SPACES AND GROUPS WITH CONFORMAL DIMENSION GREATER THAN ONE

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Abstract. We show that if a complete, doubling metric space is annularly linearly connected then its conformal dimension is greater than one, quantitatively. As a consequence, we answer a question of Bonk and Kleiner: if the boundary of a one-ended hyperbolic group has no local cut points, then its conformal dimension is greater than one.

1. Introduction

A standard quasi-symmetry invariant of a metric space (X, d) is its *conformal dimension*, introduced by Pansu [Pan89a]. It is defined as the infimal Hausdorff dimension of all metric spaces quasi-symmetric to X, denoted here by $\dim_{\mathcal{C}}(X)$.

Conformal dimension is a natural concept to consider since in some sense it measures the metric dimension of the best shape of X; see [BK05a] for discussion and references for this kind of uniformization problem. A key application of the definition (and its original motivation) is in the study of the conformal structure of the boundary at infinity of a negatively curved space.

Besides the trivial bound given by the topological dimension of a metric space, the conformal dimension is often difficult to estimate from below. In this paper we give such a bound for an interesting class of metric spaces.

Theorem 1.1. *Suppose (X, d) is a complete metric space which is doubling and annularly linearly connected. Then the conformal dimension $\dim_{\mathcal{C}}(X)$ is at least $C > 1$, where C depends only on the the constants associated to the two conditions above.*

Recall that a metric space is *N-doubling* if every ball can be covered using N balls of half the radius. The annularly linearly connected condition is a quantitative analogue of the topological conditions of

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being locally connected and having no local cut points. This is made precise in Definition 3.2 and the subsequent discussion. For now, a good motivating example of a space satisfying our hypotheses is the standard square Sierpiński carpet, denoted by S.

The original motivation to study such spaces was given by a particular application of Theorem 1.1. Each Gromov hyperbolic group G has an associated boundary at infinity $\partial_\infty G$, a geometric object much studied in its relationship to the group structure of G (e.g. [Gro87, Bow98, Kle06]). The boundary carries a canonical family of metrics that are pairwise quasi-symmetric, and so any quasi-symmetry invariant of metric spaces, such as conformal dimension, will give a quasi-isometry invariant of G.

If the boundary of a hyperbolic group is connected and has no local cut points, for example if it is homeomorphic to the Sierpiński carpet or the Menger curve, then its self-similarity implies that it will satisfy the (a priori stronger) hypotheses of Theorem 1.1. Thus as a corollary we answer a question of Bonk and Kleiner [BK05a, Problem 6.2].

Corollary 1.2. *Suppose G is a hyperbolic group whose boundary is non-empty, connected and has no local cut points. Then the conformal dimension of $\partial_\infty G$ is strictly greater than one.*

These topological conditions on the boundary of a hyperbolic group correspond to algebraic properties of the group itself. The boundary is non-empty when the group is infinite. Using Stallings' theorem on the ends of a group, one sees that the boundary is connected if and only if the group does not split over a finite group [Sta68].

More recently, work of Bowditch [Bow98, Theorem 6.2] shows that if $\partial_\infty G$ is connected and not homeomorphic to S^1, then G splits over a virtually cyclic subgroup if and only if $\partial_\infty G$ has a local cut point.

Using these results, we note a more algebraic version of Corollary 1.2.

Corollary 1.3. *Suppose G is a one-ended hyperbolic group whose boundary has conformal dimension one. Then either the boundary of G is homeomorphic to S^1 (and hence G is virtually Fuchsian), or G splits over a virtually cyclic subgroup.*

Outline of proof. Let us return to the example of the standard Sierpiński carpet, S. Since S has topological dimension equal to one, we need to rely on the metric structure of S to prove Theorem 1.1. It is clear that S contains an isometrically embedded copy of $C \times [0,1]$, where C equals the standard one third Cantor set. By a lemma of

Pansu, the conformal dimension of $C \times [0,1]$ equals the Hausdorff dimension of C plus one, and so we have that the conformal dimension of S is greater than one; see, for example, [Pan89b, Example 4.3].

In general, we do not have a product structure to exploit. Nevertheless, we construct a family of arcs in our space X akin to the product of an interval and a regular Cantor set (of controlled dimension), and then Pansu's lemma completes the proof.

Let us consider an example of extending a topological statement to a quantitative metric analogue. It is well known that a connected, locally connected, complete metric space X is arc-wise connected. Less well known is Tukia's analogous metric result (Theorem 2.1): a linearly connected, doubling and complete metric space is connected by quasi-arcs. (See Section 2 for definitions.)

If we now further assume that X has no local cut points – as in the situation of Theorem 1.1 – then a topological argument shows that the product of a Cantor set and the unit interval embeds homeomorphically into X. A weaker statement is that there exists a collection of arcs $\{J_\sigma\}$ in X such that, under the topology induced by the Hausdorff metric, the set $\{J_\sigma\}$ is a topological Cantor set.

We will show a quantitatively controlled analogue of this weaker statement. First, let $(\mathcal{M}(X), d_{\mathcal{H}})$ be the (complete) metric space consisting of all closed subsets of X with the Hausdorff metric $d_{\mathcal{H}}$. For each $\sigma > 0$, we shall denote by Z_σ a standard Ahlfors regular Cantor set of Hausdorff dimension σ; this is defined precisely in Section 3.

Theorem 1.4. *For all $L \geq 1$ and $N \geq 1$, there exist $C \geq 1$, $\sigma > 0$ and $\lambda' \geq 1$ such that if X is an L-annularly linearly connected, N-doubling, complete metric space of diameter at least one, then there exists a C-bi-Lipschitz embedding of Z_σ into $\mathcal{M}(X)$, where each point in the image is a λ'-quasi-arc of diameter at least $\frac{1}{C}$. Moreover, on the image the Hausdorff metric and minimum distance metric are comparable with constant C.*

So, how do we create such a good collection of arcs? First, use the topological properties of the space to split one arc into two arcs and apply Tukia's theorem (Theorem 2.1) to straighten these arcs into uniformly local quasi-arcs. Second, repeat this procedure in a controlled way by using the compactness properties of the quasi-arcs and spaces. This process gives four arcs, then eight, and so on, limiting to a collection of arcs indexed by a Cantor set. This process is described in Sections 3 and 4.

