1 The Heisenberg group does not admit a bi-Lipschitz embedding into L^1

after J. Cheeger and B. Kleiner [CK06, CK09]
A summary written by John Mackay

Abstract

We show that the Heisenberg group, with its Carnot-Caratheodory metric, does not admit a bi-Lipschitz embedding into L^1.

1.1 Introduction

Our goal is to understand the proof of the following result of Cheeger and Kleiner [CK06, CK09]:

Theorem 1. *The Heisenberg group $\mathbb{H}$ (with its usual Carnot group structure) does not admit a bi-Lipschitz embedding into L^1.*

(For definitions, see the next section and the other presentations.)

This result is interesting for its own sake, but for us its importance is that it, combined with work of Lee and Naor, gives a natural counterexample to the Goemans-Linial conjecture. To be precise, the Heisenberg group carries a metric d (comparable to the usual Carnot metric) so that $(\mathbb{H}, \sqrt{d})$ admits an isometric embedding into Hilbert space [LN06], but $(\mathbb{H}, d)$ does not admit a bi-Lipschitz embedding into L^1 [CK06, CK09].

We will actually prove a stronger statement: Lipschitz maps from $\mathbb{H}$ into L^1 collapse almost everywhere in the direction of the center. (Recall that $\mathrm{Center}(\mathbb{H}) = \{\exp(tZ) \mid t \in \mathbb{R}\}$; see the next section for details.)

Theorem 2 (Theorem 6.1 of [CK09])**.** *If $f : \mathbb{H} \to L^1$ is a Lipschitz map, then for a full measure set of points $p \in \mathbb{H}$,*

$$\liminf_{t \to 0} \frac{d(f(p), f(p \exp tZ))}{d(p, p \exp tZ)} = 0. \tag{1}$$

Of course, if there existed a C-bi-Lipschitz map $f : \mathbb{H} \to L^1$, then for every $p \in \mathbb{H}$ the quantity in equation (1) would be bounded below by $\frac{1}{C}$, a contradiction.

We are going to follow the proof of this theorem given in [CK09]. The earlier proof in [CK06] was more involved, and also used work of Franchi, Serapioni and Serra-Cassano [FSSC01] on the structure of finite perimeter sets in the Heisenberg group.

I thank Bruce Kleiner for helpful comments.

Outline: We collect some preliminary results in section 1.2, before indicating the proof of Theorem 2 in section 1.3.

1.2 Preliminary results

The Heisenberg group:

As discussed elsewhere, the Heisenberg group is the (Lie) group

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\},$$

with Lie algebra generated by

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is diffeomorphic to $\mathbb{R}^3 = \{(a, b, c) \,|\, a, b, c \in \mathbb{R}\}$, and has a Carnot group structure generated by the left invariant distribution of planes Δ, where the plane at the identity is spanned by X and Y. Putting a left-invariant norm on Δ gives a Carnot-Caratheodory metric on $\mathbb{H}$ that we denote by $d^{\mathbb{H}}$.

The center of $\mathbb{H}$ (those elements that commute with all other elements) is the subgroup

$$\text{Center}(\mathbb{H}) = \{\exp(tZ) | t \in \mathbb{R}\}.$$

(Recall that $\exp(A) = I + A + \frac{1}{2!}A^2 + \cdots$.)

Every Lie group projects onto its abelianization. In the case of $\mathbb{H}$, the map $\pi : \mathbb{H} \to \mathbb{H}/[\mathbb{H}, \mathbb{H}] = \mathbb{R}^2$ corresponds to the map $\pi((a, b, c)) = (a, b)$.

A *line* in $\mathbb{H}$ is a horizontal path (i.e., one tangent to Δ) that projects to a straight line in $\mathbb{R}^2$. Let $\mathbb{L}(\mathbb{H})$ be the collection of all lines in $\mathbb{H}$. ($\mathbb{L}(\mathbb{H})$ has a natural smooth structure.) The collection of all pairs of points that can be joined by a line is denoted by $\text{hor}(\mathbb{H}) \subset \mathbb{H} \times \mathbb{H}$.

The collection of all lines through a point $p \in \mathbb{H}$ is a *horizontal plane (centered at p)*. A *vertical plane* is the set $\pi^{-1}(L)$, where L is a line in the plane. A *half-space* is a connected component of $\mathbb{H} \setminus P$, where P is a plane.

