leq |\beta|$ such that $d(\gamma_t,\beta_u) < \frac{t_0-t+u}{L'} - A'$. Now, $d(\beta_u,\gamma_t)\geq d(\beta_u,\gamma_{t_0}) = u$, which implies $u < \frac{L'}{L'-1}(\frac{t_0-t}{L'} - A')$. But then:

$$\begin{aligned}\frac{t_0-t}{L} - A \leq d(\gamma_t,\gamma_{t_0}) &\leq d(\gamma_t,\beta_u) + d(\beta_u,\gamma_{t_0})\\ &\leq \left(\frac{t_0-t+u}{L'} - A'\right) + u\end{aligned}$$

Plugging in the values of L' and A' and the bound for u yields a contradiction.

The same argument shows $\beta + \alpha^{\xi_s}_{[r_0,\infty)}$ is a $(3,0)$–quasi-geodesic.

Now consider points γ_t and $\alpha^{\xi_s}_r$ for $t\leq t_0$ and $r\geq r_0$. Both γ and α^{ξ_s} are contained in $N_{\kappa'(\rho_\eta,L,A)}\alpha^\eta$ for distance at least S. In particular, there are points x and y on α^{ξ_s} at distance at most $3\kappa'(\rho_\eta,L,A)$ from γ_t and γ_{t_0}, respectively.

Therefore:

$$\begin{aligned} d(\gamma_t, \alpha^{\xi_s}_r) &\geq d(x, \alpha^{\xi_s}_r) - 3\kappa'(\rho_\eta, L, A) \\ &= d(x,y) + d(y, \alpha^{\xi_s}_{r_0}) + r - r_0 - 3\kappa'(\rho_\eta, L, A) \\ &\geq d(\gamma_t, \gamma_{t_0}) - 6\kappa'(\rho_\eta, L, A) + |\beta| - 3\kappa'(\rho_\eta, L, A) \\ &\qquad + r - r_0 - 3\kappa'(\rho_\eta, L, A) \\ &\geq \frac{t_0 - t}{L} - A + |\beta| + r - r_0 - 12\kappa'(\rho_\eta, L, A) \\ &> \frac{t_0 - t + |\beta| + r - r_0}{2L+1} - A + \frac{2L}{2L+1}|\beta| - 12\kappa'(\rho_\eta, L, A) \\ &\geq \frac{t_0 - t + |\beta| + r - r_0}{2L+1} - A + \frac{2L}{2L+1}(S-T) - 12\kappa'(\rho_\eta, L, A) \end{aligned}$$

If $\frac{2L}{2L+1}(S-T) - 12\kappa'(\rho_\eta, L, A) \geq 0$ then $d(\gamma_t, \alpha^{\xi_s}_r) \geq \frac{t_0 - t + |\beta| + r - r_0}{2L+1} - A$. We conclude that having chosen S sufficiently large, the resulting δ is a continuous $(2L+1, A)$–quasi-geodesic. Since the tail of δ agrees with α^{ξ_s}, we have $\delta \in \xi_s \subset U(\zeta, R)$, so δ must come $\kappa(\rho_\zeta, 2L+1, A)$–close to a point of α^ζ outside $N_R o$. Our choice of T says that only the subsegment of δ that agrees with γ could possibly do so. By Lemma 4.5, if γ comes within $\kappa(\rho_\zeta, 2L+1, A)$ of α^ζ outside $N_R o$ and does not come within $\kappa(\rho_\zeta, L, A)$ of α^ζ outside $N_r o$ then:

$$\begin{aligned} R &< r + \Lambda\kappa(\rho_\zeta, 2L+1, A) + \lambda(\rho_\zeta, L, A) \\ &\leq r + \Lambda\kappa(\rho_\zeta, 2\sqrt{r/3}+1, r/3) + \lambda(\rho_\zeta, \sqrt{r/3}, r/3) \end{aligned}$$

Conversely, given any $\zeta \in \partial_c^{\mathcal{FQ}} X$ and $r \geq 1$, for

$$R \geq r + \Lambda\kappa(\rho_\zeta, 2\sqrt{r/3}+1, r/3) + \lambda(\rho_\zeta, \sqrt{r/3}, r/3)$$

we have $\overline{U(\zeta, R)} \subset U(\zeta, r)$, as desired. □

6. Quasi-isometry invariance

Theorem 6.1. *A quasi-isometry $\phi\colon X \to X'$ between proper geodesic metric spaces induces a homeomorphism $\partial_c\phi\colon \partial_c^{\mathcal{FQ}} X \to \partial_c^{\mathcal{FQ}} X'$.*

Proof. Since the topology is basepoint invariant we choose $o \in X$ and let $o' := \phi(o) \in X'$.

The contracting property is equivalent to the Morse property [2], which is invariant under quasi-isometries. The property of being a quasi-geodesic and the property of two quasi-geodesics being asymptotic are also preserved by quasi-isometry, so ϕ naturally induces a bijection $\partial_c\phi\colon \partial_c X \to \partial_c X'$. A quasi-isometry inverse $\bar\phi$ of ϕ induces a bijection $\partial_c\bar\phi = (\partial_c\phi)^{-1}$, so we only need to show that $\partial_c\phi$ is continuous.