In Section 4 we use Pansu's lemma to complete the proof of Theorem 1.1. Corollary 1.2 follows from a short dynamical argument similar to one given by Bonk and Kleiner in [BK05b].

As a final remark, we emphasize that the work here is to show the existence of a *uniform* lower bound, greater than one, on the Hausdorff dimension of any quasi-symmetrically equivalent metric. Pansu gave examples of hyperbolic groups which do not have this property: the canonical family of (quasi-symmetrically equivalent) metrics on the boundary contains metrics whose Hausdorff dimension is arbitrarily close to, but not equal to, one. These groups are the fundamental groups of spaces obtained by gluing together two closed hyperbolic surfaces along an embedded geodesic of equal length in each, corresponding to an amalgamation of the two surface groups along embedded cyclic subgroups. Of course, the boundaries of such groups contain local cut points.

For more discussion on conformal dimension, we refer the reader to the Bonk and Kleiner [BK05a] and Kleiner [Kle06]. Note that these authors work with the Ahlfors regular conformal dimension; since this infimum is taken over a more restricted class of spaces, it is bounded below by the conformal dimension, and thus our result still applies.

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2. Background

2.1. **Quasi-arcs and arc straightening.**

Basic analytic definitions and results are contained in [Hei01]. Although conformal dimension is defined using quasi-symmetric mappings, we will primarily use geometric arguments inside metric spaces.

We will need some notation. A metric space (X, d) is said to be *L-linearly connected* for some $L \geq 1$ if for all $x, y \in X$ there exists a continuum $J \ni x, y$ of diameter less than or equal to $Ld(x, y)$. This is also known as the LLC(1) or BT (bounded turning) condition. We can actually assume that J is an arc, at the cost of increasing L by an arbitrarily small amount.

As already mentioned, X is *doubling* if there exists a constant N such that every ball can be covered by at most N balls of half the radius. Note that a complete, doubling metric space is proper: closed balls are compact.

A key tool in creating the collection of arcs in Theorem 1.4 is a result of Tukia that straightens arcs into local quasi-arcs. Before describing it we need some language to deal with embedded arcs. Denote the sub-arc of an arc A between x and y in A by $A[x,y]$. We say that an arc A in a doubling and complete metric space is an *ϵ-local λ-quasi-arc* if $\operatorname{diam}(A[x,y]) \leq \lambda d(x,y)$ for all $x,y \in A$ such that $d(x,y) \leq \epsilon$. If this holds for all $\epsilon > 0$, then we say A is a *λ-quasi-arc*. The terminology is natural since, by a result of Tukia and Väisälä [TV80], such an arc is (locally) the image of a quasi-symmetric embedding of the unit interval.

One non-standard definition will be useful to us: we say that an arc B *ϵ-follows* an arc A if there exists a (not necessarily continuous) map $p : B \to A$ such that for all $x,y \in B$, $B[x,y]$ is in the ϵ-neighborhood of $A[p(x),p(y)]$; in particular, p displaces points at most ϵ.

We can now state Tukia's theorem.

Theorem 2.1 ([Tuk96, Theorem 1B]). *Suppose (X,d) is a L-linearly connected, N-doubling, complete metric space. For every arc A in X and every $\epsilon > 0$, there is an arc J in the ϵ-neighborhood of A which ϵ-follows A, has the same endpoints as A, and is an $\alpha\epsilon$-local λ-quasi-arc, where $\lambda = \lambda(L,N) \geq 1$ and $\alpha = \alpha(L,N) > 0$.*

Tukia's original statement concerned subsets of $\mathbb{R}^n$. Bonk and Kleiner [BK05b, Proposition 3] used Assouad's embedding theorem to translate it into this language. For a shorter proof, see [Mac08, Theorem 1.1].

As mentioned in the introduction, this theorem has the following independently interesting corollary:

Corollary 2.2 (Tukia [Tuk96, Theorem 1A]). *Every pair of points in a L-linearly connected, N-doubling, complete metric space is connected by a λ-quasi-arc, where $\lambda = \lambda(L,N) \geq 1$.*

2.2. **Hausdorff distance and Gromov-Hausdorff convergence.** We recall some standard definitions and results (for example, see [BBI01, Chapters 7,8]).

Suppose (X,d) is a metric space. We define the distance between $x \in X$ and $U \subset X$ as

$$d(x,U) = \inf\{d(x,y) : y \in U\}.$$

The r-neighborhood of U is the set $N(U,r) = \{x : d(x,U) < r\}$, where $r \geq 0$. The Hausdorff distance $d_{\mathcal{H}}$ between $U,V \subset X$ is

$$d_{\mathcal{H}}(U,V) = \inf\{r \geq 0 : U \subset N(V,r), V \subset N(U,r)\}.$$

We say that a sequence of compact metric spaces $\{X_i\}$, $i \in \mathbb{N}$, converges to a metric space X in the Gromov-Hausdorff topology if there exist $f_i : X \to X_i$ and $\epsilon_i \geq 0$ so that f_i distorts distance by at most an additive error of ϵ_i, $N(f_i(X), \epsilon_i)$ equals X_i, and $\epsilon_i \to 0$. (This is equivalent to the usual definition of convergence with respect to the Gromov-Hausdorff metric.)