For each $\lambda \in (0, \infty)$, let $s_\lambda : \mathbb{H} \to \mathbb{H}$ be the automorphism that scales $d^{\mathbb{H}}$ by λ, i.e., for all $g, h \in \mathbb{H}$, $d^{\mathbb{H}}(s_\lambda(g), s_\lambda(h)) = \lambda d^{\mathbb{H}}(g, h)$.

For each $g \in \mathbb{H}$, let $l_g : \mathbb{H} \to \mathbb{H}$ be the automorphism that left translates by g, i.e., for all $h \in \mathbb{H}$, $l_g(h) = gh$.

Differentiation of functions with metric space targets:

If $X = (X, d)$ is a metric space with metric d, and $\lambda > 0$, we let λX denote the metric space $(X, \lambda d)$.

If $f : X \to Y$ is *C-Lipschitz* (i.e. $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for all $x, y \in X$), then the *pullback* of the distance function d_Y to X is the function $\rho : X \times X \to [0, \infty)$, where $\rho(x, y) = f^* d_Y(x, y) = d_Y(f(x), f(y))$.

Rademacher's theorem states that if you have a Lipschitz map $f : \mathbb{R}^m \to \mathbb{R}^n$, then it is differentiable almost everywhere (a.e.). One consequence is that if $\rho : \mathbb{R}^m \times \mathbb{R}^m \to [0, \infty)$ is the pullback of the metric on $\mathbb{R}^n$ by f, then for almost every $x \in \mathbb{R}^m$, rescalings of ρ converge uniformly on compact subsets of $\mathbb{R}^m \times \mathbb{R}^m$ to give a pseudo-metric induced by a Riemannian semi-norm on $\mathbb{R}^m$.

To be precise, there is some Riemannian semi-norm on $\mathbb{R}^m$ with induced metric $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ so that for all compact sets $K \subset \mathbb{R}^m \times \mathbb{R}^m$,

$$\frac{1}{\lambda} \rho\big(x + \lambda(\cdot), x + \lambda(\cdot)\big) \to \alpha(\cdot, \cdot)$$

uniformly on K as $\lambda \to 0$. (Note that the function on the left here is $\frac{1}{\lambda}(l_x \circ s_\lambda)^* \rho$, where l_x and s_λ denote left translation by x and rescaling by λ respectively.)

Pauls proved an analogous result for Lipschitz functions $f : \mathbb{H} \to Y$, where Y is some metric space. Let $\rho = f^* d_Y$ be the associated pullback distance function.

Theorem 3 (Pauls, [Pau01], Thm. 4.7, Prop. 5.1; [CK09], Thm. 2.5)**.** *For almost every $g \in \mathbb{H}$, rescalings of ρ at g (i.e. $\frac{1}{\lambda}(l_g \circ s_\lambda)^* \rho$) converge uniformly on compact subsets of* $\mathrm{hor}(\mathbb{H}) \subset \mathbb{H} \times \mathbb{H}$ *to a left invariant Carnot (pseudo)distance $\alpha : \mathbb{H} \times \mathbb{H} \to [0, \infty)$ induced by some Finsler semi-norm on Δ.*

In the case of Rademacher's theorem, we can zoom in on f at a point of differentiability by rescaling the source and target simultaneously at x and

$f(x)$: $(\frac{1}{\lambda}\mathbb{R}^m, x) \xrightarrow{f} (\frac{1}{\lambda}\mathbb{R}^n, f(x))$. These functions are all C-Lipschitz, and as $\lambda \to 0$, we can take a limit to get a Lipschitz map $f_\omega : \mathbb{R}^m \to \mathbb{R}^n$: the derivative of f at x. Moreover, α is the pullback metric induced by f_ω^*.

In an analogous way, using ultralimits, we can do the same thing for $\mathbb{H}$:

Corollary 4 ([CK09], Cor. 2.7)**.** *Suppose $g \in \mathbb{H}$ satisfies the conclusion of Theorem 3. Then α is the metric induced by some $f_\omega : \mathbb{H} \to L^1$, and is geodesic on lines.*

Maps to L^1 and cuts:

Suppose (X, μ) is a locally compact metric measure space, where μ is a Borel measure and is finite on compact subsets of X.

