CONFORMAL DIMENSION AND HYPERBOLIC GROUPS

JOHN M. MACKAY

ABSTRACT. These notes are based on a minicourse given in June 2022 at Institut Henri Poincaré, Paris as part of the thematic trimester programme *Groups acting on fractals, hyperbolicity and self-similarity*.

I thank the referee for helpful comments.

Conformal dimension is an invariant of metric spaces introduced by Pansu in 1989, motivated by studying the boundary at infinity of rank one symmetric spaces [Pan89]. It has subsequently proved of importance in the study of boundaries of (Gromov) hyperbolic groups.

In this minicourse we will consider the following topics, aiming to give motivation and an idea of some topics of current interest, without getting into all the technical details.

CONTENTS

1. What is conformal dimension?

1.1. **Background.** Before we dive in to study Gromov hyperbolic groups, let's remind ourselves of some classical ideas which serve as basic examples and inspiration for the general case.

In the Poincaré ball model for (real) hyperbolic space $\mathbb{H}^n$, $n \geq 2$, we naturally see the unit sphere $\mathbb{S}^{n-1}$ as the "boundary at infinity" of the space, $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$, see Figure 1.

Date: January 11, 2023.

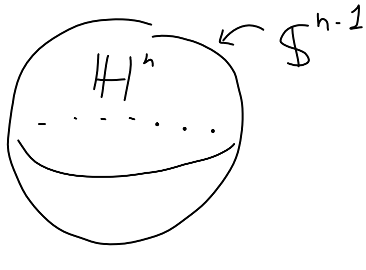

FIGURE 1. Poincaré ball model for $\mathbb{H}^n$

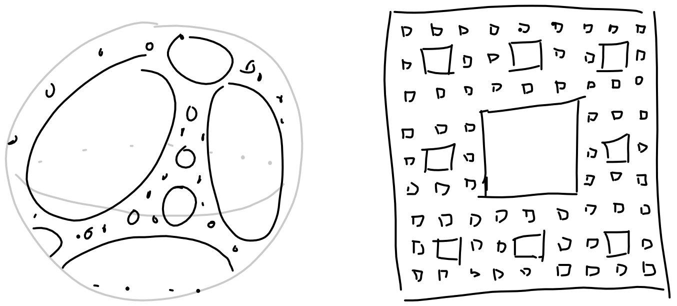

FIGURE 2. A limit set homeomorphic to the standard Sierpiński carpet

Moreover, one can identify the group $\mathrm{Isom}(\mathbb{H}^n)$ of isometries of $\mathbb{H}^n$ with the group of Möbius transformations of $\mathbb{S}^{n-1}$. Möbius transformations of $\mathbb{S}^{n-1}$ can be defined as those maps that are the composition of finitely many reflections in hyperplanes and inversions in spheres which preserve $\mathbb{S}^{n-1}$. In particular, every Möbius transformation is conformal; in fact, when $n \geq 3$, every conformal homeomorphism of $\mathbb{S}^{n-1}$ is a Möbius transformation (see e.g. [TV85]).

Given a discrete subgroup $G \leq \mathrm{Isom}(\mathbb{H}^n)$, one can consider its limit set $\Lambda G = \partial_\infty \mathbb{H}^n \cap \overline{G \cdot o}$ which is the set of points in the boundary which can be reached as a limit of point in some (any) orbit $G \cdot o$.

Example 1.1. *If $G = \pi_1(M)$ for some compact hyperbolic 3-manifold with totally geodesic boundary, then its universal cover $\tilde{M}$ is a convex subset of $\mathbb{H}^3$ with limit set ΛG homeomorphic to a Sierpiński carpet, see Figure 2.*

As in the example, the limit set is often a fractal object and, as with other fractals, its Hausdorff dimension is a useful invariant to help study the group, connected with the growth of orbits.

Keeping this background in mind, we now consider a generalisation due to Gromov.

1.2. **Gromov hyperbolic groups.** We give a quick recap of basic notions of Gromov hyperbolicity. For simplicity, we assume throughout that X is a geodesic metric space that is proper, i.e. closed balls are compact.

Definition 1.2. *We say X is* (Gromov) hyperbolic *if all geodesic triangles are δ-thin for some uniform $\delta \geq 0$, see Figure 3.*

The boundary at infinity $\partial_\infty X$ *of X is*

$$\partial_\infty X = \{\gamma : [0, \infty) \to X : \gamma \text{ is geodesic}, \gamma(0) = o\} / \sim$$

where $o \in X$ is a basepoint and $\sim$ denotes finite Hausdorff distance.

A visual metric *on $\partial_\infty X$ is a metric ρ for which there exists a* visual parameter *$\epsilon > 0$ so that for all $a, b \in \partial_\infty X$ we have $\rho(a, b) \asymp e^{-\epsilon(a \cdot b)_o}$. (Here $A \asymp B$ if $A \preceq B$ and $B \preceq A$ where $A \preceq B$ means there exists $C > 0$ so that $A \leq CB$; we may write $A \preceq_C B$ or $A \asymp_C B$.) The* Gromov product *$(a \cdot b)_o$ can be approximately defined using Figure 4.*

Remark 1.3. If X is δ-hyperbolic, for any $\epsilon \in (0, 1/4\delta)$ there exists a visual metric on $\partial_\infty X$ with visual parameter ϵ.

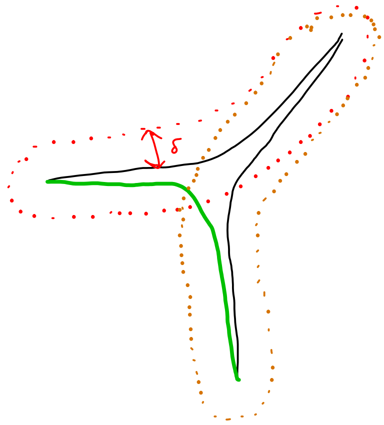

FIGURE 3. Thin triangles

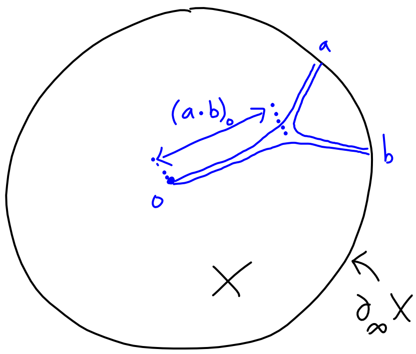

FIGURE 4. Gromov product of boundary points

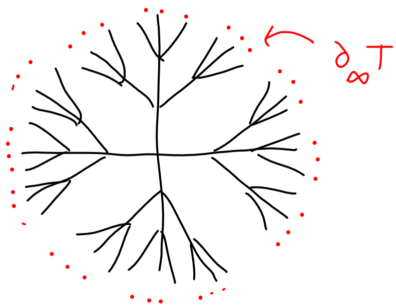

FIGURE 5. Tree with Cantor set boundary

Example 1.4. (1) *As one might expect from the Poincaré ball model, $\partial_\infty \mathbb{H}^n = \mathbb{S}^{n-1}$ where the Euclidean metric on $\mathbb{S}^{n-1}$ is a visual metric with visual parameter $\epsilon = 1$, as in Figure 1. This example helps motivate the terminology "visual metric" since the metric describes how $\partial_\infty X$ looks from the point of view of the basepoint o.*

(2) *If T is a regular tree of fixed degree $d \geq 3$, then T is 0-hyperbolic and $\partial_\infty T$ with any visual metric is a (space homeomorphic to the standard) Cantor set, see Figure 5.*

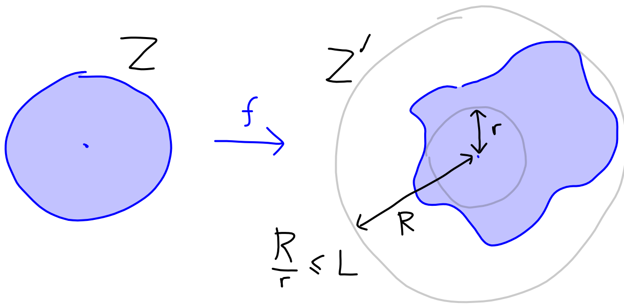

FIGURE 6. Definition of (weak) QS homeomorphism

We now consider groups.

Definition 1.5. *A finitely generated group G is* (Gromov) hyperbolic *if it acts geometrically (i.e., cocompactly, properly, by isometries) on a proper Gromov hyperbolic space X. Set $\partial_\infty G = \partial_\infty X$.*

The problem with this definition of the boundary of G is that there seem to be so many choices: X, basepoint o, ϵ, ρ. How do these affect the boundary?

By the 'Fundamental Lemma of Geometric Group Theory' (also called the Švarc–Milnor Lemma, see e.g. [BH99, I.8.19] and references therein), if such G acts geometrically on two such spaces X and X' then X and X' are *quasi-isometric*: there exists $f : X \to X'$ and $L \geq 1, C \geq 0$ so that for all $x, y \in X$,

$$\frac{1}{L}d(x,y) - C \leq d'(f(x), f(y)) \leq Ld(x,y) + C,$$

and the C-neighbourhood of $f(X)$ equals X'.

To say more, we introduce the following class of homeomorphism.