If X is N-doubling and $\epsilon > 0$, then X has a finite ϵ-net of cardinality at most $C(N, \epsilon) < \infty$. Therefore, given any sequence of N-doubling spaces their geometry on scale ϵ can be modelled using uniformly finite sets. An Arzelà-Ascoli argument gives the following result. For a proof, see [BBI01, Theorem 7.4.15].

Theorem 2.3. *Any sequence of N-doubling, complete metric spaces of diameter at most D has a subsequence that converges in the Gromov-Hausdorff topology to a complete metric space of diameter at most D.*

An analogous argument gives results when we consider configurations of subsets inside X. As a simple example, consider a sequence of pairs $\{(X_i, A_i)\}$, where each A_i is a closed subset of X_i.

We say that (X_i, A_i) converges to (X, A) in the Gromov-Hausdorff topology, where A is a closed subset of X, if, as before, there exist $f_i : X \to X_i$ and $\epsilon_i \geq 0$ so that f_i distorts distances by at most ϵ_i, $N(f_i(X), \epsilon_i) = X_i$, and $\epsilon_i \to 0$. However, we now also require that $d_{\mathcal{H}}(f_i(A), A_i) \leq \epsilon_i$. At the cost of doubling ϵ_i, we can assume that $f_i(A) \subset A_i$.

A slightly modified version of the proof of Theorem 2.3 gives the following:

Theorem 2.4. *Suppose for each $i \in \mathbb{N}$, X_i is an N-doubling metric space of diameter at most D, and A_i is an arc in X_i. Then there is a subsequence of the configurations (X_i, A_i) that converges in the Gromov-Hausdorff topology to a limit (X, A).*

Moreover, if each A_i is a λ-quasi-arc, then A will be a λ-quasi-arc also; in particular, A is an arc.

This last claim follows from an elementary argument using the definitions of convergence and quasi-arcs.

3. Unzipping arcs

Consider a complete, locally connected metric space with no local cut points, that is, no connected open set is disconnected by removing a point. In such a space it is straightforward to "unzip" a given arc A into two disjoint arcs J_1 and J_2 lying in a specified neighborhood of A. Repeating this procedure to get four arcs, then eight, and so on, it is possible, with some care, to get a limiting set homeomorphic to the product of a Cantor set and the interval. Such a limit set is useless for our purposes because there is no control on the minimum distance between two unzipped arcs, and so no way to get a lower bound on conformal dimension that is greater than one. We will use compactness type arguments to overcome this problem.

We begin by proving the topological unzipping result.

Lemma 3.1. *Given an arc A in a complete, locally connected metric space with no local cut points, and $\epsilon > 0$, it is possible to find two disjoint arcs J_1 and J_2 in $N(A, \epsilon)$ such that the endpoints of J_i are ϵ-close to the endpoints of A. Furthermore, the arcs J_i ϵ-follow the arc A.*

Proof. Here, $B_0(x, r)$ denotes the connected component of an open ball $B(x, r) \subset X$ that contains its center x. As X is locally connected, $B_0(x, r)$ is always open and connected, and, moreover, $B_0(x, r) \setminus \{x\}$ is also open and connected because x is not a local cut point. Any open and connected subset of X is arcwise connected.

Let a and b be the initial and final points of A respectively (in a fixed order given by the topology). We are going to define J_1 and J_2 inductively. There exists $w \in B_0(a, \epsilon) \setminus A$; otherwise, there would be a open set in X homeomorphic to an arc segment, violating the "no local cut point" condition. Now join w to a by an arc in $B_0(a, \epsilon)$. Stop this arc at x, the first time it meets A, and call it $J_1 = J_1[w, x]$. Set $J_2 = A[a, x]$. (Perhaps $x = a$, but this is not a problem).

Now we have two head segments for J_1 and J_2 meeting only at $x \in A$, and we want to unzip this configuration further along A. This is possible since in $B_0(x, \epsilon)$ there is a tripod type configuration with two incoming arcs J_1 and J_2 and one outgoing arc $A[x, b]$. As noted above, $B_0(x, \epsilon) \setminus \{x\}$ is arcwise connected, and so we can find an arc in this set that joins some point in J_1 (not x) to a point in $A[x, b]$ (not x). The arc may meet J_1, J_2 and $A[x, b]$ in many places but there must be some sub-arc A' joining some point in J_1 or J_2 to some point y in A with interior disjoint from them all. (See Figure 1, where A' is emphasized.) Use A' to detour one of J_1 and J_2 around x to the new unzipping point y, and extend the other J_i to y using $A[x, y]$.

What if this unzipping process approaches a limit before we are ϵ-close to the final point b in A? This cannot happen. Suppose it is not possible to unzip past $z \in A$. Since $B_0(z, \frac{\epsilon}{4}) \setminus \{z\}$ is arcwise connected, inside this set we can construct an arc A'' that detours around z, from $z_1 \in A$ to $z_2 \in A$, where $z_1 < z < z_2$ in the order on A.

Now by the limit point hypothesis, we can unzip J_1 and J_2 past z_1 to x, where $z_1 < x < z$. To continue the construction of J_1 and J_2 past z, find the arc given by following z_2 to z_1 along A'', stopping if one of

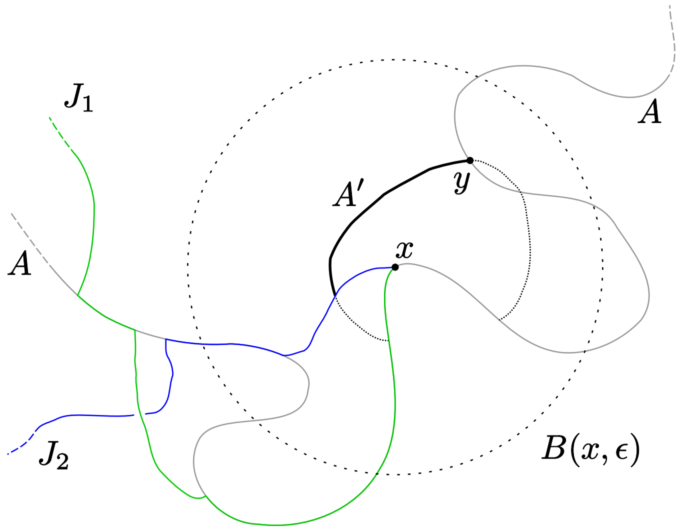

FIGURE 1. Unzipping an arc

J_1 or J_2 is met. If we reach z_1 without intersecting J_1 or J_2, as is the case in Figure 2, then continue to follow A from z_1 towards z. By the construction of J_1 and J_2, this arc will meet J_1 or J_2 before reaching z. In either case, this arc can be used as a legitimate detour around x and z, contradicting the assumption on z. Thus it is possible to continue unzipping until $x \in B(b, \frac{\epsilon}{2})$.

