QUASI-CIRCLES THROUGH PRESCRIBED POINTS

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Abstract. We show that in an L-annularly linearly connected, N-doubling, complete metric space, any n points lie on a λ-quasi-circle, where λ depends only on L, N and n. This implies, for example, that if G is a hyperbolic group that does not split over any virtually cyclic subgroup, then any geodesic line in G lies in a quasi-isometrically embedded copy of $\mathbb{H}^2$.

1. Introduction

Menger's theorem for graphs extends to the following topological result, known as the "n-Bogensatz" or n-arc connectedness theorem.

Theorem 1.1 ([Nöb32, Zip33, Why48])**.** *If X is a connected, locally connected, locally compact metric space that cannot be disconnected by removing any $n-1$ points, then any two points in X can be joined by n arcs, pairwise disjoint apart from their endpoints.*

A well known corollary of this result is that any n points in X lie on a simple closed curve (see Theorem 4.1 and remark in [TV08]).

In this paper, we prove analogues of these theorems for quasi-arcs and quasi-circles using quantitative topological arguments. Quasi-circles arise naturally in geometric function theory and in the study of boundaries of hyperbolic groups, and our results have consequences for the geometry of such groups.

1.1. **Statement of results.** In 1963, Ahlfors [Ahl63] showed that a Jordan curve $\gamma \subset \mathbb{R}^2$ is the image of $\mathbb{S}^1 \subset \mathbb{R}^2$ under some quasiconformal homeomorphism of $\mathbb{R}^2$ if and only if γ is *linearly connected*:

Definition 1.2. *A complete metric space (X, d) is L-*linearly connected*, for some $L \geq 1$, if for every $x, y \in X$, there exists a continuum J containing x and y, so that* $\operatorname{diam}(J) \leq Ld(x, y)$.

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(This is also called L-bounded turning. Note that in the above definition, we may assume J is an arc.)

More generally, a *quasi-circle* (respectively *quasi-arc*) is a quasisymmetric image of the standard Euclidean circle (respectively interval). Tukia and Väisälä showed that a metric Jordan curve (i.e., a metric space homeomorphic to $\mathbb{S}^1$) is a quasi-circle if and only if it is doubling and linearly connected [TV80], and likewise for metric Jordan arcs. Recall that a metric space (X, d) is N*-doubling*, for some $N \in \mathbb{N}$, if any ball of radius $r > 0$ in X can be covered by N balls of radius $r/2$.

In the remainder of this paper, we define a λ-quasi-circle (or λ-quasi-arc) to be a metric Jordan curve (or metric Jordan arc) that is doubling and λ-linearly connected.

The spaces we study have the property that they have no local cut points, in the following quantitatively controlled sense.

Definition 1.3. *Let (X, d) be a metric space. The annulus around x between radii r and R is denoted by $A(x, r, R) = \overline{B}(x, R) \setminus B(x, r)$.*

A metric space (X, d) is L-annularly linearly connected*, for some $L \geq 1$, if it is connected, and given $r > 0$, $p \in X$, any two points $x, y \in A(p, r, 2r)$ lie in an arc J so that $x, y \in J \subset A(p, r/L, 2Lr)$.*

(We may assume (on replacing L by $8L$) that X is also L-linearly connected.)

Our main theorems are quantitative versions of Theorems 1.1 and 4.1.

Theorem 1.4. *Let X be a N-doubling, L-annularly linearly connected, and complete metric space. For any $n \in \mathbb{N}$, there exists $\lambda = \lambda(L, N, n)$ so that any distinct $x, y \in X$ can be joined by n different λ-quasi-arcs, where the concatenation of any two forms a λ-quasi-circle.*

Theorem 1.5. *Suppose (X, d) is a non-trivial, N-doubling, L-annularly linearly connected, complete metric space. Then any finite set $T \subset X$ lies on a λ-quasi-circle $\gamma \subset X$, where $\lambda = \lambda(L, N, |T|)$. Moreover, if $|T| \geq 2$ we can ensure that* $\operatorname{diam}(\gamma) \leq \lambda \operatorname{diam}(T)$.

Note that these results apply to the boundaries of many hyperbolic groups. For example, we observe the following.

Corollary 1.6. *Suppose G is a δ-hyperbolic group, which does not virtually split over any finite or two-ended subgroup. Then any n geodesics in G lie in the image of an (L, C)-quasi-isometry $f : \mathbb{H}^2 \to G$, where L and C depend only on G and n.*

Proof. The boundary $\partial_\infty G$, given some fixed visual metric, is doubling and annularly linearly connected [Mac10, Proof of Corollary 1.2]. Let $x_1, \dots, x_{2n} \in \partial_\infty G$ be the endpoints of the geodesics.

We apply Theorem 1.5 to find a quasi-circle through $x_1, \dots, x_{2n}$, and extend this to find a quasi-isometrically embedded hyperbolic plane $f : \mathbb{H}^2 \to G$ [BS00, Theorems 7.4, 8.2]. Up to modifying f by a finite distance, we may assume that the geodesics lie in the image of f. $\square$

1.2. **Background and remarks.** Any two points in a connected, locally connected, locally compact metric space can be joined by an arc. The analogous statement for quasi-arcs was proved by Tukia, and is a key tool in this paper.

Theorem 1.7 ([Tuk96, Theorem 1A],[Mac08, Corollary 1.2])**.** *Suppose (X, d) is an N-doubling, L-linearly connected, complete metric space. Then there exists $\lambda = \lambda(N, L)$ so that any two points in X can be connected by a λ-quasi-arc.*

For quasi-circles, as far as the author knows, the only non-trivial existence result known prior to this paper is the following result of Bonk and Kleiner. (For an analogous statement for certain relatively hyperbolic groups, see [MS11, Theorem 1.3].)

Theorem 1.8 ([BK05, Theorem 1])**.** *If G is a hyperbolic group and its boundary $\partial_\infty G$ is not totally disconnected, then $\partial_\infty G$ contains a quasi-circle.*

Theorem 1.8 is motivated by the problem of finding surface subgroups in hyperbolic groups: undistorted surface subgroups give quasi-isometric embeddings of $\mathbb{H}^2$ in the group, and quasi-isometric embeddings of $\mathbb{H}^2$ exactly correspond to quasi-circles in the boundary. Theorem 1.8 showed that there is no geometric obstruction to finding such a surface subgroup once the group is not virtually free, answering a question of Papasoglu.