Suppose ϕ is an (L, A)–quasi-isometry. Suppose $\bar\phi$ is an (L, A)–quasi-isometry inverse to ϕ. We assume $\sup_{x \in X'} d(\phi \circ \bar\phi(x), x) \leq A$. Take $\zeta \in \partial_c X'$ and $r \geq 1$. By Proposition 5.6, we conclude that $\partial_c\phi$ is continuous by showing that for sufficiently large R we have $U((\partial_c\phi)^{-1}(\zeta), R) \subset (\partial_c\phi)^{-1}(U(\zeta, r))$. To prove this we play our usual game of supposing the converse, deriving a bound on R, and then choosing R to be larger than that bound.

Suppose for some $R > r$ there exists $\eta \in U((\partial_c\phi)^{-1}(\zeta), R)$ such that $\eta \notin (\partial_c\phi)^{-1}(U(\zeta, r))$. The latter implies there exist $L' \geq 1$ and $A' \geq 0$ and a continuous (L', A')–quasi-geodesic $\gamma \in \partial_c\phi(\eta)$ witnessing $\partial_c\phi(\eta) \notin U(\zeta, r)$. By Observation 5.5, it suffices to assume $(L')^2$, $A' < r/3$. Pulling γ back by $\bar{\phi}$ gives a, possibly not continuous, $(LL', LA' + A)$–quasi-geodesic in η. Tame $\bar{\phi}(\gamma)$ to get a continuous $(LL', 2L(L' + A') + 2A)$–quasi-geodesic $\hat{\gamma}$ such that the Hausdorff distance between $\hat{\gamma}$ and $\bar{\phi}(\gamma)$ is at most $L(L' + A') + A$. Define $L'' := LL'$ and $A'' := 2L(L' + A') + 2A$.

Let $\alpha := \alpha^{(\partial_c\phi)^{-1}(\zeta)}$. Since $\hat{\gamma} \in \eta \in U((\partial_c\phi)^{-1}(\zeta), R)$, there exists $x \in \alpha$ such that $d(o, x) \geq R$ and $d(x, \hat{\gamma}) \leq \kappa(\rho_{(\partial_c\phi)^{-1}(\zeta)}, L'', A'')$. This means $d(\bar{\phi}(\gamma), x) \leq J_0 := \kappa(\rho_{(\partial_c\phi)^{-1}(\zeta)}, L'', A'') + L(L' + A') + A$, so:

$$\begin{aligned} d(\gamma, \phi(x)) \leq J_1 &:= LJ_0 + 2A \\ &= L\left(\kappa(\rho_{(\partial_c\phi)^{-1}(\zeta)}, L'', A'') + A''/2\right) + 2A \end{aligned}$$

We also know $d(\phi(x), o') \geq R/L - A$.

The quasi-isometry ϕ sends the geodesic α to an (L, A)–quasi-geodesic $\phi(\alpha)$ asymptotic to $\alpha' := \alpha^\zeta$ with $\phi(\alpha)_0 = \phi(o) = o'$. Tame $\phi(\alpha)$ to produce a continuous $(L, 2L+2A)$–quasi-geodesic $\hat{\alpha}$ at Hausdorff distance at most $L+A$ from $\phi(\alpha)$. Since $\hat{\alpha} \in \zeta$ we have that $\hat{\alpha}$ is contained in the $\kappa'(\rho_\zeta, L, 2L + 2A)$–neighborhood of α', so $\phi(\alpha)$ is contained in the J_2–neighborhood of α' for $J_2 := \kappa'(\rho_\zeta, L, 2L+2A)+L+A$. In particular, $d(\phi(x), \alpha') \leq J_2$. Let x'' be the closest point of α' to $\phi(x)$, so that $d(\gamma, x'') \leq J_1 + J_2$ and $d(o', x'') \geq R/L - A - J_2$. By Lemma 4.5, since γ is an (L', A')–quasi-geodesic, if y is the last point of α' such that $d(\gamma, y) = \kappa(\rho_\zeta, L', A')$ then:

$$d(o', y) \geq R/L - A - J_2 - \Lambda(J_1 + J_2) - \lambda(\rho_\zeta, L', A') \tag{6}$$