Definition 1.6. *A homeomorphism $f : Z \to Z'$ between metric spaces $(Z,d), (Z',d')$ is* (weakly) quasisymmetric (QS) *if there exists $L \geq 1$ so that balls get sent to L quasi-balls, see Figure 6. In equations, for all $x, y, z \in Z$, if $d(x,y) \leq d(x,z)$ then $d'(f(x), f(y)) \leq Ld'(f(x), f(z))$. If such an f exists, we say Z and Z' are* quasisymmetric *and write $Z \overset{\text{QS}}{\cong} Z'$.*

Remark 1.7. The definition above refers to "weakly" QS homeomorphisms. The full definition requires control on the distortion of annuli not just balls, but for the spaces we consider the two notions are equivalent.

Example 1.8. *Examples of QS maps include:*

1. *isometries, and bi-Lipschitz homeomorphisms; i.e. maps $f : (Z,d) \to (Z',d')$ for which there exists $L \geq 1$ so that for all $x, y \in Z$, we have $d'(f(x), f(y)) \asymp_L d(x,y)$.*
2. *Snowflake transformations* $\mathrm{id} : (Z,d) \to (Z,d^\epsilon)$*, where $\epsilon > 0$ and d^ϵ is a metric; this is always true for $\epsilon \in (0,1]$.*
3. *Möbius transformations of $\mathbb{S}^n$, $n \geq 2$.*

Since bi-Lipschitz and snowflake transformations are QS, we see that any two choices of visual metric are QS. With a little more work one sees that the

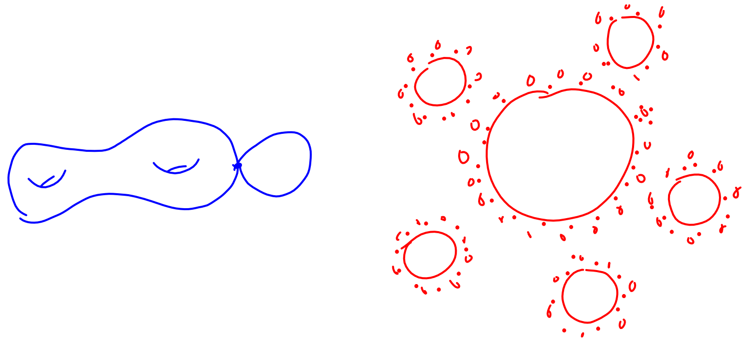

FIGURE 7. Example 1.10(4)

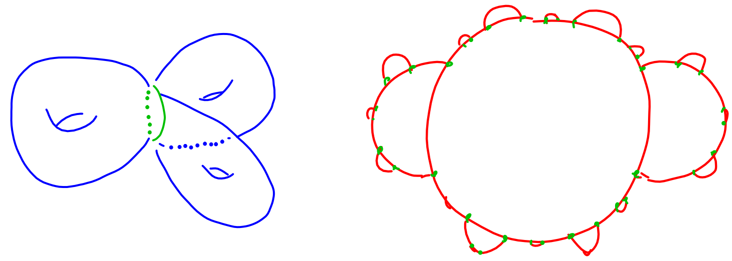

FIGURE 8. Example 1.10(5)

basepoint choice also is just a QS change, so for a given Gromov hyperbolic space X, $\partial_\infty X$ is well-defined up to QS. More generally we have:

Theorem 1.9 ($\Rightarrow$ Efremovitch–Tichonirova, Mostow, Gromov, Paulin; $\Leftarrow$ Paulin, see also Bonk–Schramm [Pau96, BS00]). *Suppose G and G' are hyperbolic groups acting geometrically on hyperbolic spaces X, X'. Then $X \overset{\mathrm{QI}}{\simeq} X'$ if and only if $\partial_\infty X \overset{\mathrm{QS}}{\cong} \partial_\infty X'$. In particular, $\partial_\infty G$ is well-defined up to QS homeomorphism.*

Now we know the boundary is well-defined, let's consider more examples.

Example 1.10. *In the following examples we have $G = \pi_1(X)$ a hyperbolic group with $G \overset{\mathrm{QI}}{\simeq} \tilde{X}$, where $\tilde{X}$ is the universal cover of X, and $\partial_\infty G \overset{\mathrm{QS}}{\cong} \partial_\infty \tilde{X}$.*

1. *If X is a closed hyperbolic n-manifold, then $\partial_\infty G \overset{\mathrm{QS}}{\cong} \mathbb{S}^{n-1}$ with the Euclidean metric, as in Figure 1.*
2. *If X is a figure 8 graph, then $\partial_\infty G$ is a Cantor set as in Figure 5.*
3. *If X is a compact hyperbolic 3-manifold with totally geodesic boundary, then $\partial_\infty G$ looks like the Sierpiński carpet limit set of Figure 2. More generally, for a convex cocompact Kleinian group G, G is Gromov hyperbolic and $\partial_\infty G$ is (QS to) the limit set ΛG.*
4. *If X is a wedge of a genus 2 surface and circle (Figure 7, left), then $\tilde{X}$ is a tree of hyperbolic planes, with $\partial_\infty G$ a "Cantor set of circles" (Figure 7, right).*
5. *If X is three tori with one boundary component all glued together along their boundary circles (Figure 8, left), then $\partial_\infty G$ is a limit of branching circles glued at pairs of points (Figure 8, right).*

The last two groups have 'graph of groups decompositions' or 'splittings': in example 4, the group is the free product of the surface group and $\mathbb{Z}$ (and the boundary is disconnected). In example 5, the group splits over the copy of $\mathbb{Z}$ coming from the circle, and one sees circles glued in the boundary at pairs of points corresponding to ends of conjugates of that $\mathbb{Z}$.

What is known about the relationship between the algebraic properties of a hyperbolic group G and the topological or metric properties of $\partial_\infty G$? In low-dimensions, quite a lot.

Theorem 1.11 (Hopf, Stallings, Dunwoody)**.** *If G is a hyperbolic group, then $\partial_\infty G$ has topological dimension* 0 *if and only if $\partial_\infty G$ is totally disconnected, if and only if $\partial_\infty G$ is homeomorphic to a Cantor set or two point set, if and only if G is virtually free.*

This theorem follows from Stallings' work on ends of groups and Dunwoody's accessiblity for finitely presented groups, combined with a correspondence between connected components of the boundary and ends of G.

Next up from zero-dimensional boundaries, we have:

Theorem 1.12 (Tukia, Casson–Jungreis, Gabai)**.** *If G is a hyperbolic group, then $\partial_\infty G$ is homeomorphic to $\mathbb{S}^1$ if and only if G is virtually cocompact Fuchsian.*

Conjecture 1.13 (Cannon)**.** *If G is a hyperbolic group, then $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$ if and only if G is virtually cocompact Kleinian.*

Away from classification questions, we will use the following result of Bowditch.

Theorem 1.14 (Bowditch)**.** *Suppose G is a one-ended hyperbolic group that is not virtually Fuchsian. Then G splits over a virtually $\mathbb{Z}$ subgroup if and only if there exists a local cut point in $\partial_\infty G$.*

Recall that a local cut point in a metric space Z is a point $z \in Z$ with a connected neighbourhood U so that $U \setminus \{z\}$ is disconnected. In Example 1.10(5) we saw examples of local cut points in $\partial_\infty G$ corresponding to the limit points of $\mathbb{Z}$ subgroups in G.

1.3. **Conformal dimension.** We've discussed the topology of $\partial_\infty G$, but we have extra information coming from the (QS class of the) metric, does this tell us more? The answer is yes, and conformal dimension is one invariant that can show this. Before defining it, we introduce a property of certain metric spaces.

Definition 1.15. *A metric space Z is* (Ahlfors) Q-regular *if there exists a Borel measure μ on Z such that for all $z \in Z$ and $r \in (0, \operatorname{diam} Z)$ we have $\mu(B(z,r)) \asymp r^Q$.*

One can show that such μ must be comparable to Hausdorff Q-measure, and Q is the Hausdorff dimension of Z.

Example 1.16. (1) *$\mathbb{R}^n$ is n-regular, using Lebesgue measure.*

(2) *C, the usual $\frac{1}{3}$ Cantor set, is $Q = \frac{\log 2}{\log 3}$-regular, as can be seen by considering the probability measure which puts $\frac{1}{2}$ mass on each of*

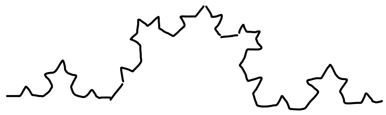

FIGURE 9. von Koch snowflake

$C\cap[0,\frac{1}{3}]$ and $C\cap[\frac{2}{3},1]$, then subdividing to $\frac{1}{4}$ mass on the subintervals of length $\frac{1}{9}$, and so on.