A *cut* in X is a measurable subset of X. We identify cuts $E, E' \subset X$ if $\mu(E \setminus E') = \mu(E' \setminus E) = 0$. The collection of cuts $\mathrm{Cut}(X)$ is can be embedded in L^1_{loc} by mapping E to its characteristic function χ_E, and it inherits a topology from this embedding.

The *elementary cut metric* $d_E : X \times X \to [0, \infty)$ associated to a cut $E \in \mathrm{Cut}(X)$ is defined by $d_E(x_1, x_2) = |\chi_E(x_1) - \chi_E(x_2)|$.

A *cut measure* on X is a Borel measure Σ on $\mathrm{Cut}(X)$ so that

$$\int_{\mathrm{Cut}(X)} \mu(E \cap K) d\Sigma(E) < \infty$$

holds for every compact $K \subset X$.

The *cut metric* on X associated to a cut measure Σ is given by averaging the elementary cut metrics according to Σ:

$$d_\Sigma(x_1, x_2) = \int_{\mathrm{Cut}(X)} d_E(x_1, x_2) d\Sigma(E).$$

(For a.e. $x_1, x_2 \in X$ this agrees with other variations on this definition.)

Now suppose that $f : (X, \mu) \to L^1(Y, \nu)$ is an L^1_{loc} map, where (Y, ν) is a σ-finite measure space (i.e., it is a countable union of finite measure sets). Let $\rho = f^* d_{L^1}$ as before.

Theorem 5 ([CK09], Thm. 2.9; [CK06], Prop. 3.40)**.** *There is a cut measure Σ and a full measure subset $Z \subset X$, so that the cut metric d_Σ equals ρ on $Z \times Z$.*

Monotone cuts:

A cut $E \subset \mathbb{R}$ is *monotone* if it is equivalent to a measurable subset that is connected with connected complement, i.e., $\emptyset, \mathbb{R}$ or a half-infinite ray.

We say that $f : \mathbb{R} \to L^1$ is a *geodesic map* if there is a set $Z \subset \mathbb{R}$ of full measure so that for all $z_1 \le z_2 \le z_3$ in Z,

$$d_{L^1}(f(z_1), f(z_3)) = d_{L^1}(f(z_1), f(z_2)) + d_{L^1}(f(z_2), f(z_3)).$$

Given any metric α on $\mathbb{R}$, for $a \le b \le c$ in $\mathbb{R}$ we let $\text{excess}(\alpha)\{a, b, c\} = \alpha(a, b) + \alpha(b, c) - \alpha(a, c) \ge 0$. If d_E is an elementary cut metric, then $\text{excess}(\alpha)$ equals zero for almost every $\{a, b, c\}$ if and only if E is monotone. Averaging this result over a cut measure, we get:

Lemma 6. *If $f : \mathbb{R} \to L^1$ is an L^1_{loc} mapping, and Σ its associated cut measure, then the following are equivalent:*

- *f is a geodesic map, and*
- *Σ-a.e. cut E is monotone.*

We now extend this discussion to $\mathbb{H}$. Recall that $\mathbb{L}(\mathbb{H})$ is the space of lines in $\mathbb{H}$. Let $\mathbb{P}(\mathbb{H})$ be the space of unit-speed parametrized lines in $\mathbb{H}$. We relate $\mathbb{P}(\mathbb{H})$ to $\mathbb{L}(\mathbb{H})$ using the natural fibration $\mathbb{P}(\mathbb{H}) \to \mathbb{L}(\mathbb{H})$ that maps a parametrized line to the line it represents. We also have the map $\Gamma : \mathbb{R} \times \mathbb{P}(\mathbb{H}) \to \mathbb{H}$, where $\Gamma(t, p) = p(t)$.

We say that a cut (i.e., a measurable subset) $E \in \text{Cut}(\mathbb{H})$ is *monotone* if $E \cap L$ is a monotone subset of $L \simeq \mathbb{R}$ for almost every $L \in \mathbb{L}(\mathbb{H})$.