This result is proved by using Theorem 1.7, a dynamical argument, and Arzelà-Ascoli; the indirectness involved means that this method cannot show that every point in $\partial_\infty G$ lies in a quasi-circle.

We now consider Theorems 1.4 and 1.5. In these results, we cannot weaken the annular linear connectedness condition to the topological condition of Nöbeling. For example, the set $X = \{(x, y) : 0 \leq x \leq 1, |y| \leq x^2\}$ is doubling, linearly connected, and has no local cut points, but there is no quasi-circle in X that contains the point $(0, 0)$.

One might hope for a stronger result than Theorem 1.5, where rather than a quantitative no local cut points condition, we merely assume a quantitative version of "cannot be disconnected by removing N points." For example, perhaps in a doubling, LLC, complete metric space, any two points lie on a quasi-circle.

However, our arguments fail in this case, as we strongly use rescaling and Gromov-Hausdorff limits of sequences of spaces. The LLC(2) condition need not be preserved under such limits: consider a sequence of larger and larger circles converging to a line.

On the other hand, Theorem 1.5 is sharp in the following two senses. First, the hypotheses of this theorem do not suffice to ensure that $x_1, \dots x_n$ lie on γ in the cyclic order given. (Consider the closed unit square, which is doubling and annularly linearly connected, and label the four corners clock-wise x_1, x_2, x_3, x_4. There is no topologically embedded circle containing these points in cyclic order x_1, x_3, x_2, x_4.)

Second, λ must depend on n, otherwise one could take increasingly dense subsets of the sphere and find uniform quasi-circles through these sets. In the limit, this gives a contradiction.

The key technical tool that we use in this paper is a new method of joining two quasi-arcs together to make a quasi-arc. This is described in Section 2. The "quasi-arc n-Bogensatz" Theorem 1.4 is proved in Section 3. Finally, we prove Theorem 1.5 in Section 4.

1.3. **Notation.** We denote balls in a metric space (X, d) by $B(x, r) = \{y \in X : d(x, y) < r\}$. The open neighbourhood of $A \subset X$ of size r is $N(A, r) = \{y \in X : d(y, A) < r\}$. If $B = B(x, r)$, and $t > 0$, then $tB = B(x, tr)$. Similarly, if $V = N(A, r)$, and $t > 0$, then $tV = N(A, tr)$.

If C is a constant depending only on C_1, C_2, then we write $C = C(C_1, C_2)$.

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2. Joining together quasi-arcs

Any arc in a doubling, linearly connected space can be straightened into a quasi-arc.

Theorem 2.1 ([Tuk96, Theorem 1B], [Mac08, Theorem 1.1]). *Suppose (X, d) is a L-linearly connected, N-doubling, complete metric space. For every arc A in X and every $\epsilon > 0$, there is an arc J that ϵ-follows A, has the same endpoints as A, and is an $\alpha\epsilon$-local λ-quasi-arc, where $\lambda = \lambda(L, N) \geq 1$ and $\alpha = \alpha(L, N) > 0$. That is, for any $x, y \in J$ with $d(x, y) \leq \alpha\epsilon$, we have $\operatorname{diam}(J[x, y]) \leq \lambda d(x, y)$.*

The notation we use for arcs is described below.

Definition 2.2 ([Mac08])**.** *For any x and y in an embedded arc A, let $A[x,y]$ be the closed, possibly trivial, subarc of A that lies between them. Let $A[x,y) = A[x,y] \setminus \{y\}$, and so on.*

*An arc B ι-*follows *an arc A if there exists a (not necessarily continuous) map $p : B \to A$, sending endpoints to endpoints, such that for all $x, y \in B$, $B[x,y]$ is in the ι-neighbourhood of $A[p(x), p(y)]$; in particular, p displaces points at most ι.*

The goal of this section is to refine Theorem 2.1 to the following situation. Suppose an arc I is formed from two quasi-arcs joined by an arc $I' \subset I$. We show how to modify I only near I' to create a quasi-arc.

Theorem 2.3. *Let X be an N-doubling, L-linearly connected, complete metric space. Let A be an arc formed of three consecutive subarcs $A_1 = A[a_0, a_1]$, $A_2 = A[a_1, a_2]$ and $A_3 = A[a_2, a_3]$. Suppose that A_1 and A_3 are ϵ-local λ-quasi-arcs, and $d(A_1, A_3) \geq 2\epsilon$.*

Then there exists an $\alpha\epsilon$-local λ'-quasi-arc J that $\alpha\epsilon$-follows A, for $\alpha = \alpha(L,N,\lambda) > 0$ and $\lambda' = \lambda'(L,N,\lambda) > 1$. Moreover, J contains the initial and final connected components of $A \setminus N(A_2, 2\epsilon)$.

This theorem follows the proof of Theorem 2.1 given in [Mac08] verbatim, once we establish the following modified version of [Mac08, Proposition 2.1].

Proposition 2.4. *We assume the hypotheses of Theorem 2.3. There exists constants $s = s(L,N,\lambda) > 0$ and $S = S(L,N,\lambda) > 0$ with the following property: for each $\iota \in (0, \epsilon)$ there exists an arc J that ι-follows A, contains the initial and final connected components of $A \setminus N(A_2, 2\iota)$, and satisfies*

$$(*) \qquad \forall x, y \in J,\ d(x,y) < s\iota \implies \operatorname{diam}(J[x,y]) < S\iota.$$

Proof. We modify the proof of [Mac08, Proposition 2.1]. To simplify notation, we replace L by $\max\{L, \lambda\}$.

Let $r = \iota/20L$, and let $\mathcal{N}$ be a maximal r-separated net in X containing a_0 and a_3. Then there exists $\delta = \delta(L,N,\lambda) \in (0,1)$ and a collection of sets $\{V_x\}_{x \in \mathcal{N}}$ so that each V_x is a union of finitely many (closed) arcs in X, and for all $x, y \in \mathcal{N}$:

(1) $d(x,y) \leq 2r \implies y \in V_x$.
(2) $\operatorname{diam}(V_x) \leq 5Lr$.
(3) $V_x \cap V_y = \emptyset \implies d(V_x, V_y) > \delta r$.
(4) $B(x,r) \cap (A_1 \cup A_3) \subset V_x$.

