QUASI-HYPERBOLIC PLANES IN RELATIVELY HYPERBOLIC GROUPS

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Abstract. We show that any group that is hyperbolic relative to virtually nilpotent subgroups, and does not admit certain splittings, contains a quasi-isometrically embedded copy of the hyperbolic plane. The specific embeddings we find remain quasi-isometric embeddings when composed with the natural map from the Cayley graph to the coned-off graph, as well as when composed with the quotient map to "almost every" peripheral (Dehn) filling.

We apply our theorem to study the same question for fundamental groups of 3-manifolds.

The proofs of these theorems involve quantitative geometric properties of the boundaries of relatively hyperbolic groups, such as linear connectedness. In particular, we prove a new existence result for quasi-arcs that avoid obstacles.

1. Introduction

A well known question of Gromov asks whether every (Gromov) hyperbolic group which is not virtually free contains a surface group. While this question is still open, its geometric analogue has a complete solution. Bonk and Kleiner [BK05], answering a question of Papasoglu, showed the following.

Theorem 1.1 (Bonk-Kleiner [BK05]). *A hyperbolic group G contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if it is not virtually free.*

In this paper, we study when a relatively hyperbolic group admits a quasi-isometrically embedded copy of $\mathbb{H}^2$. Our main result is the following.

Theorem 1.2. *Let $(G, \mathcal{P})$ be a finitely generated relatively hyperbolic group, where all peripheral subgroups are virtually nilpotent. Then there is a quasi-isometrically embedded copy of $\mathbb{H}^2$ in G if and only if G does*

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not split as a non-trivial graph of groups where the edge groups are finite and the vertex groups virtually nilpotent.

This result generalizes Theorem 1.1 since a finitely generated group is virtually free if and only if it admits a graph of groups decomposition with all vertex and edge groups finite.

For a hyperbolic group G, quasi-isometrically embedded copies of $\mathbb{H}^2$ in G correspond to quasi-symmetrically embedded copies of the circle $S^1 = \partial_\infty \mathbb{H}^2$ in the boundary of the group. Bonk and Kleiner build such a circle when a hyperbolic group has connected boundary by observing that the boundary is *doubling* (there exists N so that any ball can be covered by N balls of half the radius) and *linearly connected* (there exists L so that any points x and y can be joined by a continuum of diameter at most $Ld(x, y)$). For such spaces, a theorem of Tukia applies to find quasi-symmetrically embedded arcs, or *quasi-arcs* [Tuk96].

We note that this proof relies on the local connectedness of boundaries of one ended hyperbolic groups, a deep result following from work of Bestvina and Mess, and Bowditch and Swarup [BM91, Proposition 3.3], [Bow98, Theorem 9.3], [Bow99b, Corollary 0.3], [Swa96].

Our strategy is similar to that of Bonk and Kleiner, but to implement this we have to prove several basic results regarding the geometry of the boundary of a relatively hyperbolic group, which we believe are of independent interest.

The model for the boundary that we use is due to Bowditch, who builds a model space $X(G, \mathcal{P})$ by gluing horoballs into a Cayley graph for G, and setting $\partial_\infty(G, \mathcal{P}) = \partial_\infty X(G, \mathcal{P})$ [Bow99c] (see also [GM08]).

We fix a choice of $X(G, \mathcal{P})$ and, for suitable conditions on the peripheral subgroups, we show that the boundary $\partial_\infty(G, \mathcal{P})$ has good geometric properties. For example, using work of Dahmani and Yaman, such boundaries will be doubling if and only if the peripheral subgroups are virtually nilpotent. We establish linear connectedness when the peripheral subgroups are one ended. (See Section 4 for precise statements.)

At this point, by Tukia's theorem, we can find quasi-isometrically embedded copies of $\mathbb{H}^2$ in $X(G, \mathcal{P})$, but these can stray far away from G into horoballs in $X(G, \mathcal{P})$. To prove our theorem we must ensure that this does not occur, and we do so by building a quasi-arc in the boundary that suitably avoids the parabolic points. This requires both additional geometric properties of the boundary (see Section 5) and the following generalisation of Tukia's theorem which builds quasi-arcs that avoid certain kinds of obstacles. (For definitions, see Section 6.)