(3) *Coornaert* [Coo93] *shows that if a hyperbolic group G acts geometrically on a geodesic space X, then $\partial_\infty X$ with a visual metric is Ahlfors regular. (Beware, the cocompactness of the action is important! Many examples of non-cocompact Kleinian groups will not have Ahlfors regular limit sets.)*

(4) *If (Z,d) is Q-regular then (Z,d^ϵ) is (Q/ϵ)-regular. For example, if d_{Euc} is the usual metric on $[0,1]$ then $([0,1],d_{\mathrm{Euc}}^{\log_4 3})$, which is bi-Lipschitz to the von Koch snowflake (Figure 9), is $\frac{\log 4}{\log 3}$-regular.*

This last example shows that if $Z \overset{\mathrm{QS}}{\cong} Z'$ with Z Q-regular and Z' Q'-regular, we may have $Q \neq Q'$, so the Hausdorff dimension is not a QS invariant, and indeed can be made arbitrarily large. However, one can try to make the dimension as small as we can.

Definition 1.17 (Pansu [Pan89], compare Bourdon–Pajot [BP03])**.** *The* (Ahlfors regular) conformal dimension *of a metric space Z is*

$$\mathrm{Confdim}(Z) = \inf\left\{ Q' : \exists Q'\text{-}regular\ Z' \overset{\mathrm{QS}}{\cong} Z \right\}.$$

Since being QS is an equivalence relation, conformal definition is a QS invariant, and thus $\mathrm{Confdim}(\partial_\infty G)$ is a well-defined quasi-isometry invariant of a hyperbolic group G.

There are other variations on conformal dimension, but for the study of boundaries of hyperbolic groups, the Ahlfors regular conformal dimension is natural since such boundaries do admit such nice metrics (Example 1.16(3)), and these metrics have an analytically nice structure to work with. For less regular spaces which do not have any Ahlfors regular metric in their QS class, such as $Z = \{0\} \cup [1,2]$, a different variation may be appropriate, such as instead infimizing the Hausdorff dimension over all $Z' \overset{\mathrm{QS}}{\cong} Z$ (whereas with our definition, $\mathrm{Confdim}(Z) = \infty$ as no such Ahlfors regular Z' exists).

Remark 1.18. There are two easy ways to estimate $\mathrm{Confdim}(Z)$. First, if Z is Q-regular, $\mathrm{Confdim}(Z) \leq Q$. Second, since any Q-regular space has Hausdorff dimension Q, and Hausdorff dimension is always at least the topological dimension, $\mathrm{Confdim}(Z) \geq \dim_{\mathrm{top}}(Z)$.

Example 1.19. (1) $\mathrm{Confdim}(\partial_\infty F_n) = 0$ *for the free group F_n by choosing visual metrics on the tree with visual parameter ϵ arbitrarily large. In fact for a hyperbolic group* $\mathrm{Confdim}(\partial_\infty G) = 0$ *if and only if* $\mathrm{Confdim}(\partial_\infty G) < 1$*, if and only if G is virtually free, as follows from Theorem 1.12.*

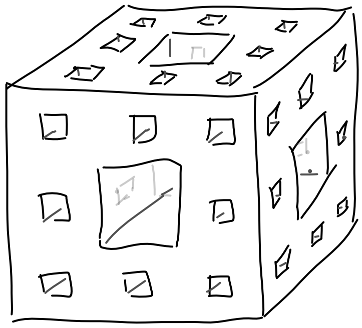

FIGURE 10. Menger sponge

(2) $\mathrm{Confdim}(\partial_\infty \mathbb{H}^n) = \mathrm{Confdim}(\mathbb{S}^{n-1}) = n-1$ *since the Euclidean metric on* $\mathbb{S}^{n-1}$ *is* $(n-1)$*-regular and the topological dimension is* $n-1$.

(3) *If* Z *is* Q*-regular, then* $\mathrm{Confdim}(Z \times [0,1]) = Q+1$. *This is our first non-trivial estimate, and follows from a short modulus argument, which in this course we'll treat as a black box (see e.g.* [MT10, §5.1]*).*

(4) *If* S *is the usual Sierpiński carpet, then since* Z *contains an embedded copy of* $C \times [0,1]$*, with* C *the usual* $\frac{1}{3}$*-Cantor set,* $\mathrm{Confdim}(S) \geq \mathrm{Confdim}(C \times [0,1]) = \log_3(2) + 1$. *(Here we use that* Confdim *is monotone with respect to subsets, which is a little subtle to show.) This estimate is not sharp, but it's surprisingly hard to get a better lower bound, and the exact value of* $\mathrm{Confdim}(S)$ *remains unknown. See Kwapisz* [Kwa20] *for the best known estimates.*

(5) $\mathrm{Confdim}(\partial_\infty \mathbb{H}^2_{\mathbb{C}}) = 4$. *Here* $\mathbb{H}^2_{\mathbb{C}}$ *is the complex hyperbolic space of complex dimension* 2*, real dimension* 4*, which has boundary a topological* 3*-sphere. In a half-space model, the metric on* $\partial_\infty \mathbb{H}^4_{\mathbb{R}}$ *comes from the Euclidean metric on* $\mathbb{R}^3$*, however, the metric on* $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$ *comes from the Carnot–Carathéodory metric on the Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Here, roughly the x *and* y *directions look like Euclidean directions of dimension* 1*, while the* z *direction looks like the Euclidean metric raised to the power* $\frac{1}{2}$*, so of dimension* 2*. So the whole space is* 4*-regular, and moreover one can find a family of curves in the* x*-direction parametrized by a* 3*-regular subset in the* y *and* z *directions which, by example (3) shows the conformal dimension is* $3+1=4$. *See* [Pan89].

This is gives our first example of hyperbolic groups which have homeomorphic boundaries that are not QS, hence the groups are not QI.

(6) *Bourdon* [Bou97] *has found an infinite family of hyperbolic groups with boundaries all homeomorphic to the Menger sponge (Figure 10), whose conformal dimensions take a dense set of values in* $(1, \infty)$.

(7) *Random groups can sometimes be distinguished by conformal dimension, see later.*

In relation to Cannon's conjecture, Bonk and Kleiner showed:

Theorem 1.20 (Bonk–Kleiner [BK05]). *If G is a hyperbolic group with $\partial_\infty G$ homeomorphic to $\mathbb{S}^2$, and $\mathrm{Confdim}(\mathbb{S}^2)$ is attained, then G is virtually cocompact Kleinian.*

In that same paper, they asked if lower-dimensional case were approachable:

Question 1.21 (Bonk–Kleiner [BK05, Problem 6.1]). *Can one algebraically characterize the hyperbolic groups G with $\mathrm{Confdim}(\partial_\infty G) = 1$?*

We will consider this in the next section.

1.4. **References.** For background and further discussion on Gromov hyperbolic spaces and groups, see for example [Gro87, CDP90, GdlH90, BH99]. For background and more on conformal dimension see for example [Hei01, MT10].

2. Splittings and conformal dimension

This section outlines joint work with Matias Carrasco [CM22], where we were able to give an answer to Bonk and Kleiner's Question 1.21.

2.1. **Results.** The key step is the following:

Theorem 2.1 (Carrasco–M. [CM22]). *Suppose G is a hyperbolic group with a graph of groups decomposition over elementary (i.e., finite or virtually $\mathbb{Z}$) subgroups, where the vertex groups are $\{G_i\}$ and G is not virtually free. Then*

$$\mathrm{Confdim}(\partial_\infty G) = \max\{1\} \cup \{\mathrm{Confdim}\, \partial_\infty G_i\}.$$

(To simplify the statement, suppose $\mathrm{Confdim}\, \emptyset = -\infty$.)

We can also characterise attainment in this context, using work of Bonk–Kleiner in the case $\mathrm{Confdim}(\partial_\infty G) = 1$ is attained [BK02].

Theorem 2.2 (Carrasco–M. [CM22]). *For G as above, or G virtually free, then $\mathrm{Confdim}(\partial_\infty G)$ is attained if and only if exactly one of the following holds:*

- $\mathrm{Confdim}(\partial_\infty G) = 0$ *and G is virtually $\mathbb{Z}$, or*
- $\mathrm{Confdim}(\partial_\infty G) = 1$ *and G is virtually cocompact Fuchsian, or*
- *$G = G_i$ for some i with $\partial_\infty G_i$ attaining its conformal dimension > 1.*

Let us now answer Question 1.21.

Corollary 2.3 (Carrasco–M. [CM22]). *Suppose a hyperbolic group G is not virtually free, and has no 2-torsion. Then $\mathrm{Confdim}(\partial_\infty G) = 1$ if and only if G can be formed from finite and virtually cocompact Fuchsian groups by amalgamating over elementary subgroups finitely many times.*

Proof. $\Rightarrow$: This follows from older work. If $\mathrm{Confdim}(\partial_\infty G) = 1$ then [Mac10] shows that G splits over a virtually $\mathbb{Z}$ subgroup. If a resulting vertex group is disconnected, split over a finite subgroup. Repeat this process, and after finitely many steps you stop thanks to Louder–Touikan accessibility [LT17].

$\Leftarrow$: Apply Theorem 2.1 repeatedly. $\square$

2.2. **Combinatorial modulus.** The key tool used to prove Theorem 2.1 is combinatorial modulus. This has been developed by many authors: Pansu, Cannon, Heinonen–Koskela, Keith–Laakso, Kleiner, Bourdon, Carrasco, to name but a few. See [Kle06, CP13] and references therein.