To show this, we follow the proof of [Mac08, Lemma 3.1], with the exception that when we construct $V_x^{(0)}$, we also add closed arcs from $B(x, 2r) \cap (A_1 \cup A_3)$ so that the hypotheses of (4) are satisfied, and arcs joining them to x in $B(x, 2rL)$. Observe that $\operatorname{diam}(V_x^{(0)}) \leq 4Lr$, so the rest of the proof of [Mac08, Lemma 3.1] follows unchanged.

Now cover A_2 by connected open arcs which lie in some $B(z,r), z \in \mathcal{N}$, and take a finite subcover of A_2. Let $y_1, y_2, \ldots, y_m$ be points in A_2 lying in the arcs corresponding to this cover, in the order given by A_2, and let $z_1, z_2, \ldots, z_m \subset \mathcal{N}$ be the centres of the associated balls.

The collection of sets $\{V_x\}$ is locally finite, and each V_x is compact, so there exists a point $q_0 \in A_1$ that is the first point in A_1 to be contained in some V_{w_0} which meets $\bigcup_j V_{z_j}$.

Let K be the union of V_x so that $V_x \cap A_3 \neq \emptyset$. Define w_i inductively as follows, for $i > 0$. If $V_{w_{i-1}} \cap K \neq \emptyset$, set $n = i$, and stop. Otherwise, let $k_i = \max\{j : V_{w_{i-1}} \cap V_{z_j} \neq \emptyset\}$, set $w_i = z_{k_i}$, and continue.

Finally, let q_{n+1} be the last point in A_3 to be contained in some V_{w_n} meeting $V_{w_{n-1}}$. By (1), this process is well defined.

We use this sequence to build our path J in stages. Set $J_{-1} = A_1[a_0, q_0]$. Let J_0 be an arc in V_{w_0} that joins q_0 to $q_1 \in V_{w_1}$, where $J_0 \cap A[a_0, q_0] = \{q_0\}$ and $J_0 \cap V_{w_1} = \{q_1\}$. Now for i from 1 to $n-1$, let J_i be an arc in V_{w_i} that joins q_i in V_{w_i} to some $q_{i+1} \in V_{w_{i+1}}$, where q_{i+1} is the first point of J_i to meet $V_{w_{i+1}}$. Let J_n be an arc in V_{w_n} that joins q_n to $q_{n+1} \in A[q_{n+1}, a_3]$, where $J_n \cap A[q_{n+1}, a_3] = \{q_{n+1}\}$. To finish, we set $J_{n+1} = A[q_{n+1}, a_3] = A_3[q_{n+1}, a_3]$.

We claim that the arc $J = J_{-1} \cup J_0 \cup \cdots \cup J_{n+1}$ satisfies our conclusions, for suitable s and S.

To show that J ι-follows A, define a coarse map $f : J \to A$ as follows. If $x \in J_{-1} \cup J_{n+1}$, let $f(x) = x$. If $x \in J[q_0, q_1)$, set $f(x) = q_0$, and if $x \in J[q_n, q_{n+1}]$, set $f(x) = q_{n+1}$, For $x \in J[q_i, q_{i+1})$, $i = 1, \ldots, n-1$, set $f(x) = y_{k_i} \in A_2 \cap B(z_{k_i}, r)$.

It is straightforward to check that f satisfies the definition of ι-following. For example, suppose $y \in J_i, y' \in J_{i'}$ and $1 \leq i \leq i' \leq n-1$. Then

$$\begin{aligned} J[y,y'] &\subset J[q_i, q_{i'+1}] \subset N(\{w_i, \ldots, w_{i'}\}, 5Lr) \\ &\subset N(\{y_{k_i}, \ldots, y_{k_{i'}}\}, 5Lr + r) \subset N(A[y_{k_i}, y_{k_{i'}}], 5Lr + r) \\ &\subset N(A[f(y), f(y')], \iota). \end{aligned}$$

The other cases follow in similar fashion.

As f is the identity on $J_{-1} \cup J_{n+1}$, and $d(q_0, A_2), d(q_{n+1}, A_2) < \iota$, J contains the required components of A_1 and A_3.

All that remains is to show that J satisfies $(*)$. Suppose that $y \in J_i, y' \in J_{i'}$, with $y < y'$ in J, and $d(y,y') < r\delta$. There are four cases.

(i) If $i = i' = -1$ or $i = i' = n+1$, we have $\operatorname{diam}(J[y,y']) \leq \lambda d(y,y') \leq Lr\delta$.

(ii) If $0 \le i, i' \le n$, then $d(V_{w_i}, V_{w_{i'}}) < r\delta$ so by construction and (3) we have $|i - i'| \le 1$ and thus by (2), $\mathrm{diam}(J[y, y']) \le 10Lr$.

(iii) If $i = -1, i' \ge 0$ then by (4), y lies in some V_x, and $d(V_x, V_{w_{i'}}) \le d(y, y') < \delta r$, so by (3) and construction, $i' = 0$. Thus $d(y', q_0) \le \mathrm{diam}(V_{w_0}) \le 5Lr$, and $d(y, q_0) \le 5Lr + d(y, y') < (5L + \delta)r$. So

$$\mathrm{diam}(J[y, y']) = \mathrm{diam}(J[y, q_0] \cup J[q_0, y']) \le \lambda(5L + \delta)r + 5Lr \le 11L^2 r.$$

(iv) The case $i \le n$, $i' = n + 1$ follows similarly to (iii).

We let $s = \delta r/\iota = \delta/20L$ and $S = \max\{Lr\delta, 10Lr, 11L^2 r\}/\iota = 11L/20$, and have proven the proposition. □

3. Many quasi-arcs between two points

Our goal in this section is the following theorem.

Theorem 1.4 *Let X be a N-doubling, L-annularly linearly connected, and complete metric space. For any $n \in \mathbb{N}$, there exists $\lambda = \lambda(L, N, n)$ so that any distinct $x, y \in X$ can be joined by n different λ-quasi-arcs, so that the concatenation of any two forms a λ-quasi-circle.*

The key part of this theorem is the following proposition that splits a quasi-arc into two relatively close and separated quasi-arcs. This uses arguments similar to [Mac10, Section 3].