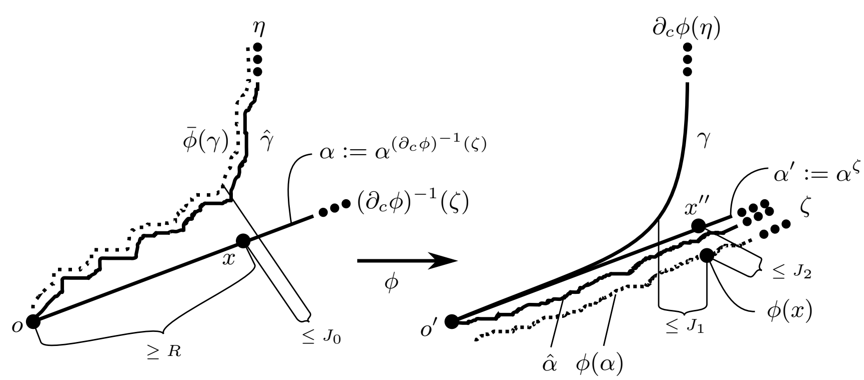

FIGURE 3. Setup for Theorem 6.1

The contraction function for a set determines a bound for a Morse function, and vice versa [2], so there is some μ depending on ρ_ζ such that α^ζ is μ–Morse. The Morse function for $\alpha^{(\partial_c\phi)^{-1}(\zeta)}$ can then be bounded in terms of L, A, and μ, which means that $\rho_{(\partial_c\phi)^{-1}(\zeta)}$ can be bounded in terms of L, A and ρ_ζ. Therefore, everything except R in (6) can be bounded in terms of L, A, r, and ρ_ζ, so, given L, A, r, and ρ_ζ we can choose R large enough to guarantee $d(o', y) > r$. For such an R, we have $\partial_c\phi(\eta) \in U(\zeta, r)$ for every $\eta \in U((\partial_c\phi)^{-1}(\zeta), R)$. ☐

7. Comparison to other topologies

Definition 7.1. Let X be a proper geodesic metric space. Take $\zeta \in \partial_c X$. Fix a geodesic ray $\alpha \in \zeta$. For each $r \geq 1$ define $V(\zeta, r)$ to be the set of points $\eta \in \partial_c X$ such that for every geodesic ray $\beta \in \eta$ we have $d(\beta, \alpha \cap N_r^c o) \leq \kappa(\rho_\zeta, 1, 0)$.

The same argument as Proposition 5.6 shows that $\{V(\zeta, n) \mid n \in \mathbb{N}\}$ gives a neighborhood basis at ζ for a topology $\mathcal{FG}$ on $\partial_c X$. We call $\mathcal{FG}$ the *topology of fellow-travelling geodesics*. It is immediate from the definitions that $\mathcal{FQ}$ is a refinement of $\mathcal{FG}$. The topology $\mathcal{FG}$ need not be preserved by quasi-isometries of X [7]. It is an open question whether $\mathcal{FG}$ is preserved by quasi-isometries when X is the Cayley graph of a finitely generated group.

One might also try to take $V'(\zeta, r)$ to be the set of points $\eta \in \partial_c X$ such that for *some* geodesic ray $\beta \in \eta$ we have $d(\beta, \alpha \cap N_r^c o) \leq \kappa(\rho_\zeta, 1, 0)$. Let $\mathcal{FG}'$ denote the resulting topology. Beware that in general $\{V'(\zeta, r) \mid r \geq 1\}$ is only a filter base converging to ζ, not necessarily a neighborhood base of ζ in $\mathcal{FG}'$; the sets $V'(\zeta, r)$ might not be neighborhoods of ζ.

Lemma 7.2. *Let X be a proper geodesic metric space. Let $\partial_\rho X = \{\zeta \in \partial_c X \mid \rho_\zeta \leq \rho\}$, i.e. $\zeta \in \partial_\rho X$ if all geodesics $\alpha \in \zeta$ are ρ–contracting. The topologies on $\partial_\rho X$ generated by taking, for each $\zeta \in \partial_\rho X$ and $r \geq 1$, the sets $U(\zeta, r) \cap \partial_\rho X$, $V(\zeta, r) \cap \partial_\rho X$, or $V'(\zeta, r) \cap \partial_\rho X$, are equivalent.*

Proof. For each ζ and r we have $U(\zeta, r) \cap \partial_\rho X \subset V(\zeta, r) \cap \partial_\rho X \subset V'(\zeta, r) \cap \partial_\rho X$ by definition.

Given that points in $V'(\zeta, r) \cap \partial_\rho X$ and $U(\zeta, r) \cap \partial_\rho X$ are uniformly contracting, a straightforward application of Lemma 4.5 shows that for all ζ and r, for all sufficiently large R we have $V'(\zeta, R) \cap \partial_\rho X \subset U(\zeta, r) \cap \partial_\rho X$. Also since points of $V'(\zeta, R) \cap \partial_\rho X$ are uniformly contracting, these do, in fact, give a neighborhood basis at ζ for the induced topology, as in Proposition 5.6. □

Proposition 7.3. *Let X be a proper geodesic metric space. If X is hyperbolic then $\partial_c^{\mathcal{FQ}} X \cong \partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$, and these are homeomorphic to the Gromov boundary. If X is CAT(0) then $\partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$, and these are homeomorphic to the subset of the visual boundary of X consisting of endpoints of contracting geodesic rays, topologized as a subspace of the visual boundary.*

Proof. For a description of a neighborhood basis for points in the Gromov or visual boundary see [6, III.H.3.6] and [6, II.8.6], respectively. Note that these are equivalent to the neighborhood bases for $\mathcal{FG}'$.

The claim for hyperbolic spaces follows from Lemma 7.2, because geodesics in a hyperbolic space are uniformly contracting.