The idea is that we approximate our space by balls and assign a new size for the balls, and if this can be done in an appropriately controlled way it is possible to define a corresponding QS map. Let us be a bit more formal, fixing $Z = \partial_\infty X = \partial_\infty G$ with a visual metric for a hyperbolic group G acting geometrically on X. (More generally, Z can be any "approximately self-similar space".)

- Fix $a \geq 2$ ($a = 2$ suffices for us), and for each $n \in \mathbb{N}$ let $\mathcal{S}_n = \{B(x_{n,i}, a^{-n})\}$ be a cover of Z by a^{-n}-balls centred on points in a maximal separated a^{-n}-net.
- Fix $\delta_0 > 0$ small, and say $\rho_n : \mathcal{S}_n \to (0,\infty)$ is *admissible* (for the family of all paths of diameter $\geq \delta_0$) if for any curve γ in Z of diameter $\geq \delta_0$, one has
$$\sum_{A \in \mathcal{S}_n, A\cap\gamma\neq\emptyset} \rho_n(A) \geq 1.$$
- Let $\mathrm{Vol}^p(\rho_n) = \sum_{A\in\mathcal{S}_n} \rho_n(A)^p$, and define the *combinatorial p-modulus*
$$\mathrm{Mod}_p\, \mathcal{S}_n = \inf\left\{\mathrm{Vol}^p(\rho_n) : \rho_n \text{ admissible }\right\}.$$

In a sense, $\mathrm{Vol}^p(\rho_n)$ measures the "p-size" of the assigned weights ρ_n, and $\mathrm{Mod}_p\, \mathcal{S}_n$ asks for the p-size of the most efficient way to assign new radii such that paths with big diameter stay big.

Theorem 2.4 (Carrasco [CP13], but see work cited there of Keith–Laakso, Keith–Kleiner, Bourdon–Kleiner)**.** *For Z as above, there exists $\delta_0 > 0$ so that*
$$\mathrm{Confdim}(Z) = \inf\left\{p : \lim_{n\to\infty} \mathrm{Mod}_p\, \mathcal{S}_n = 0\right\}.$$

Let's see the power of this theorem in action.

Example 2.5. *Let Z be the Sierpiński gasket. (Although not a boundary, it is approximately self-similar and the theorem works.) To simplify, let's consider blocking the paths which meet two of the three corners, and let S_n consist of the collection of triangles on scale 2^{-n}.*

We can put weights of $\rho_n = \frac{1}{n}$ on the $6n-3$ red places indicated in Figure 11, and $\rho_n = 0$ otherwise. This is certainly admissible as paths connecting the corners have to pass through the cut pairs. For this choice of ρ_n, we have $\mathrm{Vol}^p(\rho_n) = \frac{6n-3}{n^p} \to 0$ as $n\to\infty$, for any fixed $p > 1$. Thus by Theorem 2.4 we have $\mathrm{Confdim}(Z) = 1$.

This particular example was worked out already by Laakso, and can be deformed explicitly as shown by Tyson–Wu [TW06], but it does illustrate an important idea which we will use. Namely, the geometric sequence of scales around the corners are given equal weights which has the effect of sending them to an arithmetic sequence of scales.

What made the Sierpiński gasket calculation easy was that it was easy to cut, "WS" in Carrasco's terminology: Z has the property that for any $\epsilon > 0$, one can delete finitely many points so that all connected components

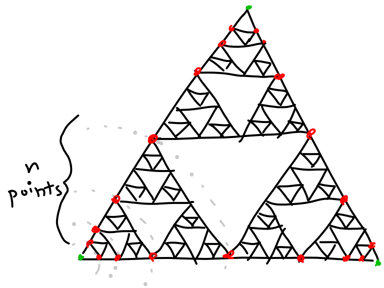

FIGURE 11. Sierpiński gasket

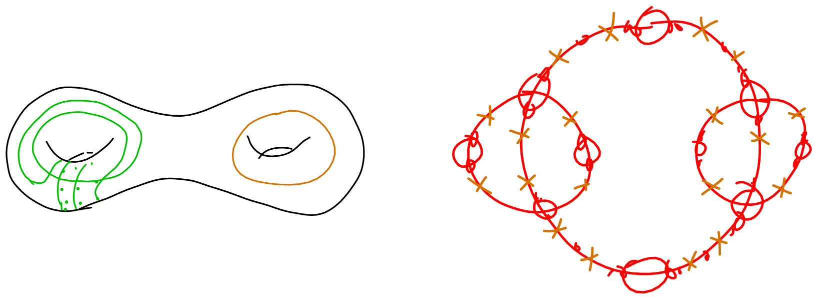

FIGURE 12. Non-filling surface double

of the remainder have diameter $\leq \epsilon$. An example of such a situation is when $G = H *_{\mathbb{Z}} H$ is the amalgam of two copies of a surface group over a cyclic group corresponding to a non-filling curve (green in Figure 12). Then the (orange) endpoints of the conjugates of an essential curve in the complement can be used to disconnect the boundary, shown on right of Figure 12, so $\mathrm{Confdim}(\partial_\infty G) = 1$.

2.3. **Toy example.** In general, the boundaries in Theorem 2.1 will not have property WS, so we do not have cut points which disconnect the space into finitely many pieces. However, we do have lots of cut pairs. One can think of the boundary $\partial_\infty G$ as consisting of copies of the boundaries of the (conjugates of the) vertex groups $\partial_\infty G_i$, one for each vertex in the Bass–Serre tree, joined at pairs of points coming from the limit points of the corresponding edge groups. (There are also points coming from the boundary of the Bass–Serre tree, but these aren't important once we take combinatorial approximations.)

The strategy of the proof is to assign weights in an inductive way which relatively deform each $\partial_\infty G_i$ copy with "geometric scales" around the "parent" cut points re-weighted to equal "arithmetic scales".

Rather than get lost in the details, let's consider the following toy example, which is also discussed in [CM22, §1.3]. This roughly corresponds to the boundary of $G = H *_{\mathbb{Z}} H$ where H is a surface group and $\mathbb{Z}$ corresponds to a filling curve in the surface.

We simultaneously define our space Z as a limit of spaces Z_n and a sequence of weights ρ_n on Z_n. Again, rather than trying to block all paths in Z_n we simplify by just blocking all paths connecting a pair of points. The

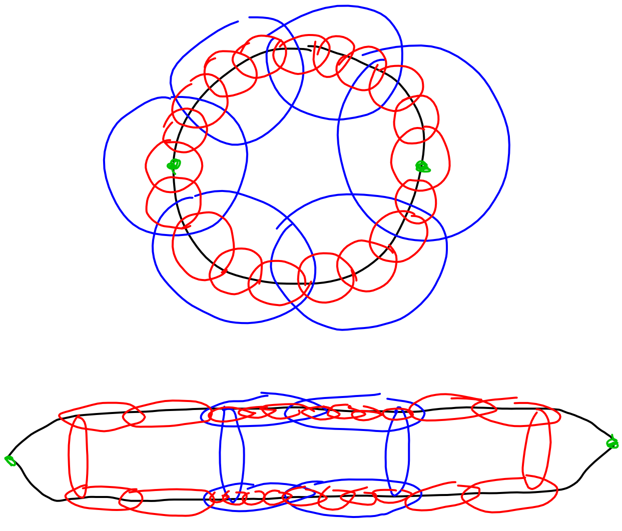

FIGURE 13. Toy example, before and after

first three steps of the example are shown in Figure 13, in black, blue and red respectively, along with a cartoon of the resulting deformation.

Let Z_0 be a (black) unit circle, and try to block all paths connecting two (green) fixed antipodal points. Put $\rho_0 = 1$ on a single set covering Z_0, so $\mathrm{Vol}^p(\rho_0) = 1$.

For the inductive step, suppose for any $k < n$ we have defined Z_k and ρ_k, so that any path connecting the special pair of points in Z_k has total ρ_k values at least 1. To define Z_n, start with the unit circle and fixed antipodal points p_-, p_+. For each $j = 1, \ldots, n$, at $10 \cdot 3^j$ pairs of points spread out around the circle glue in copies of Z_{n-k}, each scaled to size 3^{-j}.

Now to define the weight, we consider each annulus $B(p_\pm, 3^{-i}) \backslash B(p_\pm, 3^{-i-1})$. This contains $\preceq (3^{-i}/3^{-j})$ copies of Z_{n-j}. We want the annulus thickness to go from $3^{-i} - 3^{-i-1} \asymp 3^{-i}$ to $1/n$, so the new diameter of these copies of Z_{n-j} should be $3^{-j} \cdot (1/n)/3^{-i} = 3^{i-j}/n$. This gives an inductive way to define ρ_n, with the following bound on Vol^p:

$$\begin{aligned}
\mathrm{Vol}^p(\rho_n) &\preceq \underbrace{\sum_{i=0}^{n-1}}_{\text{annuli size}} \underbrace{\sum_{j=i+1}^{n}}_{\text{children size}} \underbrace{\frac{3^{-i}}{3^{-j}}}_{\#\text{ children}} \cdot \underbrace{\left(\frac{3^{i-j}}{n}\right)^p \mathrm{Vol}^p(\rho_{n-j})}_{\text{scaled }\mathrm{Vol}^p} \\
&= \frac{1}{n^p} \sum_{j=1}^{n} \underbrace{\sum_{i=1}^{j-1} 3^{(i-j)(p-1)}}_{\preceq 1} \cdot \mathrm{Vol}^p(\rho_{n-j}) \\
&\preceq \frac{1}{n^{p-1}} \max\left\{\mathrm{Vol}^p(\rho_0), \ldots, \mathrm{Vol}^p(\rho_{n-1})\right\}.
\end{aligned}$$

Thus for any $p > 1$ the sequence $(\mathrm{Vol}^p(\rho_n))$ is eventually non-increasing, hence is bounded, hence by the same inequality $\mathrm{Vol}^p(\rho_n) \to 0$. So by Theorem 2.4 we have $\mathrm{Confdim}(Z) = 1$.