Proposition 3.1. *Given $\lambda_0 \ge 1$, $\epsilon > 0$, there exists $\lambda = \lambda(L, N, \lambda_0, \epsilon) \ge 1$ and $\eta = \eta(L, N, \lambda_0, \epsilon) > 0$ with the following property:*

For any λ_0-quasi-arc $A = A[a, b]$ in an N-doubling, L-annularly linearly connected, complete metric space X, there exist two λ-quasi-arcs $J = J[a, b]$ and $J' = J'[a, b]$ with the following properties:

For all $z \in (J \cup J') \setminus \{a, b\}$,

$$d(z, A) \le \epsilon\, d(z, \{a, b\}), \text{ and} \tag{3.2}$$

$$\max\{d(z, J), d(z, J')\} \ge \eta\, d(z, \{a, b\}). \tag{3.3}$$

Proof. We may rescale so that $d(a, b) = 1$, and assume that $\epsilon < 1$. Let $\delta = 1/10\lambda_0$.

We consider A in the natural order from a to b. For each $i \in \mathbb{N}$, let x_{-i} be the first point in A at distance δ^i from a, and let x_i be the last point in A at distance δ^i from b.

Let $D_1 = \epsilon\delta/3\lambda_0$, and for $i \in \mathbb{Z} \setminus \{0\}$, let $B_i = B(x_i, D_1\delta^{|i|})$.

For $i < 0$ let $A_i = A[x_{i-1}, x_i]$, let $A_0 = A[x_{-1}, x_1]$, and for $i > 0$ let $A_i = A[x_i, x_{i+1}]$. Set $D_2 = D_1\delta/10\lambda_0 L$, and for $i \in \mathbb{Z}$ let $V_i = N(A_i, D_2\delta^{|i|})$.

Lemma 3.4. *These neighbourhoods have the following properties:*

(1) If $i \ne j$, then $B_i \cap B_j = \emptyset$.

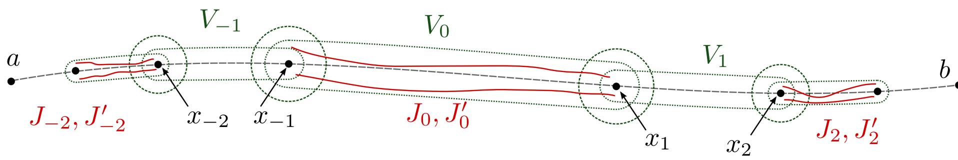

FIGURE 1. Splitting a quasi-arc into a quasi-circle

(2) If $i<0$ and $j<i$, then $V_{i+1} \cap B_j = V_{i+1} \cap V_j = B_i \cap V_j = \emptyset$.
(3) If $i<0$, then $(V_i \cap V_{i-1}) \subset B_{i-1}$.

Proof. (1) This is immediate.

(2) This follows from the following claim. If for some $i<0$, we have $z \in A[a, x_{i-1}], z' \in A[x_i, x_{i+1}]$ then $d(z,z') \geq D_1\delta^{|i-1|} + D_1\delta^{|i+1|}$, for otherwise

$$\operatorname{diam}(A[z,z']) \leq \lambda_0(D_1\delta^{|i-1|} + D_2\delta^{|i+1|}) \leq \frac{1}{2}\delta^{|i|},$$

but

$$\operatorname{diam}(A[z,z']) \geq \operatorname{diam}(A[x_{i-1},x_i]) \geq \delta^{|i|} - \delta^{|i-1|} \geq \frac{2}{3}\delta^{|i|}.$$

(3) If not, there exist $z \in A_{i-1}, z' \in A_i$ outside $\frac{1}{2}B_{i-1}$, so that $d(z,z') \leq D_2\delta^{|i-1|} + D_2\delta^{|i|}$. Therefore

$$\operatorname{diam}(A[z,z']) \leq \lambda_0 D_2 \delta^{|i-1|}(1+\delta^{-1}) \leq \frac{1}{10} D_1 \delta^{|i-1|}$$

but $A[z,z']$ must pass through the centre of $\frac{1}{2}B_{i-1}$, so $\operatorname{diam}(A[z,z']) \geq \frac{1}{2}D_1\delta^{|i-1|}$, a contradiction. □

We now split A into two disjoint arcs along the subarcs A_{2i}, $i \in \mathbb{Z}$. See Figure 1.

Lemma 3.5. *For $i \in \mathbb{Z}$ we can find two λ_1-quasi-arcs J_{2i}, J'_{2i} that $\frac{1}{2}D_2\delta^{|2i|}$-follow A_{2i}, and are $\eta_1\delta^{|2i|}$ separated, where $\eta_1 = \eta_1(L,N,\lambda_0,\epsilon) > 0$ and $\lambda_1 = \lambda_1(L,N,\lambda_0,\epsilon) > 1$.*

Proof. Using [Mac10, Lemma 3.3] with "ϵ" equal to $\frac{1}{4}D_2\delta^{|2i|}$, we split A_{2i} into two $2\eta_1\delta^{|2i|}$ separated arcs that $\frac{1}{4}D_2\delta^{|2i|}$-follow A_{2i}, for some $\eta_1 = \eta_1(L,N,\epsilon,\lambda_0) \in (0, \frac{1}{4}D_2)$. We then apply Theorem 2.1 to these arcs with "ϵ" equal to $\frac{1}{2}\eta_1\delta^{|2i|}$, to get two $\frac{1}{2}\alpha\eta_1\delta^{|2i|}$-local λ_1'-quasi-arcs J_{2i}, J'_{2i} that $\frac{1}{2}D_2\delta^{|2i|}$-follow $A[x_1,y_1]$ and are $\eta_1\delta^{|2i|}$-separated, for suitable $\alpha = \alpha(L,N)$ and $\lambda_1' = \lambda_1'(L,N)$.

Every β-local μ-quasi-arc of diameter D is a $\max\{\mu, D/\beta\}$-quasi-arc, and these arcs have diameter at most $2\lambda_0\delta^{|2i|}$, so J_{2i}, J'_{2i} are λ_1-quasi-arcs, for $\lambda_1 = \lambda_1(\alpha,\eta_1,\lambda_0,\lambda_1')$. □

Now, following the proof of [Mac10, Lemma 3.5], for each $i \in \mathbb{Z}$, one can join the pair of arcs J_{2i}, J'_{2i} to the arcs J_{2i-2}, J'_{2i-2} inside $\frac{1}{2}B_{2i-1} \cup \frac{1}{2}V_{2i-1} \cup \frac{1}{2}B_{2i-2}$, with control on the separation between the resulting arcs. We prove this in the case $i \leq 0$; $i > 0$ is handled similarly.