It remains to find labellings $f_i : J_i \to A$, for $i = 1, 2$. Define f_i to be the identity on $J_i \cap A$. Each element v of $J_i \setminus A$ was created to detour around some point $x \in A$; define $f_i(v)$ to equal x. This labelling coarsely preserves order as desired. $\square$

We would like to give a lower bound for the distance between the two split arcs. To do this we need a quantitative metric version of being locally connected with no local cut points. Let $A(p, r, R)$ be the annulus $\overline{B}(p, R) \setminus B(p, r)$.

Definition 3.2. We say a metric space X is *(L-)annularly linearly connected* for some $L \geq 1$ if whenever $p \in X$, and $x, y \in A(p, r, 2r)$ for some $r > 0$, there exists an arc J joining x to y that lies in the annulus $A(p, \frac{r}{L}, 2Lr)$. Furthermore, we assume that X is connected and complete.

At the cost of replacing L by $8L$, we may assume that such a space is also L-linearly connected.

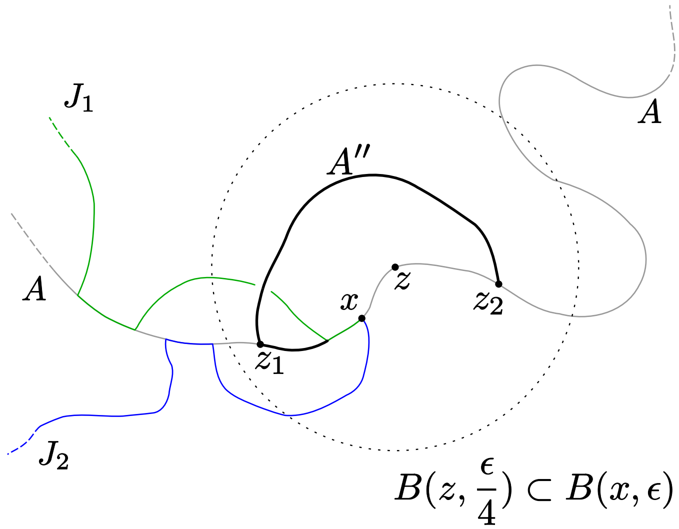

FIGURE 2. Avoiding a limit point

This condition is stronger than the usual LLC (linearly locally connected) condition [HK98, Definition 3.12], and is mentioned in [HK98, Remark 3.19]. It is called LLC (linearly locally convex) in [BMS01, Section 2]; the authors of this paper use this condition in the context of spaces that satisfy a Poincaré inequality.

The key feature of Definition 3.2 is that, unlike the usual LLC condition, it preserved under Gromov-Hausdorff convergence. To be precise, if $\{X_i\}$ is a sequence of L-annularly linearly connected, uniformly doubling, complete metric spaces and $X_i \to X_\infty$ in the Gromov-Hausdorff topology, then X_∞ is L'-annularly linearly connected for any $L' > L$. (We need to increase L slightly to allow ourselves to connect by arcs rather than just continua.) Furthermore, annularly linearly connected implies that there are no local cut points.

As a side remark, let us note that we do need a stronger condition than no local cut points as a hypothesis for Theorem 1.1: it is straightforward to modify the Sierpiński carpet construction to get a doubling, linearly connected, complete metric space with no local cut points whose *Hausdorff* dimension is one. Therefore, its conformal dimension is also one.

Now for the remainder of this section we will assume that L and N are fixed constants, and $\lambda \geq 1$, $\alpha \in (0, 1]$ are as given by Theorem 2.1. Consider the collection $\mathcal{C}$ of all λ-quasi-arcs A in any complete metric space X that is L-annularly linearly connected and N-doubling, and whose endpoints a and b satisfy $d(a,b) \in [\frac{1}{R}, R]$ for some $R \geq 1$. Fix $\epsilon > 0$, and consider the supremum of possible separations of two arcs split from A by the topological lemma above. Call this δ_A ($\delta_A > 0$).

Lemma 3.3. *There exists $\delta^\star = \delta^\star(\lambda, L, N, \epsilon, R) > 0$ such that for all $A \in \mathcal{C}$, $\delta_A > \delta^\star$.*

Proof. If not, then we can find a sequence of arcs $A_i \subset X_i$ such that $\delta_{A_i} < \frac{1}{i}$. Let a_i and b_i denote the endpoints of A_i. We are only interested in what happens inside the ball $B_i := B(a_i, 10L(\lambda R + \epsilon))$. As the sequence of configurations (B_i, A_i, a_i, b_i) is precompact in the Gromov-Hausdorff topology, by an argument similar to Theorem 2.4 we can take a subsequence converging to $(B_\infty, A_\infty, a_\infty, b_\infty)$, where A_∞ is a λ-quasi-arc inside B_∞ with endpoints a_∞ and b_∞.

Convergence here means that there exist constants $C_i \to 0$ and maps $f_i : B_\infty \to B_i$ such that f_i distorts distances by an additive error of at most C_i, and every point of B_i is within C_i of $f_i(B_\infty)$. Furthermore, $f_i(A_\infty) \subset A_i$, $f_i(a_\infty) = a_i$ and $f_i(b_\infty) = b_i$.