Theorem 6.2. *Let (Z,ρ) be a compact, N-doubling and L-linearly connected metric space. Suppose $\mathcal{V}$ is a collection of compact subsets of Z with scale function $D : \mathcal{V} \to (0,\infty)$.*

Suppose $\mathcal{V}$ is L-separated, and each $V \in \mathcal{V}$ is both L-porous and L-avoidable on scales below $D(V)$. Then for a constant $\lambda = \lambda(N,L)$ there exists a λ-quasi-arc γ in Z which satisfies $\operatorname{diam}(\gamma) \geq \frac{1}{2}\operatorname{diam}(Z)$*, and $\rho(\gamma, V) \geq \frac{1}{\lambda}D(V)$ for each $V \in \mathcal{V}$.*

Our methods are able to find embeddings that avoid more subgroups than just virtually nilpotent peripheral groups.

Theorem 1.3. *Suppose both $(G,\mathcal{P}_1)$ and $(G,\mathcal{P}_1 \cup \mathcal{P}_2)$ are finitely generated relatively hyperbolic groups, where all peripheral subgroups in $\mathcal{P}_1$ are virtually nilpotent and non-elementary, and all peripheral subgroups in $\mathcal{P}_2$ are hyperbolic. Suppose G is one-ended and does not split over a subgroup of a conjugate of some $P \in \mathcal{P}_1$.*

Finally, suppose that $\partial_\infty H \subset \partial_\infty(G,\mathcal{P}_1)$ does not locally disconnect the boundary, for any $H \in \mathcal{P}_2$ (see Definition 4.2). Then there is a transversal *quasi-isometric embedding of $\mathbb{H}^2$ in G.*

If both $\mathcal{P}_1$ and $\mathcal{P}_2$ are empty the group is hyperbolic and the result is a corollary of Theorem 1.1. If $\mathcal{P}_1$ is empty, but $\mathcal{P}_2$ is not, then the group is hyperbolic, but the quasi-isometric embeddings we find avoid the hyperbolic subgroups conjugate to those in $\mathcal{P}_2$.

Example 1.4. *Let M be a compact hyperbolic* 3*-manifold with a single, totally geodesic surface as boundary ∂M. The fundamental group $G = \pi_1(M)$ is hyperbolic, and also is hyperbolic relative to $F = \pi_1(\partial M)$ (see, for example,* [Bel07, Proposition 13.1]*).*

The hypotheses of Theorem 1.3 are satisfied for $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2 = \{F\}$, since $\partial_\infty G = \partial_\infty(G,\emptyset)$ is a Sierpinski carpet, with the boundary of conjugates of F corresponding to the peripheral circles of the carpet. Thus, we find a transversal quasi-isometric embedding of $\mathbb{H}^2$ into G.

Roughly speaking, a quasi-isometric embedding fails to be transversal if the image contains points which are far apart from each other, but both close to the same left coset of a peripheral subgroup (see Definition 2.7). The notion of transversality is interesting for us in view of the following results, which show that the embeddings constructed above persist both in the coned-off (or "electrified") graph of G, and in certain peripheral fillings of G.

Proposition 2.10. *Let $\hat{\Gamma}$ be the coned-off graph of a relatively hyperbolic group $(G,\mathcal{P})$ and let $c : \Gamma \to \hat{\Gamma}$ be the natural map. Suppose Z is a geodesic metric space. If a quasi-isometric embedding $f : Z \to G$ is transversal then $c \circ f : Z \to \hat{\Gamma}$ is a quasi-isometric embedding.*

Proposition 2.11. *Let G be hyperbolic relative to $P_1, \dots P_n$ (with a fixed system of generators) and let $N_i \lhd P_i$. Let Z be a geodesic metric space, and suppose that $f : Z \to G$ is a transversal quasi-isometric embedding. There exists R_0, depending only on suitable data, so that if $B_{R_0}(e) \cap N_i = \{e\}$ for each i, then $p \circ f : Z \to G/\ll\{N_i\}\gg$ is a quasi-isometric embedding, where $p : G \to G/\ll\{N_i\}\gg$ is the quotient map.*

Recall that $G/\ll\{N_i\}\gg$ is usually referred to as a peripheral (or Dehn) filling of G, and these are relatively hyperbolic (with peripheral groups $\{P_i/N_i\}$) under the hypotheses described above [Osi07, GM08] (see Section 2.3).