The separation properties of Lemma 3.4 ensure that the following process can be applied independently in each location.

Topological joining: Join the endpoints of J_{2i}, J'_{2i} to A inside the ball $B(x_{2i-1}, \frac{1}{2}LD_2\delta^{|2i|}) = (LD_2/2D_1\delta)B_{2i-1}$. Similarly, join the endpoints of J_{2i-2}, J'_{2i-2} to A inside $(LD_2/2D_1)B_{2i-2}$.

We "unzip" A along this segment to join the two pairs of arcs J_{2i}, J'_{2i} and J_{2i-2}, J'_{2i-2} by two disjoint arcs $\tilde{J}_{2i-1}, \tilde{J}'_{2i-1}$ in $\frac{1}{4}V_{2i-1}$ (see [Mac10, Lemma 3.1, Lemma 3.5]). This involves discarding the ends of the four given arcs, but all changes take place inside $\frac{1}{4}B_{2i-2} \cup \frac{1}{4}V_{2i-1} \cup \frac{1}{4}B_{2i-1}$.

Quantitative control: As in [Mac10, Lemma 3.5], compactness arguments ensure that $d(\tilde{J}_{2i-1}, \tilde{J}'_{2i-1}) \geq 2\eta_2\delta^{|2i-1|}$, for some value $\eta_2 = \eta_2(L, N, \epsilon, \lambda_0, \lambda_1) > 0$.

We now straighten these separated arcs into quasi-arcs.

Straightening: We assume, after swapping J_*, J'_* if necessary, that $\tilde{J}_{2i-1}$ joins J_{2i} and J_{2i-2}, and that $\tilde{J}'_{2i-1}$ joins J'_{2i} and J'_{2i-2}.

We apply Theorem 2.3 to $J_{2i-2} \cup \tilde{J}_{2i-1} \cup J_{2i}$ with "ϵ" equal to $\frac{1}{2}\eta_2\delta^{|2i-1|}$, to straighten the arc into a λ_2-quasi-arc $J_{2i-2} \cup J_{2i-1} \cup J_{2i}$, making changes only in $\frac{1}{2}B_{2i-2} \cup \frac{1}{2}V_{2i-1} \cup \frac{1}{2}B_{2i-1}$. Here $\lambda_2 = \lambda_2(L, N, \eta_2, \lambda_1) \geq \lambda_1$. Again, we may discard ends of $J_{2i}, J'_{2i}, J_{2i-2}, J'_{2i-2}$ in $\frac{1}{2}B_{2i-2} \cup \frac{1}{2}B_{2i}$.

We claim that the arcs $J = \bigcup_{i \in \mathbb{Z}} J_i$ and $J' = \bigcup_{i \in \mathbb{Z}} J'_i$ satisfy our requirements.

Lemma 3.6. *J and J' are λ-quasi-arcs, for $\lambda = \lambda(L, N, \lambda_0, \epsilon)$.*

Proof. Suppose $x, y \in J$, where $x \in J_i$ and $y \in J_j$, $i \leq j$. It suffices to consider the following three cases.

If $|i - j| \leq 1$, then $\operatorname{diam}(J[x, y]) \leq \lambda_2 d(x, y)$.

If $i < 0 < j$ then $d(a, x), d(b, y) \leq \frac{1}{5}$, so $\operatorname{diam}(J[x, y]) \leq \operatorname{diam}(J) \leq 2\lambda_0 \leq 4\lambda_0 d(x, y)$.

If $i+1 < j \leq 0$, then $x \in \frac{1}{2}B_{i-1} \cup \frac{1}{2}V_i \cup \frac{1}{2}B_i$, and $y \in \frac{1}{2}B_{j-1} \cup \frac{1}{2}V_j \cup \frac{1}{2}B_j$ (where $B_0 = B_1$). Thus by Lemma 3.4, $d(x, y) \geq \frac{1}{2}D_2\delta^{|j|}$, so

$$\operatorname{diam}(J[x, y]) \leq \operatorname{diam}(J[a, x_j]) \leq 2\lambda_0\delta^{|j|} \leq \frac{4\lambda_0}{D_2} d(x, y).$$

We set $\lambda = \max\{\lambda_2, 4\lambda_0/D_2\}$, and are done. □

It remains to check the neighbourhood and separation conditions.

Lemma 3.7. *J and J' satisfy* (3.2).

Proof. It suffices to consider $z \in J_i$, $i \leq 0$. If $z \in \frac{1}{2}B_{i-1}$, then $d(z, A) < \frac{1}{2}D_1\delta^{|i-1|}$, and $d(z, a) > \delta^{|i-1|}(1 - \frac{1}{2}D_1) > \frac{1}{2}\delta^{|i-1|}$. Therefore, as $D_1 < \epsilon$, (3.2) holds. Similarly, if $z \in \frac{1}{2}B_i$, (3.2) holds.

It remains to check when $z \in \frac{1}{2}V_i$. Then there exists some $z' \in A_i$ so that $d(z, z') \leq \frac{1}{2}D_2\delta^{|i|}$. Thus as $d(a, A_i) \geq \delta^{|i-1|}/\lambda_0$,

$$d(z, a) \geq d(z', a) - \frac{1}{2}D_2\delta^{|i|} \geq \delta^{|i|}(\frac{\delta}{\lambda_0} - \frac{D_2}{2}) \geq \frac{1}{20\lambda_0^2}\delta^{|i|}.$$

Since $d(z, A) \leq \frac{1}{2}D_2\delta^{|i|}$, and $20\lambda_0^2 D_2/2 \leq \epsilon$, we are done. □

Lemma 3.8. *J and J' satisfy* (3.3)*, for some $\eta = \eta(L, N, \epsilon, \lambda_0)$.*

Proof. Suppose $z \in J_i \subset J$, for $i \leq 0$. Let $z' \in J'$ be the closest point to z. If $z' \in J'_{i-1} \cup J'_i \cup J'_{i+1}$, then $d(z, z') \geq \min\{\eta_1\delta^{|i|}, \eta_2\delta^{|i-1|}\}$. Otherwise, by Lemma 3.4, $d(z, z') \geq \frac{1}{2}D_2\delta^{|i|}$. □

This completes the proof of Proposition 3.1. □

Observe that the relative separation condition (3.3) proven above suffices to show that we have a quasi-circle.