Since B_∞ will be L-annularly linearly connected (away from the edge of the ball), it will have no local cut points in its interior. Consequently, we can split A_∞ into two arcs J_1 and J_2 using Lemma 3.1 inside an $\frac{\epsilon}{3}$-neighborhood of A_∞. These arcs are disjoint so they are separated by some distance $0 < \delta' \leq \frac{\epsilon}{3}$. The remainder of the proof consists of showing that this contradicts the assumption on $A_i \subset B_i$ for some large i.

For sufficiently large i, $C_i \leq \frac{\delta'}{8L}$ because $C_i \to 0$ as $i \to \infty$. For $j = 1, 2$, the arc J_j in B_∞ contains a discrete path D_j with C_i-sized jumps that corresponds to a discrete path $D_j' = f_i(D_j)$ in X_i with $2C_i \leq \frac{\delta'}{4L}$ jumps. The L-linearly connected condition can then be used to join each D_j' up into a continuous arc J_j'.

To be precise, if $D_j' = \{p_1, \ldots, p_M\}$, join p_1 to p_2 by an arc J_j' of diameter at most $2C_i L \leq \frac{\delta'}{4}$. Assume that, at a stage k, we have an arc J_j' from p_1 to p_k. There is an arc I of diameter at most $\frac{\delta'}{4}$ joining p_{k+1} to p_k. We extend J_j' to p_{k+1} by following I from p_{k+1} to p_k, stopping at x, the first time it meets J_j', and gluing together $J_j'[p_1, x]$ and $I[x, p_{k+1}]$ to make a new arc J_j', and we repeat this until $k = M$. Define a map $h_j : J_j' \to D_j'$ that sends each of the points added at stage k to the point p_k. Note that for all $x, y \in J_j'$, $J_j'[x,y] \subset N(D_j'[h_j(x), h_j(y)], \frac{\delta'}{4})$; in a coarse sense, J_j' $\frac{\delta'}{4}$-follows D_j'.

By construction, J_1' and J_2' are $\frac{\delta'}{4}$-separated and ϵ-close to A_i, but to get a contradiction we need them to ϵ-follow A_i.

Since A_∞ and A_i are both λ-quasi-arcs, Lemma 3.4 below implies that for all $x, y \in A_\infty$, $f_i(A_\infty[x,y])$ is contained in the $((2C_i\lambda + C_i)\lambda + C_i)$-neighborhood of $A_i[f_i(x), f_i(y)]$. For each j, we can lift the map $h_j : J'_j \to D'_j$ to a map $h'_j : J'_j \to D_j \subset B_\infty$. By Lemma 3.1, D_j $\frac{\epsilon}{3}$-follows A_∞, so further compose with the associated map $D_j \to A_\infty$. Finally, compose with $f_i : A_\infty \to A_i$.

The composed maps $J'_j \to D_j \to A_\infty \to A_i$, for each j, show that each J'_j follows A_i with constant $\left(\frac{\delta'}{4} + C_i + \frac{\epsilon}{3} + (2C_i\lambda + C_i)\lambda + C_i\right)$. This is smaller than ϵ for sufficiently large i because $C_i \to 0$ as $i \to \infty$. We have contradicted our initial assumption, so the proof is complete. □

We used the following lemma in the proof:

Lemma 3.4. *If A and A' are λ-quasi-arcs, and $f : A \to A'$ is a map distorting distances by at most C, then for all x and y in A,*

$$f(A[x,y]) \subset N(A'[f(x), f(y)], (2C\lambda + C)\lambda + C).$$

Proof. Let $x = p_0 < p_1 < \cdots < p_n = y$ be a chain of points in A so that the diameter of $A[p_{i-1}, p_i]$ is less than C, for $i = 1, \ldots, n$.

Let $x' = f(x)$, $y' = f(y)$, and $p'_i = f(p_i)$. Order A' so that $x' \leq y'$. Let $l \geq 1$ be the greatest index so that $p'_l \leq x' \leq p'_{l+1}$. Let m, $l \leq m \leq n$, be the smallest index so that $p'_m \leq y' \leq p'_{m+1}$. So $p'_{l+1}, \ldots, p'_m \in A'[x', y']$.

Since $d(p'_i, p'_{i+1}) \leq 2C$, we have $d(p'_{l+1}, x')$ and $d(p'_m, y')$ are both less than or equal to $2C\lambda$. This lifts, by f, to give that $d(p_l, x)$ and $d(p_m, y)$ are both less than or equal to $2C\lambda + C$, and so

$$\operatorname{diam}(A[x, p_{l+1}]) \leq (2C\lambda + C)\lambda \quad \text{and} \quad \operatorname{diam}(A[p_m, y]) \leq (2C\lambda + C)\lambda.$$

Therefore,

$$\begin{aligned} f(A[x,y]) &\subset f(A[x, p_{l+1}] \cup A[p_{l+1}, p_m] \cup A[p_m, y]) \\ &\subset N(\{x', y'\}, (2C\lambda + C)\lambda + C) \cup N(A'[x', y'], 2C\lambda) \\ &\subset N(A'[f(x), f(y)], (2C\lambda + C)\lambda + C). \end{aligned}$$ □

The important point to note in Lemma 3.3 was the presence of the diameter constraint R allowing us to use a compactness type technique. Without this constraint we have various problems: our sequence of counterexamples still converges in some sense, but could give an unbounded arc. Topological unzipping still works but the resulting arcs would not necessarily have a positive lower bound on separation.

We can deal with the problem of no diameter bounds by dividing the problem into two collections of non-interacting smaller problems. To be precise, given a λ-quasi-arc A, or even just a local λ-quasi-arc, we can use Lemma 3.3 on uniformly spaced out small subarcs of A (that are genuine λ-quasi-arcs) with a sufficiently small ϵ value – this is the first collection of problems.