When combined, Theorem 1.3 and Proposition 2.11 provide interesting examples of relatively hyperbolic groups containing quasi-isometrically embedded copies of $\mathbb{H}^2$ that do not have virtually nilpotent peripheral subgroups. A key point here is that Theorem 1.3 provides embeddings transversal to hyperbolic subgroups, and so one can find many interesting peripheral fillings. See Example 2.13 for details.

Using our results, we give a simple proof showing when the fundamental group of a closed, oriented 3-manifold contains a quasi-isometrically embedded copy of $\mathbb{H}^2$. Determining which 3-manifolds (virtually) contain immersed or embedded π_1-injective surfaces [KM09, CLR97, Lac10, BS04] is a very difficult problem. The following theorem essentially follows from known results, in particular work of Masters and Zhang [MZ08, MZ11]. However, our proof is a simple consequence of Theorem 1.2 and the geometrisation theorem.

Theorem 8.2. *Let M be a closed 3-manifold. Then $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if M does not split as the connected sum of manifolds each with geometry $S^3, \mathbb{R}^3, S^2 \times \mathbb{R}$ or* Nil.

Notice that the geometries mentioned above are exactly those that give virtually nilpotent fundamental groups.

As a final observation, we note that Leininger and Schleimer recently proved a result similar to Theorem 1.2 for Teichmüller spaces [LS11], using very different techniques.

1.1. **Outline.**

In Section 2 we define relatively hyperbolic groups and their boundaries, and discuss transversality and its consequences. In Section 3 we give preliminary results linking the geometry of the boundary of a relatively hyperbolic group to that of its model space. Further results on the boundary itself are found in Sections 4 and 5, in particular, how sets can be connected (and avoided) in a controlled manner.

The existence of quasi-arcs that avoid obstacles is proved in Section 6. The proofs of Theorems 1.2 and 1.3 are given in Section 7. Finally, connections with 3-manifold groups are explored in Section 8.

1.2. **Notation.** The notation $x \gtrsim_C y$ (occasionally abbreviated to $x \gtrsim y$) signifies $x \geq y - C$. Similarly, $x \lesssim_C y$ signifies $x \leq y + C$. If $x \lesssim_C y$ and $x \gtrsim_C y$ we write $x \approx_C y$.

Throughout, C, C_1, C_2, etc., will refer to appropriately chosen constants. The notation $C_3 = C_3(C_1, C_2)$ indicates that C_3 depends on the choices of C_1 and C_2.

For a metric space (Z, d), the *open ball* with centre $z \in Z$ and radius $r > 0$ is denoted by $B_r(z)$ or $B(z, r)$. The *closed ball* with the same centre and radius is denoted by $\overline{B}_r(z) = \overline{B}(z, r)$. We write $d(z, V)$ for the infimal distance between a subset $V \subset Z$ and a point $z \in Z$. The *open neighbourhood* of $V \subset Z$ of radius $r > 0$ is the set

$$N_r(V) = N(V, r) = \{z \in Z : d(z, V) < r\}.$$

The closed neighbourhood $\overline{N}_r(V) = \overline{N}(V, r)$ is defined analogously.

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2. Relatively hyperbolic groups and transversality

In this section we define relatively hyperbolic groups and their (Bowditch) boundaries. We introduce the notion of a transversal embedding, and show that such embeddings persist into the coned-off graph of a relatively hyperbolic group, or into suitable peripheral fillings of the same.

2.1. **Basic definitions.** There are many definitions of relatively hyperbolic groups. We give one here in terms of actions on a cusped space. First we define our model of a horoball.

Definition 2.1. *Suppose Γ is a connected graph with vertex set V and edge set E, where every edge has length one. Let T be the strip $[0, 1] \times [1, \infty)$ in the upper half-plane model of $\mathbb{H}^2$. Glue a copy of T to each edge in E along $[0, 1] \times \{1\}$, and identify the rays $\{v\} \times [1, \infty)$ for every $v \in V$. The resulting space with its path metric is the* horoball *$\mathcal{B}(\Gamma)$.*

These horoballs are hyperbolic with boundary a single point. See discussion following [Bow99c, Theorem 3.8]. Moreover, it is easy to estimate distances in horoballs.