Lemma 3.9. *If J, J' are two λ-quasi-arcs with the same endpoints a, b, and satisfying* (3.3) *for some $\eta \in (0, 1)$, then $\gamma = J \cup J'$ is a $6\lambda/\eta$-quasi-circle.*

Proof. Clearly γ is a topological circle. Let $x, y \in \gamma$ be two points we wish to check for linear connectivity. The only non-trivial case is when (up to relabelling) $x \in J \setminus \{a, b\}$ and $y \in J' \setminus \{a, b\}$.

First suppose that $d(\{x, y\}, \{a, b\}) \geq \frac{1}{2}d(a, b)$, Then by (3.3), we have $d(x, y) \geq \eta \cdot \frac{1}{2}d(a, b)$, so

$$\operatorname{diam}(\gamma[x, y]) \leq \operatorname{diam}(\gamma) \leq 2\lambda d(a, b) \leq \frac{4\lambda}{\eta}d(x, y), \tag{3.10}$$

where $\gamma[x, y]$ denotes an appropriate subarc of γ.

Otherwise, we may suppose that $d(x, a) \leq \frac{1}{2}d(a, b)$, and so $d(x, y) \geq \eta \cdot d(x, a)$. If $d(x, y) \geq \frac{1}{3}d(a, b)$ then as in (3.10) we have $\operatorname{diam}(\gamma) \leq 6\lambda d(x, y)$. So we assume that $d(x, y) \leq \frac{1}{3}d(a, b)$, giving that $d(y, a) \leq \frac{5}{6}d(a, b)$, hence $d(y, b) \geq \frac{1}{5}d(y, a)$. Thus $d(x, y) \geq \eta \cdot \frac{1}{5}d(y, a)$. Therefore,

$$\begin{aligned}\operatorname{diam}(\gamma[x, y]) &\leq \operatorname{diam}(J[a, x]) + \operatorname{diam}(J'[a, y]) \leq \lambda d(a, x) + \lambda d(a, y) \\ &\leq \frac{\lambda}{\eta}d(x, y) + \frac{5\lambda}{\eta}d(x, y) = \frac{6\lambda}{\eta}d(x, y).\end{aligned}$$ □

We now complete the proof of the "quasi-arc n-Bogensatz."

Proof of Theorem 1.4. We may assume that $n = 2^m$.

We claim that by induction on m, we can find $\kappa_m = \kappa_m(L, N) \geq 1$ and $\eta_m = \eta_m(L, N) \in (0, 1)$ so that there are 2^m different κ_m-quasi-arcs from x to y that pairwise satisfy (3.3) with $\eta = \eta_m$.

The $m = 0$ case follows from Theorem 1.7, finding a κ_0-quasi-arc between x and y, where $\kappa_0 = \kappa_0(L, N)$. We set $\eta_0 = 1$.

For the induction step with $m \geq 1$, apply Proposition 3.1 to each of the 2^{m-1} previous κ_{m-1}-quasi-arcs, with $\epsilon_m = \frac{1}{4}\eta_{m-1}$. This results in 2^m different κ_m-quasi-arcs, pairwise satisfying (3.3) for some value of η which we denote by η_m. (Here $\kappa_m = \kappa_m(L, N, \kappa_{m-1}, \epsilon_m) > 1$, and $\eta_m = \eta_m(L, N, \kappa_{m-1}, \epsilon_m) > 0$.)

Finally, Lemma 3.9 completes the proof. □

4. Quasi-circles through n points

The following corollary of the n-Bogensatz is well known. To motivate the proof of Theorem 1.5, we include a short proof.

Theorem 4.1. *Let X be a connected, locally connected and locally compact metric space. If $n \geq 2$ and X is not disconnected by the removal of any $n-1$ points, then any n points in X lie on a simple closed curve.*

Proof. The $n = 2$ case is just a restatement of Theorem 1.1.

We prove the $n > 2$ case by induction. Suppose $x_1, \ldots, x_n$ are given. By induction, we can find a simple closed curve γ containing $x_1, \ldots, x_{n-1}$; we relabel so that they are in the cyclic order $x_1, \ldots x_{n-1}$. Let D be a closed disc with centre labelled x_*, and choose subsets $\{y_1, \ldots y_n\} \subset \partial D$ and $\{y'_1, \ldots, y'_n\} \subset \gamma$. Let Y be the topological space formed from D and X by gluing together y_i and y'_i for each $1 \leq i \leq n$.

The space Y is connected, locally connected, locally compact, and cannot be disconnected by the removal of any $n-1$ points. Thus Theorem 1.1 gives n disjoint arcs $\alpha_1, \ldots, \alpha_n$ from x_* to x_n in Y. For each $i = 1, \ldots, n$, let β_i be the closed, connected subarc of α_i which contains x_n and exactly one point z_i of γ. Each point z_i lies in one of $\gamma[x_1, x_2), \gamma[x_2, x_3), \ldots \gamma[x_{n-1}, x_1)$. By the pigeonhole principle, two of the points lie in the same interval, and so we use these two β arcs to find a simple closed curve containing $x_1, \ldots, x_n$. □

This proof cannot be used directly in the quasi-arc case: the space Y has local cut points. Moreover, to apply the straightening techniques of Theorem 2.3 we need a quasi-arc of controlled size through each x_i. In adapting this proof, the following corollary of the n-Bogensatz, due to Zippin, will be useful.

Theorem 4.2 ([Zip33, Corollary 9]). *Let X be a connected, locally connected, locally compact, separable metric space. If $A, B \subset X$ are compact subsets of size at least n, and there is no subset $S \subset X$ of size at most $n-1$ so that $A \setminus S$ and $B \setminus S$ lie in different components of $X \setminus S$, then there exists n disjoint arcs joining A and B.*

Proof of Theorem 1.5. The $n = 1$ and $n = 2$ cases follow from Theorem 1.4. We prove the $n > 2$ case by strong induction.