Now the second collection of independent problems is how to join together two of these small splittings with two disjoint arcs having uniform bound on their separation – but this a problem with bounded diameter! So compactness arguments allow us to fix this and to remove the dependence of $\delta^\star$ on R in Lemma 3.3.

Lemma 3.5. *Given* $0 < \epsilon \leq \operatorname{diam}(X)$ *and an* $\alpha\epsilon$*-local* λ*-quasi-arc* A *in* X*, where* $\alpha \in (0, 1]$ *is a constant, there exists* $\delta^\star = \delta^\star(\lambda, L, N, \alpha) > 0$ *such that for all* $\delta < \delta^\star$ *we can split* A *into two arcs that* ϵ*-follow* A *and that are* $\delta\epsilon$*-separated.*

Proof. Without loss of generality we can rescale to $\epsilon = 1$. As before, choose a linear order on A compatible with its topology. Let x_0 be the first point in A, and y_0 be the first point at distance $D_1 = \frac{\alpha}{5\lambda}$ from x_0. (If there is no such point, $\operatorname{diam}(A) \leq \frac{1}{5} = \frac{\epsilon}{5}$ and so we can split A into two points that are $\frac{\epsilon}{2}$-separated.) Label the next point at distance D_1 from y_0 by x_1. Continue in this manner with all jumps D_1 until the last label y_n, with $d(x_n, y_n) \in [D_1, 3D_1)$.

Let $D_2 = \frac{1}{4}D_1 = \frac{\alpha}{20\lambda}$, and $D_3 = \frac{1}{2\lambda(L\lambda+2)}D_2$. We can control the interactions of the collection of sub-arcs of types $A[x_i, y_i]$ and $A[y_i, x_{i+1}]$: the D_3 neighborhoods of two different such sub-arcs are disjoint outside the collection of balls $\{B(x_i, D_2)\} \cup \{B(y_i, D_2)\}$. This is because otherwise there are points z and z' in two different sub-arcs that satisfy $d(z, z') \leq 2D_3 < \alpha$, so the diameter of $A[z, z']$ is less than $2\lambda D_3 < \frac{1}{2}D_2$ – but $A[z, z']$ has to pass through the center of a D_2-ball that does not contain z or z', which is a contradiction.

Now $A[x_i, y_i]$ is a λ-quasi-arc, and we use Lemma 3.3 to create J_i and J_i' in a $\frac{1}{2}D_3$ neighborhood of $A[x_i, y_i]$ that are $\frac{1}{2}\delta_0$-separated for some $\delta_0 = \delta_0(\lambda, L, N, D_3) > 0$. By applying Theorem 2.1 to straighten the arcs, we may assume that they are λ'-quasi-arcs in a D_3 neighborhood of $A[x_i, y_i]$ that are $\frac{1}{4}\delta_0$-separated, where $\lambda' = \lambda'(L, N, \delta_0, D_3)$.

We want to join up the pair of arcs J_i and J_i' ending in $B(y_i, D_3)$ to the arcs J_{i+1} and J_{i+1}' starting in $B(x_{i+1}, D_3)$, without altering the setup outside the set $\operatorname{Join}(i) = B(y_i, D_2) \cup N(A[y_i, x_{i+1}], D_3) \cup B(x_{i+1}, D_2)$. Figure 3 shows this configuration. We will do this joining in two stages: first, a topological joining that keeps the arcs disjoint, and second, a quantitative version that controls the separation of the arcs in the joining.

Topological joining: Join the endpoints of J_i and J_i' to the arc A in the ball $B(y_i, LD_3)$ and the endpoints of J_{i+1} and J_{i+1}' to A in the ball

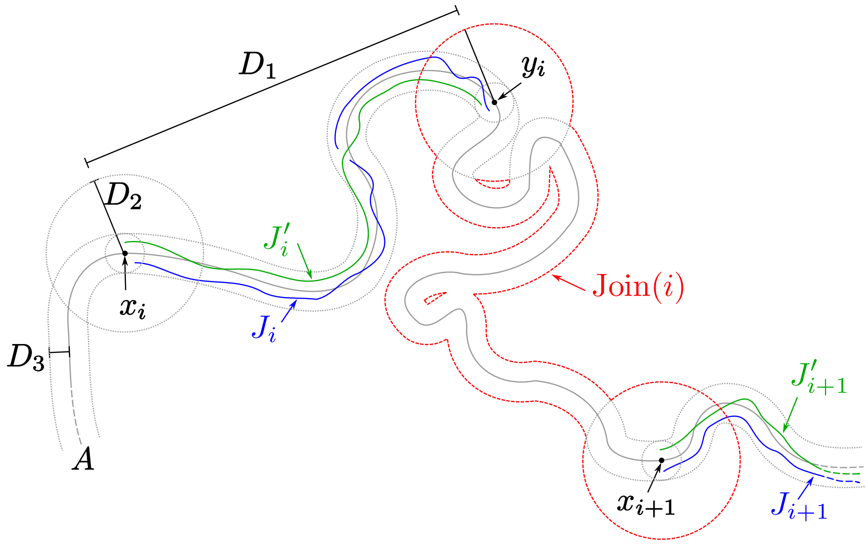

FIGURE 3. Joining unzipped arcs

$B(x_{i+1}, LD_3)$. Use the topological unzipping argument of Lemma 3.1 to unzip A along this segment resulting in 'wiring' the pair (J_i, J_i') to the pair (J_{i+1}, J_{i+1}') (not necessarily in that order) inside Join(i). These arcs are disjoint, and so separated by some distance $\delta > 0$.