If X is CAT(0) then $\partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$ because there is a unique geodesic ray in each asymptotic equivalence class. □

More generally, $\partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$ if X is a proper geodesic metric space with the property that every geodesic ray in X is either not contracting or has contraction function bounded by a constant. This follows by the same argument as in [7].

Next, we recall the *direct limit topology*, $\mathcal{DL}$, on $\partial_c X$ of Charney and Sultan [8] and Cordes [9].

For a given contraction function ρ consider the set $\partial_\rho X$ of points ζ in $\partial_c X$ such that one can take $\rho_\zeta \leq \rho$; in other words, each geodesic ray $\alpha \in \zeta$ is ρ–contracting. The topologies $\mathcal{FQ}$, $\mathcal{FG}$, and $\mathcal{FG}'$ on $\partial_\rho X$ coincide by Lemma 7.2. For $\rho \leq \rho'$ the inclusion $\partial_\rho X \hookrightarrow \partial_{\rho'} X$ is continuous, and $\partial_c X$, as a set, is the direct limit of this system of inclusions over all contraction functions.

Let $\mathcal{DL}$ be the direct limit topology on $\partial_c X$, that is, the finest topology on $\partial_c X$ such that all of the inclusion maps $\partial_\rho X \hookrightarrow \partial_c X$ are continuous.

Proposition 7.4. *$\mathcal{DL}$ is a refinement of $\mathcal{FQ}$.*

Proof. The universal property of the direct limit topology says that a map from the direct limit is continuous if and only if the precomposition with each inclusion map is continuous. Thus, it suffices to show the inclusion $\partial_\rho X \hookrightarrow \partial_c^{\mathcal{FQ}} X$ is continuous. This is clear from Lemma 7.2, since we can take the topology on $\partial_\rho X$ to be the subspace topology induced from $\partial_\rho X \hookrightarrow \partial_c^{\mathcal{FQ}} X$. □

Lemma 7.5. *$\partial_c^{\mathcal{DL}} X$ is homeomorphic to Cordes's Morse boundary.*

Proof. Cordes considers Morse geodesic rays, and defines the Morse boundary to be the set of asymptotic equivalence classes of Morse geodesic rays based at o, topologized by taking the direct limit topology of the system of uniformly Morse subsets.

Arzhantseva, Cashen, Gruber, and Hume [2] show that a set is Morse if and only if it is contracting, and bound the Morse function in terms of the contraction function and vice versa. Thus, a collection of uniformly Morse rays is contained in a collection of uniformly contracting rays, and vice versa. It follows as in [9, Remark 3.4] that the direct limit topology over uniformly Morse points and the direct limit topology over uniformly contracting points agree on $\partial_c X$. □

As with the other topologies, if X is hyperbolic then $\partial_c^{\mathcal{DL}} X$ is homeomorphic to the Gromov boundary. Thus, if X is a proper geodesic hyperbolic metric space then all of the above topologies yield a compact contracting boundary. Conversely, Murray [17] showed if X is a complete CAT(0) space admitting a properly discontinuous, cocompact, isometric group action, and if $\partial_c^{\mathcal{DL}} X$ is compact and non-empty, then X is hyperbolic.

Question 7.6. *If X is a proper geodesic metric space admitting a properly discontinuous, cocompact, isometric group action and $\partial_c^{\mathcal{FQ}} X$ is non-empty and compact, must X be hyperbolic?*

8. Metrizability for group boundaries

In this section let G be a finitely generated group with nonempty contracting boundary. Consider the Cayley graph of G with respect to some fixed generating set, which is a proper geodesic metric space we again denote G, and take the basepoint to be the vertex $\mathbb{1}$ corresponding to the identity element of the group.

There is a natural action of G on $\partial_c^{\mathcal{FQ}} G$ by homeomorphisms defined by sending $g \in G$ to the map that takes $\zeta \in \partial_c G$ to the equivalence class of the quasi-geodesic that is the concatenation of a geodesic from $\mathbb{1}$ to g and the geodesic $g\alpha^\zeta$.

The following two results generalize results of Murray [17] for the case of $\partial_c^{\mathcal{DL}} X$ when X is CAT(0). See also [15].

Proposition 8.1. *G is virtually cyclic if and only if $G \curvearrowright \partial_c^{\mathcal{FQ}} G$ has a finite orbit.*

Proof. If G is virtually cyclic then $|\partial_c G| = 2$ and every orbit is finite.

Conversely, if G has a finite orbit then it has a finite index subgroup that fixes a point in $\partial_c G$. The inclusion of a finite index subgroup is a quasi-isometry, so we may assume that G fixes a point $\zeta \in \partial_c G$.