By induction, there exists $\lambda_1 = \lambda_1(L, N, n-1)$ so that any set T of at most $n-1$ points in an N-doubling, L-annularly linearly connected, complete metric space X must lie in a λ_1-quasi-circle γ with $\operatorname{diam}(\gamma) \leq \lambda_1 \operatorname{diam}(T)$.

Suppose $x_1, \ldots, x_n$ are given. Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_i, x_j)$ for all $i \neq j$, and that $d(x_1, x_i) \leq d(x_1, x_{i+1})$ for $i = 2, \ldots, n-1$. We rescale so that $d(x_1, x_n) = 1$.

Let $s = d(x_1, x_2)$, $S = \operatorname{diam}(\{x_1, \ldots, x_n\}) \in [1, 2]$, and set $\delta = 1/200L^2\lambda_1^3$.

The proof splits into two cases.

Case 1: Suppose $s \geq \delta^{n-1}$.

By induction, there exists a λ_1-quasi-circle α_1 through $x_2, \ldots, x_n$ of diameter at most $2\lambda_1$, and at least s. We relabel $x_2, \ldots, x_n$ so that they lie in α_1 in this cyclic order.

Now suppose $d(x_1, \alpha_1) \leq s/10L\lambda_1$. Then one can alter α_1 using a detour in $A(x_1, s/10L^2\lambda_1, s/5\lambda_1)$ to find a simple closed curve α_2 which does not meet $B(x_1, s/10L^2\lambda_1)$. Since this only cuts out loops of α_1 in $B(x_1, s/5)$, α_2 agrees with α_1 outside $B(x_1, s/5)$, and is a λ_1-quasi-arc there. Therefore we can apply Theorem 2.3 with $\epsilon = s/100L^2\lambda_1$ to straighten α_2 into a λ_2-quasi-circle β_1, which passes through $x_2, \ldots, x_n$, and does not meet $B(x_1, s/20L^2\lambda_1)$, for $\lambda_2 = \lambda_2(L, N, \lambda_1, s/S) \geq \lambda_1$.

If $d(x_1, \alpha_1) \geq s/10L\lambda_1$, then we set $\beta_1 = \alpha_1$ and continue.

By the $n = 2$ case of the theorem, we find a λ_1-quasi-circle β_2 through x_1 of diameter at least $s/50L^2\lambda_1^2$, inside $B(x_1, s/40L^2\lambda_1)$.

As X has no local cut points, no two disjoint compacta can be separated by removing any finite number of points. Therefore, Theorem 4.2 implies that we can join β_1 to β_2 by $2n$ disjoint arcs inside $B(x_1, 4LS)$. We can control the separation of these arcs.

Lemma 4.3 (Cf. [Mac10, Lemma 3.3]). *We can join β_1 to β_2 by $2n$ arcs in $B(x_1, 4\lambda_1 LS)$ that are $\delta_* S$-separated, for $\delta_* = \delta_*(L, N, \lambda_1, \lambda_2, s/S) > 0$.*

Proof. This follows from a compactness argument: if not, there is a sequence of configurations giving counterexamples. To be precise, we can find (on rescaling to $S = 1$), a sequence

$$\{\mathcal{C}^i = (X^{(i)}, x_1^{(i)}, \beta_1^{(i)}, \beta_2^{(i)})\}_{i\in\mathbb{N}}$$

so that for each $i \in \mathbb{N}$, $X^{(i)}$ is an L-annularly linearly connected, N-doubling, complete metric space with base point $x_1^{(i)}$, and $\beta_1^{(i)}$ and $\beta_2^{(i)}$ are λ_2-quasi-circles in $B(x_1^{(i)}, 2\lambda_1 S)$, with uniformly controlled diameter and separation. Moreover, there do not exist $2n$ disjoint arcs connecting $\beta_1^{(i)}$ to $\beta_2^{(i)}$ which are $1/i$ separated.

Such configurations have a subsequence that converges to a limit configuration $(X^\infty, x_1^\infty, \beta_1^\infty, \beta_2^\infty)$ in the Gromov-Hausdorff topology. We apply Theorem 4.2 to the limit space X^∞ to find $2n$ disjoint arcs joining β_1^∞ to β_2^∞ inside $B(x_1^\infty, 3\lambda_1 LS)$. As these arcs are disjoint, they are separated by some definite distance. These arcs will then lift back to $\mathcal{C}^i$ for sufficiently large i to give a contradiction. □

Now of these $2n$ arcs, at most n of them can be $\frac{1}{2}\delta_* S$ close to any of the n different points $x_1, \dots, x_n$. Therefore, we can find n arcs $\gamma_1', \dots, \gamma_n'$ which join β_1 to β_2, are $\delta_* S$-separated, and have distance at least $\frac{1}{2}\delta_* S$ from any x_i.

By the pigeonhole principle, two of the arcs in $\{\gamma_j'\}$ must have endpoints that lie in the same arc out of $\beta_1(x_2, x_3), \dots, \beta_1(x_{n-1}, x_n)$ and $\beta_1(x_n, x_2)$. Let us call these arcs $\gamma_1 = \gamma_1[y_1, z_1]$ and $\gamma_2 = \gamma_2[y_2, z_2]$, where $y_1, y_2 \in \beta_1$, and $z_1, z_2 \in \beta_2$.

Let γ_3 be the simple closed curve formed from $\beta_1[y_1, y_2]$ (containing $x_2, \dots, x_n$), γ_1, γ_2, and $\beta_2[z_1, z_2]$ (containing x_1). As β_1, β_2 are quasi-arcs, and we have control on the distance of γ_1, γ_2 from $x_1, \dots, x_n$, we can apply Theorem 2.3 to straighten γ_2 into a λ-quasi-circle γ, where $\lambda = \lambda(L, N, \lambda_1, \lambda_2, \delta_*, s/S) = \lambda(L, N, n)$. Moreover, $\operatorname{diam}(\gamma) \leq 4\lambda_1 S$ as desired.

Case 2: Suppose $s < \delta^{n-1}$.

This case is similar to Case 1, except now s may be arbitrarily small, so we replace β_2 by a quasi-circle through x_1 and all points close to it.

Consider the set $U = \{d(x_1, x_i)\}_{i=3}^{n-1}$ of size $n-3$. One of the intersections $U \cap [\delta^{n-1}, \delta^{n-2}), \dots, U \cap [\delta^2, \delta^1)$ is empty. Thus there exists $m \in \{2, \dots, n-1\}$ so that $d(x_1, x_m) \leq \delta d(x_1, x_{m+1})$.