Quantitative bound on δ: If there is no quantitative lower bound on δ then there are configurations (relabeling for convenience our joining arcs)

$$\mathcal{C}^n = (X^n, A^n, J_1^n, J_1'^n, J_2^n, J_2'^n),$$

where the best joining of the pair J_1^n and $J_1'^n$ to the pair J_2^n, $J_2'^n$ is at most $\frac{1}{n}$-separated.

But this configuration is precompact in the Gromov-Hausdorff topology as the X^n are all N-doubling, and the arcs are all uniform quasi-arcs. (This is the importance of Tukia's theorem.) So we can take a subsequence converging to a configuration

$$\mathcal{C}^\infty = (X^\infty, A^\infty, J_1^\infty, J_1'^\infty, J_2^\infty, J_2'^\infty)$$

in a suitable ball, and join the arcs using the topological method above, giving some valid rewiring with some positive separation $\delta^\infty > 0$. Following the proof of Lemma 3.3 we can lift this to $\mathcal{C}^n$ for sufficiently large n retaining a separation of $\frac{1}{2}\delta^\infty > 0$, which is a contradiction for large n.

Now since we have some $\delta^\star > 0$ to use when joining together our wirings in the disjoint collection of all Join(i), we can apply this procedure for all i to create two arcs along A that are $\delta^\star$-separated, for $\delta^\star$ depending only on λ, L, N, and α as desired. We assumed $\epsilon = 1$, but rescaling to any ϵ gives the same conclusion with our resulting arcs $\delta^\star\epsilon$-separated. □

4. Bounding the conformal dimension from below

We now can use the unzipping results of Section 3 (Lemma 3.5) to create a Cantor set of arcs.

By a Cantor set we mean the space $Z = \{0,1\}^{\mathbb{N}}$ with an (ultra-)metric

$$d_\sigma((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = \exp(-(\log(2)/\sigma)\inf\{n | a_n \neq b_n\}),$$

where $\sigma > 0$ is a constant. (Recall that, by convention, the infimum of the empty set is positive infinity.) The space (Z, d_σ) has Hausdorff dimension σ, and is Ahlfors regular since there is a Borel probability measure ν_σ on Z that satisfies $\frac{1}{2}r^\sigma \leq \nu_\sigma(B(z,r)) \leq r^\sigma$, for all $z \in Z$ and $r \leq \operatorname{diam}(Z)$.

Returning to the metric space (X, d) of Theorem 1.1, we can now prove Theorem 1.4.

Proof of Theorem 1.4. Begin with any arc J', assume it has endpoints one unit apart and apply Theorem 2.1 to J' and $\epsilon = \frac{1}{10}$ to get $J_\emptyset$, a λ-quasi-arc on scales below $\frac{\alpha}{10}$. Let our scaling factor be $\beta = \frac{\alpha\delta^\star}{32\lambda} \leq \frac{1}{32}$.

We can assume that, for a given n, we have a collection of λ-quasi-arcs on scales below β^n, written as $\{J_{a_1a_2\ldots a_n} | a_i \in \{0,1\}, 1 \leq i \leq n\}$, and that these arcs are β^n separated.

For each $J_{a_1a_2\ldots a_n}$, we split it into two arcs using Lemma 3.5 applied to $\epsilon = \frac{1}{8}\beta^n$, then straighten each arc using Theorem 2.1 with $\epsilon = \frac{\delta^\star}{32}\beta^n$ to get two new arcs $J_{a_1a_2\ldots a_n0}$ and $J_{a_1a_2\ldots a_n1}$ that are λ-quasi-arcs on scales below $\frac{\alpha\delta^\star}{32}\beta^n \geq \beta^{n+1}$, and are $\frac{\delta^\star}{16}\beta^n \geq \beta^{n+1}$ separated. In fact, all the arcs created at this stage are β^{n+1} separated as the new arcs arising from different arcs in the previous generation can only get $2\left(\frac{1}{8}\beta^n + \frac{\delta^\star}{32}\beta^n\right) < \frac{1}{2}\beta^n$ closer, thus remaining at least β^{n+1} apart.

At this point it is useful to record the following.

Lemma 4.1. *If J is a λ-quasi-arc on scales below ϵ, and we have an arc $J' \subset N(J, \frac{\epsilon}{4})$, whose endpoints are $\frac{\epsilon}{4}$ close to those of J, then we must have $J \subset N(J', \lambda\epsilon)$. In particular, $d_{\mathcal{H}}(J, J') \leq \lambda\epsilon$.*

Given a sequence $a = (a_1, a_2, \ldots) \in \{0,1\}^{\mathbb{N}}$, the sequence of arcs $J_\emptyset$, J_{a_1}, $J_{a_1a_2}, \ldots$ is Cauchy in the Hausdorff metric (using Lemma 4.1), and hence convergent to $J_{a_1a_2\ldots} = J_a$, a set of diameter at least $\frac{1}{2}$. A priori, this set need not be an arc, but only compact and connected. (This is actually enough to apply the argument of Pansu's lemma.) However, for each n we know that $J_{a_1 a_2 \ldots a_n}$ is a β^n-local λ-quasi-arc that β^n-follows $J_{a_1 a_2 \ldots a_{n-1}}$, and we know that $\beta < \min\left\{\frac{1}{4+2\lambda}, \frac{1}{10}\right\}$. Using these facts, [Mac08, Lemma 2.2] shows that J_a is a λ'-quasi-arc, with $\lambda' = \lambda'(\beta, L, N) = \lambda'(L, N)$, that β^n-follows $J_{a_1 a_2 \ldots a_n}$ for each n.

(Finding quasi-arcs in the limit is not unexpected since on each scale the limit set will look like the quasi-arc approximation on the same scale.)

If we set $\mathcal{M}(X)$ to be the set of all closed sets in X, we can define a map $F : Z \to \mathcal{M}(X)$ by $F(a) = J_a$. Let $\mathcal{J} = F(Z)$ be the image of this map and choose the metric d_σ for Z, $\sigma = \frac{-\log(2)}{\log(\beta)} > 0$. It remains to show that $F : (Z, d_\sigma) \to (\mathcal{M}(X), d_{\mathcal{H}})$ is a bi-Lipschitz embedding.