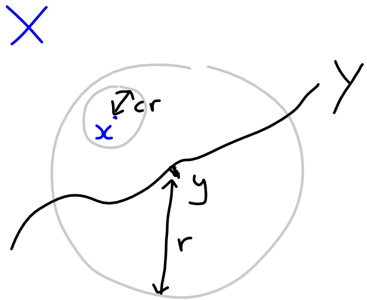

FIGURE 14. Porosity

The proof of Theorem 2.1 has a similar idea at its core, but the weights defined are layered with deformations of $\partial_\infty G_i$ near to their conformal dimension, and the argument has quite a complicated induction on scales to make everything precise.

2.4. **Attainment of conformal dimension.** The key idea here is porosity.

Definition 2.6. *A subset Y of a metric space X is* porous *if there exists $c > 0$ such that for all $y \in Y, r \in (0, \operatorname{diam} X]$ there exists $x \in X$ so that $B(x, cr) \subset B(y, r) \setminus Y$, see Figure 14.*

I'm not sure who to credit the following proposition, but it was surely known to experts before our paper.

Proposition 2.7. *Suppose $Y \subset X$ is a porous subset of a metric space, and $\operatorname{Confdim} Y = \operatorname{Confdim} X < \infty$. Then $\operatorname{Confdim} X$ is not attained.*

Sketch proof. If $\operatorname{Confdim} X$ was attained by some QS $f : X \to X'$, then $f(Y) \subset X'$ would also be porous, hence its Assouad dimension satisfies $\dim_A f(Y) < \dim_A X' = \operatorname{Confdim} Y$, a contradiction. (For more on Assouad dimension see [Hei01]). □

Now we outline the proof of the attainment theorem.

Proof of Theorem 2.2. $\Leftarrow$: this is trivial.

$\Rightarrow$: Supposing $\operatorname{Confdim}(\partial_\infty G)$ is attained, there are three cases.

- $\operatorname{Confdim}(\partial_\infty G) = 0$, then by Stallings–Dunwoody G is virtually $\mathbb{Z}$.
- $\operatorname{Confdim}(\partial_\infty G) = 1$, then by Bonk–Kleiner [BK02] G is virtually cocompact Fuchsian.
- $\operatorname{Confdim}(\partial_\infty G) > 1$, then by Theorem 2.1 we have $\operatorname{Confdim}(\partial_\infty G) = \operatorname{Confdim}(\partial_\infty G_i)$ for some vertex group G_i, hence $\partial_\infty G_i$ attains its conformal dimension. Studying the graph of groups decomposition, one can show that unless $G_i = G$ we have $[G : G_i] = \infty$, and moreover that $\partial_\infty G_i$ is porous in $\partial_\infty G$, contradicting Proposition 2.7.

□

We have discussed how conformal dimension behaves in relation to splittings of hyperbolic groups over elementary subgroups. Splittings over larger subgroups, e.g. non-abelian free groups, remain mysterious in many respects, see Bourdon–Pajot [BP03] and Bourdon–Kleiner [BK13] for further reading.

3. Random groups and conformal dimension

We now change tack and see how conformal dimension can be useful in studying "typical" groups. This section outlines some aspects of the theory of random groups based on work by many people, particularly highlighting recent work of Frost [Fro22].

3.1. **What are they?** There are a variety of models of random groups; Ollivier has a good survey on this topic [Oll07]. We'll consider two variations on models studied by Gromov [Gro93] and Żuk [Żuk03]. These are in the spirit of studies of Erdős–Renyi random graphs, where one considers graphs with say n vertices, and $M = M(n)$ edges, and considers which properties hold with probability $\to 1$ for which functions $M(n)$ as $n \to \infty$.

Definition 3.1. *Given $m \geq 2, n \geq 1, l \geq 1$, let*

$$\mathcal{G}_{m,n,l} = \{\langle s_1, s_2, \ldots, s_m | r_1, \ldots, r_n\rangle : \forall i, |r_i| = l, r_i \textit{ cyclically reduced}\},$$

viewed as a (finite) probability space with uniform distribution. Consider the following models:

- few relator model*: Fix $m \geq 2, n \geq 1$, and let $l \to \infty$;*
- Gromov density model: *Fix $m \geq 2$ and a* density $d \in (0,1)$*, then let $n = n(l) = (2m-1)^{ld}$ and let $l \to \infty$;*
- Triangular density model: *Fix $l = 3$ and a density $d \in (0,1)$, then let $n = n(m) = (2m-1)^{ld}$ and let $m \to \infty$.*

In each model, we say a property holds asymptotically almost surely (a.a.s.) *if the probability it holds converges to 1 as the free parameter (l or m) goes to infinity.*

(Note that we suppress rounding, so $n = (2m-1)^{ld}$ should be interpreted as $n = \lceil (2m-1)^{ld} \rceil$, and so on.)

The few relator model can be easier to work with, since for example a.a.s. such presentations have the $C'(\frac{1}{6})$-small cancellation property. However, in some sense they are not so natural since as $l \to \infty$, out of the $\sim (2m-1)^l$ possible relations of that length, they only choose a fixed number n, whereas the density models pick a logarithmic fraction of these relations.

Another strong motivation for the density models is the interesting thresholds for different behaviour one sees. There are also some interesting parallels between the two models. We summarise some properties:

Theorem 3.2.

- *For $0 < d < \frac{1}{2}$ in the Gromov/Triangular density models, a.a.s. a random group is non-elementary hyperbolic. For $d > \frac{1}{2}$, a.a.s. a random group is trivial (or $\mathbb{Z}/2\mathbb{Z}$) (Gromov* [Gro93]*, Żuk* [Żuk03]*, Ollivier* [Oll07]*).*
- *For $0 < d < \frac{1}{2}$ in the Gromov density model, and $\frac{1}{3} < d < \frac{1}{2}$ in the Triangular density model, a.a.s. a random group G (is hyperbolic and) has $\partial_\infty G$ homeomorphic to the Menger sponge (Figure 10). (Gromov model: Champetier* [Cha95] *for $d < \frac{1}{24}$, Dahmani–Guirardel–Przytycki* [DGP11] *for $0 < d < \frac{1}{2}$. Gromov and Triangular model at $d \geq \frac{1}{3}$: this follows from Kazhdan's Property (T)* [Żuk03, KK13]*, and Kapovich–Kleiner* [KK00]*; see Ollivier* [Oll05, §1.3.d]*)*

- *For* $0 < d < \frac{1}{3}$ *in the Triangular density model, a.a.s. a random group is free by Antoniuk–Łuczak–Świątkowski* [AŁŚ15]*), hence has Cantor set boundary.*

So, are all these groups with Menger sponge boundaries the same? No, the presentation is aspherical so the Euler characteristic is $\chi(G) = 1 - 2m + (2m-1)^{ld}$ which goes to infinity. But are they all quasi-isometric, or even commensurable?

3.2. **Conformal dimension.** The answer to the previous question is no: many of these random groups are different, and as far as I know the only way this has been shown in density models is using conformal dimension. We'll survey what is known, leading up to recent results of Oppenheim and Frost.

First, there is a straightforward upper bound.

Proposition 3.3 (cf. [Mac12, Proposition 1.7])**.** *In the Gromov/Triangular models, for* $0 < d < \frac{1}{2}$*, a.a.s. we have*

$$\mathrm{Confdim}(\partial_\infty G) \leq \frac{19 \log(2m-1) \cdot l}{1-2d}.$$

Proof. By Ollivier [Oll07] we have the linear isoperimetric inequality $|\partial D| \geq (1-2d-\epsilon)l|D|$ for all van Kampen diagrams (in both the Gromov and Triangular density models, see [AŁŚ15]). As Ollivier explains [Oll07, Proposition 15], this means the Cayley graph X is (at most) $\delta = \frac{16l}{5(1-2d)}$-hyperbolic (tweaking the proof using $20/2\pi < 16/5$ instead of < 4). There is a visual metric on $\partial_\infty X$ with $\epsilon = \frac{\log(2)}{4\delta} \geq \frac{1-2d}{19l}$ [BH99, Proposition III.H.3.21]. Thus by Coornaert [Coo93], since the volume entropy $h(X)$ is at most that of the free group on m generators,

$$\mathrm{Confdim}(\partial_\infty G) \leq \frac{1}{\epsilon} h(X) \leq \frac{19l}{1-2d} \cdot \log(2m-1). \qquad \square$$

The first results with stronger bounds were obtained at low densities, using small cancellation techniques which only hold there.