Lemma 2.2. *Suppose Γ and $\mathcal{B}(\Gamma)$ are defined as above. Let d_Γ and $d_\mathcal{B}$ denote the corresponding path metrics. Then for any distinct vertices $x, y \in \Gamma$, identified with $(x,1), (y,1) \in \mathcal{B}(\Gamma)$, we have*

$$d_\mathcal{B}(x,y) \approx_1 2\log(d_\Gamma(x,y)).$$

Proof. Any geodesic in $\mathcal{B}(\Gamma)$ will project to the image of a geodesic in Γ, so it suffices to check the bound in the hyperbolic plane, for points $(0,1)$ and $(t,1)$, with $t \geq 1$. But then $d_\mathcal{B}((0,1),(t,1)) = \operatorname{arccosh}(1+\frac{t^2}{2})$, and

$$\left|\operatorname{arccosh}\left(1+\frac{t^2}{2}\right) - 2\log(t)\right|$$

is bounded (by 1) for $t \geq 1$. □

Definition 2.3. *Suppose G is a finitely generated group, and $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ a collection of finitely generated subgroups of G. Let S be a finite generating set for G, so that $S \cap P_i$ generates P_i for each $i = 1, \ldots, n$.*

Let $\Gamma(G) = \Gamma(G,S)$ be the Cayley graph of G with respect to S. Let $X = X(G,\mathcal{P}) = X(G,\mathcal{P},S)$ be the space resulting from gluing to $\Gamma(G,S)$ a copy of $\mathcal{B}(\Gamma(P_i, S \cap P_i))$ to each coset gP_i of P_i, for each $i = 1, \ldots, n$.

We say that $(G,\mathcal{P})$ is relatively hyperbolic *if X is Gromov hyperbolic, and call the members of $\mathcal{P}$* peripheral subgroups.

This is equivalent to the other usual definitions of (strong) relative hyperbolicity; see [Bow99c], [GM08, Theorem 3.25] or [Hru10, Theorem 5.1]. (Note that the horoballs of Bowditch and of Groves-Manning are quasi-isometric.)

Let $\mathcal{O}$ be the collection of all disjoint (open) horoballs in X, that is, the collection of all connected components of $X \setminus \Gamma(G,S)$. Note that G acts properly and isometrically on X, cocompactly on $X \setminus \left(\bigcup_{O\in\mathcal{O}} O\right)$, and the stabilizers of the sets $O \in \mathcal{O}$ are precisely the conjugates of the peripheral subgroups. Subgroup of conjugates of peripheral subgroups are called *parabolic subgroups*.

The *boundary* of $(G,\mathcal{P})$ is the set $\partial_\infty(G,\mathcal{P}) = \partial_\infty X(G,\mathcal{P}) = \partial_\infty X$. Choose a basepoint $w \in X$, and denote the Gromov product in X by $(\cdot|\cdot) = (\cdot|\cdot)_w$; this can be defined as $(a|b) = \inf\{d(w,\gamma)\}$, where the infimum is taken over all (bi-infinite) geodesic lines γ from a to b; such a geodesic is denoted by (a,b).

Fix a visual metric ρ on $\partial_\infty X$ with parameters $C_0, \epsilon > 0$, that is for all $a, b \in \partial_\infty X$, we have $e^{-\epsilon(a|b)}/C_0 \leq \rho(a,b) \leq C_0 e^{-\epsilon(a|b)}$. These choices of ρ, ϵ and C_0 are fixed for the rest of the paper.

We denote the (path) metric on X by d, while the word metric on G will be denoted by d_G. Notice that $d \leq d_G$.

For each $O \in \mathcal{O}$ the set $\partial_\infty O$ consists of a single point $a_O \in \partial_\infty X$ called a *parabolic point*. We also set $d_O = d(w, O)$.

Horoballs can also be viewed as sub-level sets of Busemann functions. We will use this fact frequently in the remainder of this paper.

Definition 2.4. *Given a point $c \in \partial_\infty X$, and basepoint $w \in X$, let $\gamma : [0, \infty) \to X$ be a geodesic from w to c. The* Busemann function *corresponding to c and w is the function $\beta_c(\cdot, w) : X \to \mathbb{R}$ defined by*

$$\beta_c(x, w) = \lim_{t\to\infty} (d(\gamma(t), x) - t)$$

Note that this is well defined as it is the limit of a monotone function.