Take $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in Z$. Then $d_\sigma(a, b) \in (\beta^{n+1}, \beta^n]$ if and only if $a_i = b_i$ for $1 \le i < n$ and $a_n \ne b_n$. By construction, and a geometric series, $J_a \subset N(J_{a_1 \ldots a_n}, \frac{1}{4}\beta^n)$, and so as n stage arcs are β^n separated, we have

$$d_{\mathcal{H}}(J_a, J_b) \ge d(J_a, J_b) \ge \frac{1}{2}\beta^n \ge \frac{1}{2} d_\sigma(a, b). \tag{1}$$

Conversely, applying the triangle inequality and Lemma 4.1, we have

$$d_{\mathcal{H}}(J_a, J_b) \le d_{\mathcal{H}}(J_a, J_{a_1 \ldots a_{n-1}}) + d_{\mathcal{H}}(J_{b_1 \ldots b_{n-1}}, J_b) \le 2\lambda\beta^{n-1} \le \frac{2\lambda}{\beta^2} d_\sigma(a, b), \tag{2}$$

so F is bi-Lipschitz, quantitatively.

As a final remark, note that there is a natural measure $\mu_\sigma = F_*(\nu_\sigma)$ on $\mathcal{J}$. The estimates (1) and (2) imply that, for any ball $B(x, r) \subset X$, the set $\{J_a \in \mathcal{J} | J_a \cap B(x, r) \ne \emptyset\}$ is measurable (in fact open), and if two arcs J_a and J_b both meet this ball, we have $2r \ge d(J_a, J_b) \ge \frac{1}{2} d_\sigma(a, b)$, and so

$$\mu_\sigma\{J_a \in \mathcal{J} | J_a \cap B(x, r) \ne \emptyset\} \le 4^\sigma r^\sigma. \qquad \square$$

We now prove our main theorem.

Proof of Theorem 1.1. The construction of Theorem 1.4 gives a lower bound for conformal dimension by virtue of the following lemma of Pansu [Pan89b, Lemma 6.3]. This version is due to Bourdon.

Lemma 4.2 ([Bou95, Lemma 1.6])**.** *Suppose that (X, d) is a uniformly perfect, compact metric space containing a collection of arcs $\mathcal{C} = \{\gamma_i | i \in I\}$ whose diameters are bounded away from zero. Suppose further that we have a Borel probability measure μ on $\mathcal{C}$ and constants $A > 0$, $\sigma \ge 0$*

such that, for all balls $B(x,r)$ in X, the set $\{\gamma \in \mathcal{C} | \gamma \cap B(x,r) \neq \emptyset\}$ is μ-measurable with measure at most Ar^σ. Then the conformal dimension of X is at least $1 + \frac{\sigma}{\tau - \sigma}$, where τ is the packing dimension of X, and in fact $\tau - \sigma \geq 1$.

In our case X may be non-compact, but it is proper and all arcs $\gamma \in \mathcal{C}$ lie in some fixed (compact) ball in X. The packing dimension of X is finite and bounded from above by a constant derived from the doubling constant N. Furthermore, X is connected, so it is certainly uniformly perfect.

Following Theorem 1.4, we apply Lemma 4.2 with $\mathcal{C} = \mathcal{J}$, $\mu = \mu_\sigma$ and $A = 4^\sigma$, where σ depends only on L and N, to find a lower bound for the conformal dimension of $C = C(L,N) > 1$. □

We now apply our theorem to the case of conformal boundaries of hyperbolic groups.

Proof of Corollary 1.2. In [BK05b, Proposition 4], Bonk and Kleiner show that $\partial_\infty G$ with some visual metric d is compact, doubling and linearly connected. It remains to show that $(X,d) = (\partial_\infty G, d)$ is annularly linearly connected, but this follows by a proof similar to that of Bonk and Kleiner's proposition.

Suppose (X,d) is not annularly linearly connected. Then there is a sequence of annuli $A_n = A(z_n, r_n, 2r_n)$ containing points x_n and y_n such that there is no arc joining x_n to y_n inside $A(z_n, \frac{1}{n}r_n, 2nr_n)$. As X is compact we have $r_n \to 0$; otherwise, there would be a subsequence $n_j \to \infty$ as $j \to \infty$ with $r_{n_j} > \epsilon > 0$ for some ϵ. In this case, take further subsequences so that $r_{n_j} \to r_\infty \in [\epsilon, \operatorname{diam}(X)]$, $z_{n_j} \to z_\infty$, $x_{n_j} \to x_\infty$, and $y_{n_j} \to y_\infty$. Then a contradiction follows from the fact that z_∞ is not a local cut point, so we must have $r_n \to 0$.

Now we can consider the rescaled sequence $(X, \frac{1}{r_n}d, z_n)$. By doubling, this subconverges to a limit (W, d_W, z_∞) with respect to pointed Gromov-Hausdorff convergence. By [BK02, Lemma 5.2], W is homeomorphic to $\partial_\infty G \setminus \{p\}$ for some p, and so z_∞ cannot be a local cut point in W. So we can connect the components of $A(z_\infty, 0.9, 2.1)$ in $W \setminus z_\infty$ by finitely many compact sets, and these must lie in some $A(z_\infty, 1/M, 2M)$ for $1 \leq M < \infty$. For sufficiently large n we can lift these connecting sets to $A(z_n, \frac{1}{2M}r_n, 4Mr_n)$, contradicting our hypothesis.

In conclusion, $\partial_\infty G$ is annularly linearly connected, doubling and complete, and so Theorem 1.1 gives that the conformal dimension of $\partial_\infty G$ is strictly greater than one. □

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