Theorem 3.4 ([Mac12,Mac16])**.** *In the Gromov density model, for* $0 < d < \frac{1}{8}$ *a.a.s.*

$$\mathrm{Confdim}(\partial_\infty G) \asymp \frac{dl \log(2m-1)}{|\log d|}.$$

The next main direction to be developed was the use of spectral methods. Bourdon showed that for a hyperbolic group G, if $p > \mathrm{Confdim}(\partial_\infty G)$ then G admits a proper affine isometric action on L^p [Bou16]. Consequently, if for a given p, the group G has fixed points for any such action on L^p, we must have $\mathrm{Confdim}(\partial_\infty G) \geq p$. For example, any hyperbolic group G with Kazhdan's Property (T) has fixed points when acting on L^p for all $p \in [1, 2+\epsilon)$, so $\mathrm{Confdim}(\partial_\infty G) > 2$.

In the triangular model, a sequence of works by Druţu–Mackay [DM19], de Laat–de la Salle [dLdlS21] and finally Oppenheim [Opp22] established increasingly strong fixed point properties for random groups, resulting in:

Theorem 3.5 ([DM19, dLdlS21, Opp22]). *In the triangular density model, for $\frac{1}{3} < d < \frac{1}{2}$, there exists $C = C(d)$ such that a.a.s. we have*

$$\text{Confdim}(\partial_\infty G) \asymp_C \log(2m-1).$$

In the Gromov model, until very recently spectral methods did not give strong bounds (i.e., with $\text{Confdim}(\partial_\infty G) \to \infty$ as $l \to \infty$). This was because in the proof of fixed point properties for such groups one came across links in the Cayley complex which were bi-partite, unlike in the triangular model, for which the existing fixed point criteria did not apply.

So, what happened to conformal dimension in the Gromov model at $\frac{1}{8} \leq d < \frac{1}{2}$ was unclear, until very recent independent work of Oppenheim [Opp21] and Frost [Fro22] that gives a fairly clear picture.

Theorem 3.6 (M. [Mac16] for $0 < d < \frac{1}{8}$; Oppenheim [Opp21] for $\frac{1}{3} < d < \frac{1}{2}$; Frost [Fro22] for $\frac{1}{8} \leq d < \frac{1}{2}$). *There exists $C > 0$ so that in the Gromov model for $0 < d < \frac{1}{2}$, a.a.s. we have*

$$\frac{d \cdot l}{C\ |\log d|} \leq \text{Confdim}(\partial_\infty G) \leq \frac{Cd \cdot l}{|\log d|\ (1-2d)}.$$

In particular, at each density $0 < d < \frac{1}{2}$, as $l \to \infty$ the random groups pass through infinitely many quasi-isometry classes.

Remark 3.7.

- Oppenheim uses spectral methods which, unlike Frost, also give fixed point properties for the random groups acting on L^p spaces; they also give a better constant for the lower bound at densities close to $\frac{1}{2}$. As low density random groups do not have property (T), Oppenheim's methods would not apply at those densities.
- Frost uses a bootstrap technique to extend bounds like [Mac16] to all densities $\frac{1}{8} \leq d < \frac{1}{2}$; they also give a better lower bound constant than Oppenheim for densities close to $\frac{1}{3}$.
- The upper bounds in this theorem follow from [Mac16] at low densities and Proposition 3.3 at high densities.

I'll now give a quick overview of some of the ideas in Frost's work; for further details on Oppenheim's interesting work see the references above.

3.3. Frost's lower bound.

To explain Frost's work, first I'll mention some ideas from [Mac16], which are inspired by Gromov's "round trees" [Gro93], constructions of Champetier [Cha95], and conformal dimension estimates of Bourdon [Bou95].

For simplicity, suppose $d < \frac{1}{16}$, so that a random group in the Gromov density model has $C'(\frac{1}{8})$ small cancellation.

1. First, small cancellation gives that geodesic bigons look like "ladders", that is when one finds a van Kampen diagram bounded by two geodesic segments with the same endpoints it must look something like that in Figure 15. This also extends to geodesic triangles between $x, y, y' \in X$ where y and y' are neighbours.

2. Following Champetier, in the Cayley graph $X^{(1)}$ one can always extend geodesic segments from the identity e to any vertex x in all but ≤ 2 ways, for if this failed, there would be two neighbours y, y' of x which satisfied

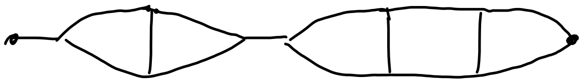

FIGURE 15. Geodesic bigon forming a ladder

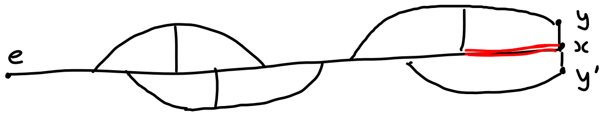

FIGURE 16. Extending geodesics

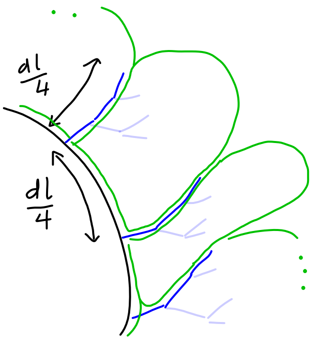

FIGURE 17. Gluing in faces

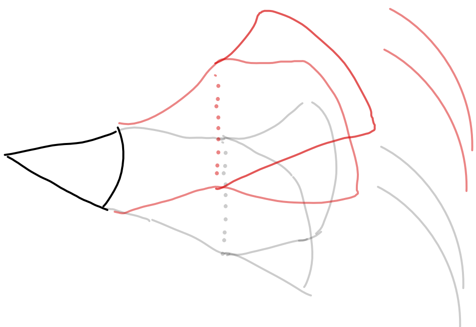

FIGURE 18. Part of the round tree, with limit set

$d(e, y), d(e, y') \leq d(e, x)$ but did not lie on the given geodesic from e. Using 1. one can find and glue together two van Kampen diagrams as in Figure 16; small cancellation then forces there to exist a reduction across the red bold edge, which implies that $y = y'$.

3. So, we can extend geodesics in at least $2m - 2 \geq 2$ ways from every point. One then does an inductive construction which builds a "round tree" Y in the Cayley complex $X^{(2)}$: at each step extend out a distance $\frac{dl}{4}$ with gaps of $\frac{dl}{4}$ along the frontiers; since every word of length $\frac{3dl}{4}$ appears in the presentation, one can fill in faces as in Figure 17.

4. By making slight restrictions on the directions one extends geodesics, one can ensure that Y is quasi-convex in X. This again uses small cancellation, but we skip the details.

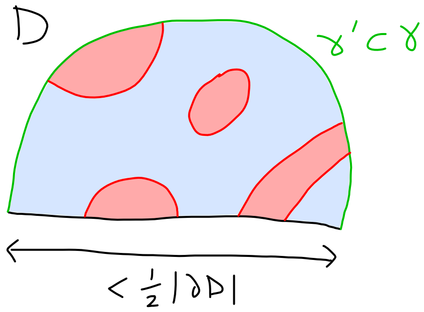

FIGURE 19. Diagram with low and high density relations

5. This complex Y gives an embedded product of a Cantor set and $[0, 1]$ in $\partial_\infty G$, see Figure 18 which has "horizontal branching" $H \sim l/(\frac{dl}{4})$ and "vertical branching" $V \sim (2m-2)^{dl/4}$. Using an argument of Bourdon, we get:

$$\mathrm{Confdim}(\partial_\infty X) \geq \mathrm{Confdim}(\partial_\infty Y) \geq 1 + \frac{\log V}{\log H} \succeq \frac{dl \log(2m-2)}{\log(4/d)}.$$

So, how to extend this to all $0 < d < \frac{1}{2}$? Frost uses a "bootstrapping" argument inspired by that which Calegari–Walker [CW15]. They build surface subgroups in random Gromov hyperbolic groups at all densities $0 < d < \frac{1}{2}$ by, for a given density $d < \frac{1}{2}$, building a surface subgroup in a random one-relator group (say the first relator of the presentation) in a way that with high probability the subgroup will remain quasiconvex and embedded at density d.

Frost takes this idea and shows that, for each $d < \frac{1}{2}$, if one takes the round tree of [Mac16] at some suitable small $0 < d' < \frac{1}{8}$, with some modifications to "thin out" the number of directions one extends in, then such complexes will remain quasiconvex at high densities.

A sketch of part of the argument is as follows. Suppose we have a random group G_d at density d, and consider the first $(2m-1)^{d'l}$ of the relators as giving a random group $G_{d'}$ at density $d' < d$. Let $X_{d'}, X_d$ be the Cayley complexes of the respective presentations, with the quotient map $G_{d'} \to G_d$ giving a natural map $X_{d'} \to X_d$. Let Y be a round tree in $X_{d'}$ constructed (roughly) as above.