Lemma 2.5. *There exists $C = C(\epsilon, C_0, X)$ so that for each $O \in \mathcal{O}$ we have*

$$\{x \in X : \beta_{a_O}(x, w) \leq -d_O - C\} \subseteq O \subseteq \{x \in X : \beta_{a_O}(x, w) \leq -d_O + C\}.$$

Proof. Note that $-\beta_{a_O}(\cdot, w)$ is a horofunction according to Bowditch's definition, and so the claim follows from the discussion before [Bow99c, Lemma 5.4]. □

Finally, we extend the distance estimate of Lemma 2.2 to X.

Lemma 2.6. *Let $(G, \mathcal{P})$ be relatively hyperbolic, and $X = X(G, \mathcal{P})$ as above. There exists $A = A(\delta_X) < \infty$ so that for any left coset gP of some $P \in \mathcal{P}$, with metric d_P, for any distinct $x, y \in gP$, we have*

$$d(x, y) \approx_A 2 \log(d_P(x, y)).$$

Proof. Follows from Lemma 2.2 and the hyperbolicity of X. □

2.2. **Transversality and coned-off graphs.** Our goal here is to define transversality of quasi-isometric embeddings, and show that a transversal quasi-isometric embedding of $\mathbb{H}^2$ in Γ will persist if we cone-off the Cayley graph.

We continue with the notation of Definition 2.3.

Definition 2.7. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group. Let Z be a geodesic metric space. Given a function $\eta : [0, \infty) \to [0, \infty)$, a quasi-isometric embedding $f : Z \to (G, d_G)$ is η-*transversal *if for all $M \geq 0$, $g \in G$, and $P \in \mathcal{P}$, we have* $\operatorname{diam}(f(Z) \cap N_M(gP)) \leq \eta(M)$.

A quasi-isometric embedding $f : Z \to G$ is transversal *if it is η-transversal for some such η.*

Let $\hat{\Gamma}$ be the coned-off graph of G and let $c : \Gamma \to \hat{\Gamma}$ be the natural map. Essentially, $\hat{\Gamma}$ is the graph obtained by adding an edge of length one between any two members of the same coset of the same peripheral subgroup, see [Far98]. Recall that a λ-quasi-geodesic is a (λ, λ)-quasi-isometric embedding of an interval.

Lemma 2.8. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group. If $\alpha : \mathbb{R} \to (G, d_G)$ is an η-transversal λ-quasi-geodesic, then $c(\alpha)$ is a λ'-quasi-geodesic where $\lambda' = \lambda'(\lambda, \eta, \hat{\Gamma})$. Moreover, α is also a λ'-quasi-geodesic in $X(G, \mathcal{P})$.*

Proof. Notice that α is coarsely Lipschitz in both $X(G)$ and $\hat{\Gamma}$, and on Γ we have $d_{\hat{\Gamma}} \leq d_{X(G)}$. Therefore, it suffices to show that for some $\lambda' = \lambda'(\lambda, \eta, \hat{\Gamma})$, for any $x_1, x_2 \in \alpha$, we have

$$d_{\hat{\Gamma}}(x_1, x_2) \geq \frac{1}{\lambda'} d_G(x_1, x_2) - \lambda'. \tag{2.9}$$

Let γ be the subgeodesic of α with endpoints x_1, x_2. Let $\hat{\gamma}$ be a geodesic in $\hat{\Gamma}$ with endpoints $c(x_1), c(x_2)$.

Now let $\pi : \hat{\Gamma} \to \hat{\gamma}$ be a closest point projection map, fixing x_1, x_2. As $\hat{\Gamma}$ is hyperbolic, such a projection map is coarsely Lipschitz: there exists $C_1 = C_1(\hat{\Gamma})$ so that for all $x, y \in \hat{\Gamma}$, $d_{\hat{\Gamma}}(\pi(x), \pi(y)) \leq C_1 d_{\hat{\Gamma}}(x, y) + C_1$.