Let $\alpha \in \zeta$ be geodesic and ρ–contracting. Let β be an arbitrary geodesic ray or segment with $\beta_0 = \mathbb{1}$. Since $G\zeta = \zeta$, for all $n \in \mathbb{N}$ the geodesic rays α and $\beta_n\alpha$ are asymptotic. By Theorem 3.4, α and $\beta_n\alpha$ eventually stay within distance κ'_ρ of one another. Truncate α and $\beta_n\alpha$ when their distance is κ'_ρ. By Lemma 3.6, these segments are contracting, and they form a geodesic almost triangle with $\beta_{[0,n]}$, so, by Lemma 3.8, $\beta_{[0,n]}$ is ρ'–contracting for some $\rho' \asymp \rho$ depending only on ρ. Since this is true uniformly for all n, β is ρ'–contracting. Since β was arbitrary and G is homogeneous, every geodesic in G is uniformly contracting, which means G is hyperbolic and $\partial_c^{\mathcal{FQ}} G$ is the Gromov boundary. If G is hyperbolic and not virtually cyclic then its boundary is uncountable and every orbit is dense, hence infinite. □

Proposition 8.2. *Suppose $|\partial_c^{\mathcal{FQ}} G| > 2$, and fix a point $\eta \in \partial_c G$. For every $\zeta \in \partial_c G$ and every $r \geq 1$ there exists an $R' \geq 1$ such that for all $R_2 \geq R_1 \geq R'$ there exist $g \in G$ such that $\zeta \in U(g\eta, R_2) \subset U(g\eta, R_1) \subset U(\zeta, r)$.*

Corollary 8.3. *$\partial_c^{\mathcal{FQ}} G$ is separable.*

Corollary 8.4. *If G is not virtually cyclic then $G \curvearrowright \partial_c^{\mathcal{FQ}} G$ is minimal, that is, every orbit is dense.*

Remark. For the corollaries we just need to know that we can push η into $U(\zeta, r)$ via the group action. The stronger statement of Proposition 8.2 is used in Proposition 8.5 to upgrade first countable and separable to second countable. The reason for having two parameters R_1 and R_2 is to account for the possibility that $U(g\eta, R_1)$ is not an open set.

Proof of Proposition 8.2. By Proposition 8.1, $G \curvearrowright \partial_c^{\mathcal{FQ}} G$ does not have a finite orbit, so there exists a $g' \in G$ with $\eta' := g'\eta \neq \eta$. Let β be a geodesic joining η' and η. It suffices to assume $\beta_0 = \mathbb{1}$; otherwise, we could consider $\beta' := \beta_0^{-1}\beta$, which is a geodesic with $\beta'_0 = \mathbb{1}$ and endpoints in $G\eta$.

Let $\alpha := \alpha^\zeta$ be the geodesic representative of ζ. Choose ρ so that α, $\beta_{[0,\infty)}$, and $\bar{\beta}_{[0,-\infty)}$ are all ρ–contracting.

When t is a non-negative integer, α_t is a vertex of the Cayley graph, which corresponds to an element of G. Let $\alpha_t\beta_{[0,\infty)}$ and $\alpha_t\bar{\beta}_{[0,-\infty)}$ denote the translates of $\beta_{[0,\infty)}$ and $\bar{\beta}_{[0,-\infty)}$, respectively, by the group element corresponding to α_t.

For each integral $t \gg 0$, at most one of $\alpha_t\beta_{[0,\infty)}$ and $\alpha_t\bar{\beta}_{[0,-\infty)}$ remains in the closed $\kappa'(\rho, 1, 0)$–neighborhood of $\alpha_{[0,t]}$ for distance greater than $2\kappa'(\rho, 1, 0)$, otherwise we contradict the fact that $\alpha_t\beta$ is a geodesic. Define $g_t := \alpha_t$ if $\alpha_t\beta_{[0,\infty)}$ does not remain in the closed $\kappa'(\rho, 1, 0)$–neighborhood of $\alpha_{[0,t]}$ for distance greater than $2\kappa'(\rho, 1, 0)$. Otherwise, define $g_t := \alpha_t g'$. For each $s \in \mathbb{N}$ consider a geodesic triangle with sides $\alpha_{[0,t]}$, $g_t\beta_{[0,s]}$, and a geodesic $\delta^{s,t}$ joining α_0 to $g_t\beta_s$. By Lemma 3.6, the first two sides are uniformly contracting, so $\delta^{s,t}$ is as well, by Lemma 3.8. Since G is proper, for each fixed t a subsequence of the $\delta^{s,t}$ converges to a contracting geodesic $\delta^t \in g_t\eta$. Moreover, since the $\delta^{s,t}$ are uniformly contracting, the contraction function for δ^t does not depend on t. By Lemma 4.4, there is a ρ' independent of t such that the geodesic representative $\alpha^{g_t\eta}$ of $g_t\eta$ is ρ'–contracting. Furthermore, the defining condition for g_t guarantees that there is a C independent of t such that the geodesic representative $\alpha^{g_t\eta}$ comes within distance $\kappa(\rho,1,0)$ of α outside of $N_{t-C}\mathbb{1}$, which implies that $\alpha^{g_t\eta}_{[0,t-C]}\subset \bar{N}_{2\kappa'(\rho,1,0)}\alpha_{[0,t-C]}$.