With this definition, Lemma 6 can be used to show the following:

Proposition 7 ([CK09], Prop. 3.5)**.** *Let $f : \mathbb{H} \to L^1$ be locally integrable, with associated cut measure Σ. Then the following are equivalent:*

- *For almost every $p \in \mathbb{P}(\mathbb{H})$, the map $\mathbb{R} \to L^1$ given by $t \mapsto f(p(t))$ is a geodesic map, and*
- *Σ-a.e. cut E is monotone.*

In particular, if $f : \mathbb{H} \to L^1$ is a geodesic map for almost every $L \in \mathbb{L}(\mathbb{H})$, then Σ-a.e. cut in $\text{Cut}(\mathbb{H})$ is monotone.

Why have we bothered to work with monotone sets? Monotone sets in $\mathbb{H}$ have a very special form:

Theorem 8 ([CK09], Theorem 5.1)**.** *If $E \subset \mathbb{H}$ is a monotone set, then E equals $\emptyset$, $\mathbb{H}$ or a half-space, up to a set of measure zero.*

Jeehyeon Seo will prove this in her presentation.

1.3 Outline of proof of Theorem 2

Assume that $f : \mathbb{H} \to L^1$ is a C-Lipschitz map, and that the theorem fails. Therefore there is a positive measure set of points $V \subset \mathbb{H}$ so that for all $p \in V$,

$$\liminf_{t\to 0} \frac{d(f(p), f(p \exp tZ))}{d(p, p \exp tZ)} > 0.$$

By countable additivity, there is a measurable set $W \subset \mathbb{H}$ of positive measure, $r > 0$, $\lambda > 0$, so that

$$d(f(p), f(p \exp tZ)) \geq \lambda d(p, p \exp tZ) \tag{2}$$

holds for all $p \in W$ and $|t| < r$.

Since Theorem 3 holds almost everywhere, and W has positive density, we can find a density point $p \in W$ where the conclusion of Theorem 3 is true. (A density point is a point p where as $r > 0$ tends to zero, the ratio of the measure of $W \cap B(p, r)$ relative to the measure of $B(p, r)$ tends to one.)

As in Theorem 3, we take a sequence $\lambda_k \to 0$ and we blow up the pullback distance $\rho = f^* d_{L^1}$ at p:

$$\rho_k = \frac{1}{\lambda_k}(l_g \circ s_\lambda)^* \rho = \frac{1}{\lambda_k}(f \circ l_g \circ s_\lambda)^* d_{L^1}.$$

Since f is C-Lipschitz, $\rho_k \leq Cd$, so the functions ρ_k are uniformly continuous and bounded on compact sets, so by Arzela-Ascoli (after we take a subsequence) they will converge uniformly on compact subsets to a pseudo-distance $\rho_\infty : \mathbb{H} \times \mathbb{H} \to [0, \infty)$.

Since p was a density point of Y, when we blow up (2) gives us

$$\rho_\infty(x, x \exp tZ) \geq \lambda d(x, x \exp tZ) \tag{3}$$

for every $x \in \mathbb{H}$ and $t \in \mathbb{R}$.

By Corollary 4, $\rho_\infty = f_\omega^* d_{L^1}$ for some Lipschitz map $f_\omega : \mathbb{H} \to L^1$, and ρ_∞ is geodesic on lines. Therefore, Proposition 7 implies that Σ-a.e. cut $E \in \mathrm{Cut}(\mathbb{H})$ is monotone, and Theorem 8 gives that Σ-a.e. cut $E \in \mathrm{Cut}(\mathbb{H})$ is a half-space.

This is good because we are interested in the behavior of ρ_∞ in the Z direction, i.e., along fibers $\pi^{-1}(z)$, for $z \in \mathbb{R}^2$. For almost every $z \in \mathbb{R}^2$, the cut measure will restrict to a well-defined cut measure on $\pi^{-1}(z)$ that is supported on monotone cuts, as the intersection of a half-space with $\pi^{-1}(z)$

is monotone. Therefore, $\rho_\infty(x_1, x_3) = \rho_\infty(x_1, x_2) + \rho_\infty(x_2, x_3)$ if x_1, x_2, x_3 are points in order in $\pi^{-1}(z)$.

So, putting it all together,

$$\rho_\infty(x, x \exp nZ) = n\rho_\infty(x, x \exp Z) \geq n\lambda d(x, x \exp Z) \simeq n,$$

using (3), but

$$\rho_\infty(x, x \exp nZ) \leq Cd(x, x \exp nZ) \simeq \sqrt{n},$$

a contradiction as $n \to \infty$.

References

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