Suppose a geodesic ray γ from e in the round tree $Y \subset X_{d'}$ is no longer quasi-convex in X_d. Then the local-to-global principle for quasi-geodesics in Gromov hyperbolic spaces implies that γ is not a Cl-local geodesic in X_d for some $C = C(d)$. This means that we can find a van Kampen diagram D bounded by a subsegment γ' of γ of length $\leq Cl$, and a geodesic in X_d of strictly smaller length. Ollivier's linear isoperimetric inequality gives a universal bound $C' = C'(d)$ on the number of faces in D. One wants to rule out the existence of such a diagram D. This is a little subtle as there are two kinds of relations, those coming from $G_{d'}$ which are fixed when we build Y, and new random relations added to give G_d, shown as red and blue respectively in Figure 19. But by making d' sufficiently small, and restricting the extension directions to make the number of labellings of the γ-segment have sufficiently small exponential growth rate, one can show that no such diagram exists.

The argument to show that Y is quasiconvex embedded in X_d is similar, but considering triangles rather than individual geodesics.

Questions remain:

Question 3.8. *In the Gromov density model as $d \to \frac{1}{2}$, is there an exact linear growth rate for the conformal dimension? In particular, can one remove the '$(1-2d)$' term from Theorem 3.6?*

The conformal dimension bounds discussed here are given up to multiplicative constants. Can this be sharpened at all? In particular,

Question 3.9 (cf. [Gro93, 9.B(g)], [Oll05, IV.b])**.** *Can one detect the density of a random group G in the Gromov density model from the isomorphism, or even quasi-isometry type of G? In particular, for each $0 < d < \frac{1}{2}$ and $\epsilon > \epsilon' > 0$ does there exist a presentation-independent (or quasi-isometry-independent) property $\mathcal{P}$ of a group so that for any $d' \in (d-\epsilon', d+\epsilon')$ a.a.s. a random group at density d' has $\mathcal{P}$, while for any $d' \notin (d-\epsilon, d+\epsilon)$ a.a.s. a random group at density d' does not have $\mathcal{P}$?*

Potentially such a property $\mathcal{P}$ could involve some function of the conformal dimension, compare [Mac16] at low densities.

4. Coarse embeddings and conformal dimension

In this final section, we'll talk a bit about conformal dimension and coarse embeddings, based on joint work with Hume and Tessera [HMT20, HMT22].

If we have two hyperbolic groups G, H and a quasi-isometric embedding $G \to H$, then by [BS00] we have a quasisymmetric embedding $\partial_\infty G \to \partial_\infty H$, thus $\mathrm{Confdim}(\partial_\infty G) \leq \mathrm{Confdim}(\partial_\infty H)$.

Thus, for example, the results of the previous section show that random groups in the Gromov model with large relator length l do not quasi-isometrically embed into small l ones.

But what about coarse embeddings? We won't define this notion here, but note that for finitely generated groups G, H if G is a subgroup of H then the inclusion map $G \to H$ is a coarse embedding. This could be a distorted subgroup. We will instead work with a more general notion of embedding.

Definition 4.1 (Benjamini–Schramm–Timár [BST12])**.** *A map $f : X \to X'$ between bounded degree graphs X, X' is* regular *if there exists C so that for all adjacent $x, y \in X$, $d_{X'}(f(x), f(y)) \leq C$ and for all $z \in X'$, $|f^{-1}(z)| \leq C$. If a regular map $X \to X'$ exists, we write $X \xrightarrow{\mathrm{reg}} X'$.*

Example 4.2.
- *For finitely generated groups G, H with $G \leq H$, the inclusion map induces* $\mathrm{Cay}(G) \xrightarrow{\mathrm{reg}} \mathrm{Cay}(H)$.
- *The function $\mathbb{Z}^2 \to \mathbb{Z}^2, (x,y) \mapsto (|x|, |y|)$ is a regular map.*

Seeking a monotone invariant of regular maps, and inspired by work of Lipton–Tarjan in the 1970s and Miller–Teng–Thurston–Vavasis in the 1990s, Benjamini–Schramm–Timár introduced the following notion.

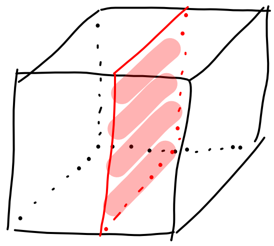

FIGURE 20. Cutting a cube

Definition 4.3 ([BST12]). *For Γ a finite graph with $|\Gamma| = |V\Gamma|$ vertices, let* $\mathrm{cut}(\Gamma) = \min |S|$, *where $S \subset V\Gamma$ satisfies that all connected components of $\Gamma \setminus S$ have at most $|\Gamma|/2$ vertices.*

For an (infinite) graph X, the separation profile $\mathrm{sep}_X : \mathbb{N} \to \mathbb{N}$ *is defined by*

$$\mathrm{sep}_X(r) = \max \{\mathrm{cut}(\Gamma) : \Gamma \subset X, |\Gamma| \leq r\}.$$

Remark 4.4. We only care about sep_X up to $\simeq$, where $f \lesssim g$ if and only if there exists $C > 0$ such that $f(r) \leq Cg(Cr + C) + C$ for all $r \in \mathbb{N}$, and $f \simeq g$ if and only if $f \lesssim g$ and $g \lesssim f$.

Proposition 4.5. *If X, Y are bounded degree graphs with $X \xrightarrow{\mathrm{reg}} Y$ then $\mathrm{sep}_X \lesssim \mathrm{sep}_Y$. In particular, for a finitely generated group G, sep_G is defined up to $\simeq$ independent of choice of generating set.*

Idea of proof. To find a good cut set for $\Gamma \subset X$, pull back an appropriate cut set for its image in Y. □

In [BST12] we find the following examples.

Example 4.6.
- $\mathrm{sep}_{\mathrm{tree}}(r) \simeq 1$.
- $\mathrm{sep}_{\mathbb{Z}^n}(r) \simeq r^{1-1/n}$. *The idea is that the hardest to cut subgraphs look like cubes (or balls), and the best way to cut them is a hyperplane, see Figure 20.*
- $\mathrm{sep}_{\mathbb{H}^n_{\mathbb{R}}}(r) \simeq \log(r)$ *for $n = 2$, but $\simeq r^{1-1/(n-1)}$ for $n \geq 3$: for $n = 2$ a hardest to cut set is a ball, while in $n \geq 3$ a sphere, or part of a horosphere are already the hardest to cut.*

This last example is really a simple case of a more general phenomenon: the separation profile is an L^1 version of more general L^p-Poincaré profiles, and the boundary of $\mathbb{H}^n_{\mathbb{R}}$ has conformal dimension $n-1$: we get different behaviour when $1 = n - 1$ from the case $1 < n - 1$, see [HMT20]. More generally,

Theorem 4.7 (Hume–M.–Tessera [HMT20]). *Let X be a (bounded degree graph approximation of a) rank 1 symmetric space, or a Bourdon building (Example 1.16(6)), whose boundary has conformal dimension Q. Then* $\mathrm{sep}_X(r) \simeq r^{1-1/Q}$.

This has the following immediate corollary. The same result was slightly earlier obtained by Pansu for a different family of maps which also includes coarse embeddings. Pansu uses different methods including ℓ^p cohomology.

Corollary 4.8 (Hume–M.–Tessera [HMT20], cf. Pansu [Pan21])**.** *For such* X, X'*, if* $X \xrightarrow{\text{reg}} X'$ *then* $Q \leq Q'$*.*

Sketch proof of Theorem 4.7. Lower bound: Such spaces X have boundaries $Z = \partial_\infty G$ which admit a 1-Poincaré inequality in the sense of Heinonen–Koskela with a particular choice of metric and Ahlfors regular measure μ. This means there exists $C > 0$ so that for any ball $B = B(z, r) \subset Z$, and any (say) Lipschitz function $f : Z \to \mathbb{R}$ with $f_B := \fint_B f \, d\mu$ we have

$$\int_B |f - f_B| \, d\mu \leq Cr \int_B |\nabla f| \, d\mu. \tag{4.9}$$

Here "$|\nabla f|$" should be interpreted as an upper gradient, or a local Lipschitz constant, but we skip the details. The point is that the average deviation of f from its average on any ball is controlled by the average of some kind of gradient on the ball.

To show the desired lower bound on the separation profile, one shows that any large sphere Γ in X is hard to cut in the required quantitative sense. Roughly, if Γ had a small cut set S, one can define a function that is locally constant on $\Gamma \setminus S$, taking values 0 and 1 on $\geq |\Gamma|/4$ of the set. This function can be pushed out to $Z = \partial_\infty G$ to define a function contradicting (4.9). Conformal dimension arises as it equals the Ahlfors regularity constant of Z and measures the exponential growth rate of the spheres in X.

Upper bound: Here the goal is to find an efficient way to cut any $\Gamma \subset X$. As in [BST12], the overall strategy is similar to Miller–Teng–Thurston–Vavasis, where one finds a "median" of the set, then (using an auxiliary embedding into some $\mathbb{H}^n_{\mathbb{R}}$) one finds a good cut through the set by averaging over all hyperplanes through the median. Conformal dimension enters into the picture via the exponential growth rate of X for suitable metrics in the quasi-isometry class. ☐

Using similar ideas, but upgraded to deal with weighted projections of sets, one can show the following non-embedding result.