First we give a condition that implies $\zeta\in U(g_t\eta,R)$. Suppose:

$$t\geq R+C+2\Lambda(\kappa'(\rho,\sqrt{R/3},R/3)+\kappa'(\rho,1,0))+\lambda(\rho',\sqrt{R/3},R/3) \tag{7}$$

Suppose that $\gamma\in\zeta$ is a continuous (L,A)–quasi-geodesic. By Observation 5.5 is suffices to consider L^2, $A<R/3$. By Corollary 4.3, $\gamma\subset\bar{N}_{\kappa'(\rho,L,A)}\alpha$, so there is a point γ_a that is $(2\kappa'(\rho,L,A)+2\kappa'(\rho,1,0))$–close to $\alpha^{g_t\eta}_{t-C}$. By Lemma 4.5, γ comes $\kappa(\rho',L,A)$–close to $\alpha^{g_t\eta}$ outside the ball around $\mathbb{1}$ of radius $t-C-\Lambda(2\kappa'(\rho,L,A)+2\kappa'(\rho,1,0))-\lambda(\rho',L,A)$. By (7), this is at least R. Since $\gamma\in\zeta$ was arbitrary, $\zeta\in U(g_t\eta,R)$.

Next, we give a condition that implies $U(g_t\eta,R)\subset U(\zeta,r)$. Suppose:

$$t-C\geq R\geq r+\Lambda(2\kappa'(\rho,1,0)+2\kappa'(\rho',\sqrt{r/3},r/3))+\lambda(\rho,\sqrt{r/3},r/3) \tag{8}$$

Suppose that γ is a continuous (L,A)–quasi-geodesic such that $[\gamma]\in U(g_t\eta,R)$. By Observation 5.5, it suffices to consider L^2, $A<r/3$. By definition, γ comes $\kappa(\rho',L,A)$ close to $\alpha^{g_t\eta}$ outside $N_R\mathbb{1}$, so some point γ_b is $2\kappa'(\rho',L,A)$–close to $\alpha^{g_t\eta}_R$, which implies that $d(\gamma_b,\alpha_R)\leq(2\kappa'(\rho',L,A)+2\kappa'(\rho,1,0))$. Now apply Lemma 4.5 to see that γ comes $\kappa(\rho,L,A)$–close to α outside the ball around $\mathbb{1}$ of radius $R-\Lambda(2\kappa'(\rho,1,0)+2\kappa'(\rho',L,A))-\lambda(\rho,L,A)$, which is at least r, by (8). Thus, $U(g_t\eta,R)\subset U(\zeta,r)$.

Equipped with these two conditions, we finish the proof. The contraction functions ρ and ρ' are determined by ζ and η. Given these and any $r\geq 1$, define R' to be the right hand side of (8). Given any $R_2\geq R_1\geq R'$, it suffices to define $g:=g_t$ for any t large enough to satisfy both condition (7) for $R=R_2$ and condition (8) for $R=R_1$. ☐

Proposition 8.5. *$\partial_c^{\mathcal{FQ}}G$ is second countable.*

Proof. If G is virtually cyclic then $\partial_c^{\mathcal{FQ}}G$ is the discrete space with two points, and we are done. Otherwise, fix any $\eta\in\partial_c G$. For each $g\in G$ and $n\in\mathbb{N}$ choose an open set $U_{g,n}$ such that $U(g\eta,\psi(\rho_{g\eta},n))\subset U_{g,n}\subset U(g\eta,n)$ as in Corollary 5.9.

Let U be a non-empty open set and let ζ be a point in U. By definition of $\mathcal{FQ}$, there exists an $r\geq 1$ such that $U(\zeta,r)\subset U$. Let R' be the constant of Proposition 8.2 for ζ and r, and let $R'\leq R_1\in\mathbb{N}$. As noted there, there exists a sublinear ρ' such that the points $g_t\eta$ in the proof of Proposition 8.2 are all ρ'–contracting. Define $R_2:=\psi(\rho',R_1)\geq\psi(\rho_{g_t\eta},R_1)$. Combining Proposition 8.2 and the definition of the sets $U_{g,n}$, there exists $g\in G$ such that:

$$\zeta\in U(g\eta,R_2)\subset U_{g,R_1}\subset U(g\eta,R_1)\subset U(\zeta,r)\subset U$$

Thus, $\mathcal{U}:=\{U_{g,n}\mid g\in G,\, n\in\mathbb{N}\}$ is a countable basis for $\partial_c^{\mathcal{FQ}}G$. ☐

Corollary 8.6. *$\partial_c^{\mathcal{FQ}}G$ is metrizable.*

Proof. $\partial_c^{\mathcal{FQ}}G$ is second countable by Proposition 8.5, regular by Proposition 5.12, and Hausdorff by Proposition 5.11. The Urysohn Metrization Theorem says every second countable, regular, Hausdorff space is metrizable. ☐

9. Dynamics

Definition 9.1. An element $g \in G$ is *contracting* if $\mathbb{Z} \to G : n \mapsto g^n$ is a quasi-isometric embedding whose image is a contracting set.

We use g^∞ and $g^{-\infty}$ to denote the equivalence classes of the contracting quasi-geodesic rays based at $\mathbb{1}$ corresponding to the non-negative powers of g and non-positive powers, respectively. These are distinct points in $\partial_c G$.