Let α_1 be a λ_1-quasi-circle through $\{x_1, x_{m+1}, x_{m+2}, \dots, x_n\}$. Similarly to case 1, use the L-annularly linearly connected property for $A(x_1, 4\lambda_1 Ld(x_1, x_m), 8\lambda_1 Ld(x_1, x_m))$ to find a circle α_2 that detours α_1 around $B(x_1, 4\lambda_1 d(x_1, x_m))$, while only cutting out loops in

$$B(x_1, 8\lambda_1^2 L^2 d(x_1, x_m)) \subset B(x_1, \tfrac{4}{5} d(x_1, x_{m+1})).$$

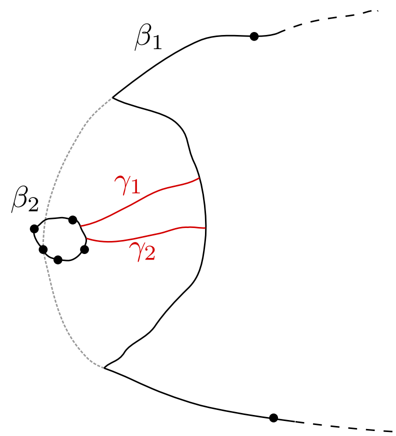

FIGURE 2. Joining two quasi-circles in case 2

In particular, α_2 contains $\{x_{m+1}, \ldots, x_n\}$, and we relabel so they are in this cyclic order.

We use Theorem 2.3 with $\epsilon = \lambda_1 d(x_1, x_m)$ to straighten α_2 into a quasi-circle β_1 which remains outside $B(x_1, 3\lambda_1 d(x_1, x_m))$. Moreover, β_1 will $9\lambda_1 L d(x_1, x_m)$-follow α_1. Inside $B(x_1, 9\lambda_1^2 L^2 d(x_1, x_m))$, β_1 is a λ_3-quasi-arc, where $\lambda_3 = \lambda_3(L, N, \lambda_1)$ is independent of $d(x_1, x_m)$.

Let β_2 be a λ_1-quasi-circle through $\{x_1, \ldots, x_m\}$, relabelled so they are in this cyclic order, of diameter at most $2\lambda_1 d(x_1, x_m)$ (see Figure 2).

As in Case 1, by Theorem 4.2 we can join β_1 to β_2 by $2n$ disjoint arcs inside $B(x_1, 10\lambda_1^2 L^2 d(x_1, x_m))$. Inside this ball we have control on the diameter of β_2, and the quasi-arc constants of β_1, β_2. Therefore, a similar argument to Lemma 4.3 gives that these arcs are $\delta_* d(x_1, x_m)$ separated, where $\delta_* = \delta_*(L, N, \lambda_1, \lambda_3)$.

As before, n of these arcs, let us call them $\gamma_1', \ldots, \gamma_n'$, will join β_1 to β_2, be $\delta_* d(x_1, x_m)$-separated, and have distance at least $\frac{1}{2}\delta_* d(x_1, x_m)$ from any x_i.

By the pigeonhole principle, two of the arcs in $\{\gamma_j'\}$ must have endpoints that lie in the same arc out of $\beta_2(x_1, x_2), \ldots, \beta_2(x_{m-1}, x_m)$ and $\beta_2(x_m, x_1)$. Let us call these arcs $\gamma_1 = \gamma_1[y_1, z_1]$ and $\gamma_2 = \gamma_2[y_2, z_2]$, where $y_1, y_2 \in \beta_1$, and $z_1, z_2 \in \beta_2$. (Again, see Figure 2.)

Using the fact that β_1 follows γ_1, we see that the diameter of the smaller arc $\beta_1[y_1, y_2]$ is at most $100\lambda_1^3 L^2 d(x_1, x_m) < \frac{1}{2} d(x_1, x_{m+1})$. Therefore, there is a subarc $\beta_1'[y_1, y_2] \subset \beta_1$ containing $x_{m+1}, \ldots, x_n$.

Let γ_3 be the simple closed curve formed from β_1', γ_1, γ_2, and $\beta_2[z_1, z_2]$ (containing $x_1, \ldots, x_m$). As β_1, β_2 are quasi-arcs, and we have control on the distance of γ_1, γ_2 from $x_1, \ldots, x_n$, we can apply Theorem 2.3 with $\epsilon = \frac{1}{2}\delta_* d(x_1, x_m)$ to straighten γ_2 into a quasi-circle γ.

Let us show that γ is a quasi-circle with controlled constant. Observe that γ D-follows α_1, where $D = 10\lambda_1^2 L^2 d(x_1, x_m)$. Let $f : \gamma \to \alpha_1$ be the associated map. Consider the following three cases.

(i) From Theorem 2.3, there exists $\lambda_4 = \lambda_4(L, N, \lambda_1, \delta_*)$ so that if $z, z' \in \gamma \cap B(x_1, 10\lambda_1 D)$, then $\mathrm{diam}(\gamma[z, z']) \leq \lambda_4 d(z, z')$.

(ii) If $\gamma[z, z'] \cap B(x_1, 2\lambda_1 D) = \emptyset$, then $\gamma[z, z'] = \alpha_1[z, z']$, so we have $\mathrm{diam}(\gamma[z, z']) \leq \lambda_1 d(z, z')$.

(iii) Otherwise, we know that $\mathrm{diam}(\gamma[z, z']) \geq 8\lambda_1 D$, so

$$\begin{aligned} \mathrm{diam}(\gamma[z, z']) &\leq 2D + \mathrm{diam}(\alpha_1[f(z), f(z')]) \leq 2D + \lambda_1 d(f(z), f(z')) \\ &\leq 2D + 2\lambda_1 D + \lambda_1 d(z, z'), \end{aligned}$$

so $\frac{1}{2}\mathrm{diam}(\gamma[z, z']) \leq \lambda_1 d(z, z')$, thus $\mathrm{diam}(\gamma[z, z']) \leq 2\lambda_1 d(z, z')$.

Therefore, γ is a λ-quasi-circle, where $\lambda = \max\{\lambda_4, 2\lambda_1\}$ depends only on L, N, n. Observe that $\mathrm{diam}(\gamma) \leq 4\lambda_1 S$, as desired. □

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