By [Hru10, Lemma 8.8], there exists $C_2 = C_2(\Gamma, \lambda)$ so that every vertex of $\hat{\gamma}$ is at most a distance C_2 from γ in Γ (not just $\hat{\Gamma}$). Let $\pi' : (\hat{\gamma}, d_{\hat{\Gamma}}) \to (\gamma, d_G)$ be a map so that for all $x \in \hat{\gamma}$, $d_G(\pi'(x), x) \leq C_2$, and assume that π' fixes x_1, x_2. This map is coarsely Lipschitz also. It suffices to check this for points $x, y \in \hat{\gamma}$ connected by an edge e. If e is an edge of Γ, then clearly $d(\pi'(x), \pi'(y)) \leq 2C_2 + 1$. Otherwise, $\pi'(x), \pi'(y)$ are both in γ and within distance C_2 of the same left coset of a peripheral group, so by transversality, $d(\pi'(x), \pi'(y)) \leq \eta(C_2)$.

Thus, for $C_3 = \max\{2C_2 + 1, \eta(C_2)\}$, (2.9) with $\lambda' = C_3 C_1$ follows from

$$\begin{aligned} d_G(x_1, x_2) = d_G(\pi' \circ \pi(x_1), \pi' \circ \pi(x_2)) &\leq C_3 d_{\hat{\Gamma}}(\pi(x_1), \pi(x_2)) \\ &\leq C_3(C_1 d_{\hat{\Gamma}}(x_1, x_2) + C_1). \end{aligned}$$ □

Proposition 2.10. *Suppose Z is a geodesic metric space, and $(G, \mathcal{P})$ a relatively hyperbolic group. If a quasi-isometric embedding $f : Z \to G$ is transversal then $c \circ f : Z \to \hat{\Gamma}$ is a quasi-isometric embedding.*

Proof. By Lemma 2.8, whenever γ is a geodesic in Z, $c \circ f(\gamma)$ is a quasi-geodesic with uniformly bounded constants in $\hat{\Gamma}$. □

2.3. **Stability under peripheral fillings.** We now consider peripheral fillings of $(G, \{P_1, \ldots, P_n\})$.

Suppose $N_i \triangleleft P_i$ are normal subgroups. The *peripheral filling* of G with respect to $\{N_i\}$ is defined as

$$G(\{N_i\}) = G/ \ll \bigcup_i N_i \gg .$$

(See [Osi07] and [GM08].) Let $p : G \to G(\{N_i\})$ be the quotient map.

Proposition 2.11. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group, and $G(\{N_i\})$ a peripheral filling of G, as defined above.*

Let Z be a geodesic metric space, and suppose that $f : Z \to G$ is an η-transversal (λ, λ)-quasi-isometric embedding. There exists $R_0 = R_0(\eta, \lambda, G, \mathcal{P})$ so that if $B_{R_0}(e) \cap N_i = \{e\}$ for each i, then $p \circ f : Z \to G(\{N_i\})$ is a quasi-isometric embedding.

Proof. We will use results from [DGO11] (see also [DG08, Cou11]). For any sufficiently large R_0 we can combine Propositions 7.6 and 5.20 in [DGO11] to show that $G(\{N_i\})$ acts on a certain δ'-hyperbolic space Y (namely $Y = X'/Rot$ for X' and $Rot = (C, \mathcal{R})$ as in Proposition 7.6), with the following properties:

(1) δ' only depends on the hyperbolicity constant of $X = X(G, \mathcal{P})$,
(2) there is a 1-Lipschitz map $\psi : G(\{N_i\}) \to Y$ (this follows from the construction of Y and the fact that the inclusion of G in X is 1-Lipschitz),
(3) there is a map $\phi : (N_L(G) \subseteq X) \to Y$ such that, for each $g \in G$, $\phi|_{B(g,L)}$ is an isometry, where $L = L(R_0, X) \to \infty$ as $R_0 \to \infty$.
(4) $\psi \circ p = \phi \circ \iota$, where $\iota : G \to X$ is the natural inclusion.

Let γ be any geodesic in Z. By Lemma 2.8, $f(\gamma)$ is a λ'-quasi-geodesic in X, for $\lambda' = \lambda'(\lambda, \eta, G, \mathcal{P})$. Let α be a geodesic in X connecting the endpoints of $f(\gamma)$. Let $C_1 = C_1(\delta', \lambda')$ bound the distance between each point on α and $G \subseteq X(G)$.