Lemma 9.2. *Given a contracting element $g \in G$, an $r \geq 1$, and a point $\zeta \in \partial_c^{\mathcal{FQ}} G \setminus \{g^\infty, g^{-\infty}\}$ there exists an $R' \geq 1$ such that for every $R \geq R'$ and every $n \in \mathbb{N}$ we have $U(\zeta, R) \subset g^{-n} U(g^n \zeta, r)$.*

Proof. Since g is contracting there is a sublinear function ρ such that all geodesic segments joining powers of g as well as geodesic rays based at $\mathbb{1}$ going to g^∞, $g^{-\infty}$, or $g^m\zeta$ for any $m \in \mathbb{Z}$ are all ρ–contracting.

Consider a geodesic triangle with sides $g^{-m}\alpha^{g^m\zeta}$, α^ζ, and a geodesic from $\mathbb{1}$ to g^{-m} for arbitrary $m \in \mathbb{Z}$. All sides are ρ–contracting, and such a triangle is B–thin for some B independent of m. Thus, for sufficiently large s', independent of m, the point $\alpha^\zeta_{s'}$ is more than B–far from $\langle g\rangle$, hence B–close to $g^{-m}\alpha^{g^m\zeta}$. Since α^ζ and $g^{-m}\alpha^{g^m\zeta}$ are asymptotic, they eventually come $\kappa(\rho,1,0)$–close and then stay $\kappa'(\rho,1,0)$–close thereafter. Theorem 3.4 says the first time they come $\kappa(\rho,1,0)$ close occurs no later than $s'+\rho'(B)$. Take $R' \geq s'+\rho'(B)$, which guarantees $d(\alpha^\zeta_s, g^{-m}\alpha^{g^m\zeta}) \leq \kappa'(\rho,1,0)$ for all $m \in \mathbb{Z}$ and all $s \geq R'$.

Let L' and A' be the constants of Lemma 4.6 for ρ, $L=\sqrt{r/3}$ and $A = r/3$, and let Λ and λ be as in Lemma 4.5. Take $T := 1 + r + \Lambda(\kappa(\rho,L',A') + \kappa'(\rho,1,0)) + \lambda(\rho,1,0)$. We require further that R' is larger than L' and A' and large enough so that for all $s \geq R'$ we have $d(\alpha^\zeta_s, \langle g\rangle) > T + \kappa'(\rho,1,0)$.

Suppose, for a contradiction, that there exist some $n \in \mathbb{N}$ and $R \geq R'$ such that there exists a point $\eta \in U(\zeta,R) \setminus g^{-n}U(g^n\zeta, r)$. Since $\eta \notin g^{-n}U(g^n\zeta,r)$, for some L, A there exists a continuous (L,A)–quasi-geodesic $\gamma \in g^n\eta$ that does not come $\kappa(\rho,L,A)$–close to $\alpha^{g^n\zeta}$ outside $N_r\mathbb{1}$. By Observation 5.5, it suffices to consider the case that $L < \sqrt{r/3}$ and $A < r/3$.

As in Lemma 4.6, construct a continuous (L',A')–quasi-geodesic δ that first follows a geodesic from $\mathbb{1}$ towards g^{-n}, then a geodesic segment of length $\kappa(\rho,L,A)$, and then follows a tail of $g^{-n}\gamma$. Since it shares a tail with $g^{-n}\gamma$, we have $\delta \in \eta$. Since $\eta \in U(\zeta,R)$, there is some $s \geq R$ such that δ comes within distance $\kappa(\rho,L',A')$ of α^ζ_s. Our choice of R' guarantees that $d(\alpha^\zeta_s, g^{-n}\alpha^{g^n\zeta}) \leq \kappa'(\rho,1,0)$ and $d(\alpha^\zeta_s,\langle g\rangle) > T+\kappa'(\rho,1,0)$. The latter implies that the point of δ close to α^ζ_s is a point of $g^{-n}\gamma$, so there is a point of $g^{-n}\gamma$ that comes within distance $\kappa(\rho,L',A')+\kappa'(\rho,1,0)$ of a point x of $g^{-n}\alpha^{g^n\zeta}$ such that $d(x,\langle g\rangle) \geq T$. Lemma 4.5 says that there is a point y on $g^{-n}\alpha^{g^n\zeta}$ such that $d(y, g^{-n}\gamma) = \kappa(\rho,L,A)$ and $d(x,y) \leq \Lambda(\kappa(\rho,L',A') + \kappa'(\rho,1,0)) + \lambda(\rho,L,A)$. The definition of T implies $d(y,g^{-n}) \geq d(y,\langle g\rangle) \geq d(x,\langle g\rangle) - d(x,y) > r$. But then $g^n y$ is a point of $\alpha^{g^n\zeta}$ with $d(g^n y, \mathbb{1}) > r$ and $d(g^n y, \gamma) = \kappa(\rho,L,A)$, contradicting the definition of γ. □