Theorem 4.10 (Hume–M.–Tessera [HMT22])**.** *If* X, X' *are as above with conformal dimensions* Q, Q'*, and* N, N' *are Lie/discrete groups of polynomial growth* d, d'*, then* $X \times N \xrightarrow{\text{reg}} X' \times N'$ *implies that* $Q \leq Q'$ *and* $d \leq d'$*.*

This answers a question of [BST12], which asked to show $\mathbb{H}^3_{\mathbb{R}} \overset{\text{reg}}{\not\longrightarrow} \mathbb{H}^2_{\mathbb{R}} \times \mathbb{R}$. In fact, it shows e.g. $\mathbb{H}^3_{\mathbb{R}} \overset{\text{reg}}{\not\longrightarrow} \mathbb{H}^2_{\mathbb{R}} \times \mathbb{R}^{1000}$.

There are many questions still open regarding separation profiles (and more generally Poincaré profiles). For example, in light of Corollary 4.8,

Question 4.11. *If* G, H *are hyperbolic groups and* $G \xrightarrow{\text{reg}} H$ *(or* $G \xrightarrow{\text{coarse}} H$*) then must we have* $\operatorname{Confdim} \partial_\infty G \leq \operatorname{Confdim} \partial_\infty H$*?*

A map $G \xrightarrow{\text{reg}} H$ need not induce a nice boundary map at all, so a positive answer to this question would mean that the conformal dimension of a hyperbolic group is not just a boundary invariant but also captures some 'internal' large-scale structure of the group. Perhaps this will open up a door to studying such structures in other, non-hyperbolic, groups. See Pansu [Pan21] for developments in this direction.

References

[AŁŚ15] S. Antoniuk, T. Łuczak, and J. Świątkowski, *Random triangular groups at density 1/3*, Compos. Math. **151** (2015), no. 1, 167–178, DOI 10.1112/S0010437X14007805.

[BST12] I. Benjamini, O. Schramm, and Á. Timár, *On the separation profile of infinite graphs*, Groups Geom. Dyn. **6** (2012), no. 4, 639–658, DOI 10.4171/GGD/168.

[BK02] M. Bonk and B. Kleiner, *Rigidity for quasi-Möbius group actions*, J. Differential Geom. **61** (2002), no. 1, 81–106.

[BK05] ———, *Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary*, Geom. Topol. **9** (2005), 219–246, DOI 10.2140/gt.2005.9.219.

[BS00] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), no. 2, 266–306, DOI 10.1007/s000390050009.

[Bou95] M. Bourdon, *Au bord de certains polyèdres hyperboliques*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 1, 119–141 (French, with English and French summaries).

[Bou97] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal. **7** (1997), no. 2, 245–268, DOI 10.1007/PL00001619 (French, with English and French summaries).

[Bou16] M. Bourdon, *Cohomologie et actions isométriques propres sur les espaces L_p*, Geometry, topology, and dynamics in negative curvature, London Math. Soc. Lecture Note Ser., vol. 425, Cambridge Univ. Press, Cambridge, 2016, pp. 84–109 (French, with English summary).

[BK13] M. Bourdon and B. Kleiner, *Combinatorial modulus, the combinatorial Loewner property, and Coxeter groups*, Groups Geom. Dyn. **7** (2013), no. 1, 39–107, DOI 10.4171/GGD/177.

[BP03] M. Bourdon and H. Pajot, *Cohomologie l_p et espaces de Besov*, J. Reine Angew. Math. **558** (2003), 85–108, DOI 10.1515/crll.2003.043 (French, with English summary).

[BH99] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[CW15] D. Calegari and A. Walker, *Random groups contain surface subgroups*, J. Amer. Math. Soc. **28** (2015), no. 2, 383–419, DOI 10.1090/S0894-0347-2014-00802-X.

[CP13] M. Carrasco Piaggio, *On the conformal gauge of a compact metric space*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 3, 495–548 (2013), DOI 10.24033/asens.2195 (English, with English and French summaries).

[CM22] M. Carrasco and J. M. Mackay, *Conformal dimension of hyperbolic groups that split over elementary subgroups*, Invent. Math. **227** (2022), no. 2, 795–854, DOI 10.1007/s00222-021-01074-w.

[Cha95] C. Champetier, *Propriétés statistiques des groupes de présentation finie*, Adv. Math. **116** (1995), no. 2, 197–262, DOI 10.1006/aima.1995.1067 (French, with English summary).

[Coo93] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. **159** (1993), no. 2, 241–270 (French, with French summary).

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990 (French). Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups]; With an English summary.

[DGP11] F. Dahmani, V. Guirardel, and P. Przytycki, *Random groups do not split*, Math. Ann. **349** (2011), no. 3, 657–673, DOI 10.1007/s00208-010-0532-4.

[DM19] C. Druţu and J. M. Mackay, *Random groups, random graphs and eigenvalues of p-Laplacians*, Adv. Math. **341** (2019), 188–254, DOI 10.1016/j.aim.2018.10.035.

[Fro22] J. Frost, *Round Trees and Conformal Dimension in Random Groups: low density to high density* (2022), arXiv 2204.05165v1

[GdlH90] É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990 (French). Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[Gro93] ———, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[Hei01] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.

[HMT20] D. Hume, J. M. Mackay, and R. Tessera, *Poincaré profiles of groups and spaces*, Rev. Mat. Iberoam. **36** (2020), no. 6, 1835–1886, DOI 10.4171/rmi/1184.

[HMT22] ———, *Poincaré profiles of Lie groups and a coarse geometric dichotomy*, Geom. Funct. Anal. (2022), to appear.

[KK00] M. Kapovich and B. Kleiner, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 647–669, DOI 10.1016/S0012-9593(00)01049-1 (English, with English and French summaries).

[Kle06] B. Kleiner, *The asymptotic geometry of negatively curved spaces: uniformization, geometrization and rigidity*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 743–768.

[KK13] M. Kotowski and M. Kotowski, *Random groups and property (T): Żuk's theorem revisited*, J. Lond. Math. Soc. (2) **88** (2013), no. 2, 396–416, DOI 10.1112/jlms/jdt024.

[Kwa20] J. Kwapisz, *Conformal dimension via* p*-resistance: Sierpiński carpet*, Ann. Acad. Sci. Fenn. Math. **45** (2020), no. 1, 3–51, DOI 10.5186/aasfm.2020.4515.

[dLdlS21] T. de Laat and M. de la Salle, *Banach space actions and L^2-spectral gap*, Anal. PDE **14** (2021), no. 1, 45–76, DOI 10.2140/apde.2021.14.45.

[LT17] L. Louder and N. Touikan, *Strong accessibility for finitely presented groups*, Geom. Topol. **21** (2017), no. 3, 1805–1835, DOI 10.2140/gt.2017.21.1805.

[Mac10] J. M. Mackay, *Spaces and groups with conformal dimension greater than one*, Duke Math. J. **153** (2010), no. 2, 211–227, DOI 10.1215/00127094-2010-023.

[Mac12] ———, *Conformal dimension and random groups*, Geom. Funct. Anal. **22** (2012), no. 1, 213–239, DOI 10.1007/s00039-012-0153-z.

[Mac16] ———, *Conformal dimension via subcomplexes for small cancellation and random groups*, Math. Ann. **364** (2016), no. 3-4, 937–982, DOI 10.1007/s00208-015-1234-8.

[MT10] J. M. Mackay and J. T. Tyson, *Conformal dimension*, University Lecture Series, vol. 54, American Mathematical Society, Providence, RI, 2010. Theory and application.

[Oll05] Y. Ollivier, *A January 2005 invitation to random groups*, Ensaios Matemáticos [Mathematical Surveys], vol. 10, Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.

[Oll07] ———, *Some small cancellation properties of random groups*, Internat. J. Algebra Comput. **17** (2007), no. 1, 37–51, DOI 10.1142/S021819670700338X.

[Opp21] I. Oppenheim, *Banach Zuk's criterion for partite complexes with application to random groups* (2021), arXiv 2112.02929v2

[Opp22] ———, *Garland's method with Banach coefficients*, Anal. PDE (2022), to appear.

[Pan89] P. Pansu, *Dimension conforme et sphère à l'infini des variétés à courbure négative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), no. 2, 177–212, DOI 10.5186/aasfm.1989.1424 (French, with English summary).

[Pan21] ———, *Large scale conformal maps*, Ann. Sci. Éc. Norm. Supér. (4) **54** (2021), no. 4, 831–887, DOI 10.24033/asens.2472 (English, with English and French summaries).

[Pau96] F. Paulin, *Un groupe hyperbolique est déterminé par son bord*, J. London Math. Soc. (2) **54** (1996), no. 1, 50–74, DOI 10.1112/jlms/54.1.50 (French, with French summary).

[TV85] P. Tukia and J. Väisälä, *A remark on* 1*-quasiconformal maps*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 561–562, DOI 10.5186/aasfm.1985.1062.

[TW06] J. T. Tyson and J.-M. Wu, *Quasiconformal dimensions of self-similar fractals*, Rev. Mat. Iberoam. **22** (2006), no. 1, 205–258, DOI 10.4171/RMI/454.

[Żuk03] A. Żuk, *Property (T) and Kazhdan constants for discrete groups*, Geom. Funct. Anal. **13** (2003), no. 3, 643–670, DOI 10.1007/s00039-003-0425-8.

School of Mathematics, University of Bristol, Bristol, UK
Email address: john.mackay@bristol.ac.uk