Lemma 9.3. *Given a contracting element $g \in G$, an $r \geq 1$, and a point $\zeta \in \partial_c^{\mathcal{FQ}} G \setminus \{g^{-\infty}\}$ there exist constants $R' \geq 1$ and N such that for all $R \geq R'$ and $n \geq N$ we have $g^n U(\zeta,R) \subset U(g^\infty, r)$.*

Proof. The lemma is easy if $\zeta = g^\infty$, so assume not. Since g is contracting the geodesics $\alpha^{g^n\zeta}$ are uniformly contracting. Let ρ be a sublinear function such that α^{g^∞}, $\alpha^{g^{-\infty}}$, and all $\alpha^{g^n\zeta}$ are ρ–contracting. Since these geodesics are uniformly contracting, ideal geodesic triangles with vertices g^∞, $g^{-\infty}$, and $g^n\zeta$ are uniformly thin, independent of n. Thus, for N sufficiently large and for all $n \geq N$ we have that $\alpha^{g^n\zeta}$ stays $\kappa'(\rho,1,0)$ close to α^{g^∞} for distance greater than $S' := 1+r+\lambda(\rho,\sqrt{r/3},r/3)+\Lambda(\kappa'(\rho,\sqrt{r/3},r/3)+\kappa'(\rho,1,0))$, where Λ and λ are as in Lemma 4.5.

Suppose that $\eta \in U(g^n\zeta, S)$ for some $n \geq N$ and $S \geq S'$. Let $\gamma \in \eta$ be a continuous (L,A)–quasi-geodesic for some L^2, $A \leq r/3$. By hypothesis, γ comes $\kappa(\rho,L,A)$–close to $\alpha^{g^n\zeta}$ outside $N_S\mathbb{1}$. Therefore, γ stays $\kappa'(\rho,L,A)$–close to $\alpha^{g^n\zeta}$ in $N_S\mathbb{1}$. By our choice of N, this implies γ stays $(\kappa'(\rho,L,A)+\kappa'(\rho,1,0))$–close to α^{g^∞} in $N_S\mathbb{1}$. By our choice of S and Lemma 4.5, γ comes $\kappa(\rho,L,A)$–close to α^{g^∞} outside the neighborhood of $\mathbb{1}$ of radius:

$$\begin{aligned} S &- \Lambda(\kappa'(\rho,L,A)+\kappa'(\rho,1,0)) - \lambda(\rho,L,A) \\ &\geq S' - \Lambda(\kappa'(\rho,\sqrt{r/3},r/3)+\kappa'(\rho,1,0)) - \lambda(\rho,\sqrt{r/3},r/3) > r \end{aligned}$$

Since γ was arbitrary, $\eta \in U(g^\infty, r)$, thus $U(g^n\zeta, S) \subset U(g^\infty, r)$.

By Lemma 9.2, given g, S', and ζ there exists an R' such that for all $R \geq R'$ and every $n \in \mathbb{N}$ we have $U(\zeta,R) \subset g^{-n}U(g^n\zeta,S')$. Thus, for this R' and N as above we have, for all $n \geq N$ and $R \geq R'$, that $g^nU(\zeta,R) \subset U(g^n\zeta,S') \subset U(g^\infty,r)$. □

Theorem 9.4 (Weak North-South dynamics for contracting elements)**.** *Let $g \in G$ be a contracting element. For every open set V containing g^∞ and every compact set $C \subset \partial_c^{\mathcal{FQ}}G\setminus\{g^{-\infty}\}$ there exists an N such that for all $n \geq N$ we have $g^nC \subset V$.*

We remark that if Theorem 9.4 were true for *closed* sets C and G contained contracting elements without common powers then we could play ping-pong to produce a free subgroup of G. Such a result cannot be true in this generality because there are Tarski Monsters such that every non-trivial element is Morse, hence, contracting [18, Theorem 1.12].

Proof of Theorem 9.4. Since V is an open set containing g^∞ there exists some $r > 0$ such that $U(g^\infty,r) \subset V$. For this r and for each $\zeta \in C$ there exist R_ζ and N_ζ as in Lemma 9.3 such that for all $n \geq N_\zeta$ we have $g^nU(\zeta,R_\zeta) \subset U(g^\infty,r)$. By Proposition 5.6, $U(\zeta,R_\zeta)$ is a neighborhood of ζ, so there exists an open set U'_ζ such that $\zeta \in U'_\zeta \subset U(\zeta,R_\zeta)$. The collection $\{U'_\zeta \mid \zeta \in C\}$ is an open cover of C, which is compact, so there exists a finite subset C' of C such that $\{U'_\zeta \mid \zeta \in C'\}$ covers C. Define $N := \max_{\zeta\in C'} N_\zeta$. For every $n \geq N$ we then have:

$$\begin{aligned} g^nC &\subset g^n(\bigcup_{\zeta\in C'} U'_\zeta) \subset g^n(\bigcup_{\zeta\in C'} U(\zeta,R_\zeta)) \\ &= \bigcup_{\zeta\in C'} g^nU(\zeta,R_\zeta) \subset U(g^\infty,r) \subset V \end{aligned}$$ □

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