Suppose that L as in (3) satisfies $L \geq C_1 + 8\delta' + 1$. Then for each $x \in \alpha$ we have that $\phi|_{B(x,8\delta'+1)}$ is an isometry, and so [BH99, Theorem III.H.1.13-(3)] gives that $\phi(\alpha)$ is a C_2-quasi-geodesic, where $C_2 = C_2(\delta')$. This implies that $(\phi \circ \iota \circ f)(\gamma)$ is a C_3-quasi-geodesic, with $C_3 = C_3(C_1, C_2)$. Let x_1, x_2 be the endpoints of γ. Using (4) above, we see that

$$\begin{aligned} d_{G(\{N_i\})}(p \circ f(x_1), p \circ f(x_2)) &\geq d_Y(\psi \circ p \circ f(x_1), \psi \circ p \circ f(x_2)) \\ &= d_Y(\phi \circ \iota \circ f(x_1), \phi \circ \iota \circ f(x_2)) \\ &\geq \tfrac{1}{C_3} d(x_1, x_2) - C_3. \end{aligned}$$

On the other hand, recall that p is 1-Lipschitz, so

$$d_{G(\{N_i\})}(p \circ f(x_1), p \circ f(x_2)) \leq d_G(f(x_1), f(x_2)) \leq \lambda d_Z(x_1, x_2) + \lambda.$$

As γ was arbitrary, we are done. $\square$

As discussed in the introduction, we can use Proposition 2.11 to find interesting examples of relatively hyperbolic groups with quasi-isometrically embedded copies of $\mathbb{H}^2$, but whose peripheral groups are not virtually nilpotent. We note the following lemma.

Lemma 2.12. *Let F_4 be the free group with four generators, and let R be fixed. Then there are normal subgroups $\{K_\alpha\}_{\alpha\in\mathbb{R}}$ of F_4 so that the quotient groups F_4/K_α are amenable but not virtually nilpotent, so that if $\alpha \neq \beta$ then F_4/K_α and F_4/K_β are not quasi-isometric, and so that $K_\alpha \cap B_R(e) = \{e\}$.*

Proof. It is shown in [Gri84] that there is an uncountable family of 4-generated groups $\{F_4/K'_\alpha\}_{\alpha\in\mathbb{R}}$ of intermediate growth with distinct growth rates. In particular, these groups are amenable, not virtually nilpotent and pairwise non-quasi-isometric. To conclude the proof, let K be a finite index normal subgroup of F_4 so that $K \cap B_R(e) = \{e\}$, and let $K_\alpha = K'_\alpha \cap K \lhd F_4$. As F_4/K_α is finite index in F_4/K'_α, it inherits all the properties above. $\square$

Example 2.13. *Let M be a closed hyperbolic 3-manifold so that $G = \pi_1(M)$ is hyperbolic relative to a subgroup $P \leq G$ that is isomorphic to F_4. For example, let M' be a compact hyperbolic manifold whose boundary $\partial M'$ is a totally geodesic surface of genus 3, and let M be the double of M' along $\partial M'$. Observe that $\pi_1(M)$ is hyperbolic relative to $\pi_1(\partial M')$, and $\pi_1(\partial M')$ is hyperbolic relative to a copy of $\pi_1(S') = F_4$, where $S' \subset \partial M'$ is a punctured genus 2 subsurface. Thus $\pi_1(M)$ is hyperbolic relative to $\pi_1(S')$.*

Since G is hyperbolic with 2-sphere boundary, and P is quasi-convex in G with Cantor set boundary, the hypotheses of Theorem 1.3 are satisfied for $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2 = \{P\}$. Therefore, we find a transversal quasi-isometric embedding of $\mathbb{H}^2$ into G.

Let R_0 be chosen by Proposition 2.11. As P is quasi-convex in G, we choose R so that for $x \in P$, if $d_P(e, x) \geq R$ then $d_G(e, x) \geq R_0$. Now let $\{K_\alpha\}$ be the subgroups constructed in Lemma 2.12. By Proposition 2.11, for each $\alpha \in \mathbb{R}$ the peripheral filling $G_\alpha = G/\ll K_\alpha \gg$ is relatively hyperbolic and contains a quasi-isometrically embedded copy of $\mathbb{H}^2$.

As P/K_α is non-virtually cyclic and amenable, it does not have a non-trivial relatively hyperbolic structure. Therefore, G_α is not hyperbolic relative to virtually nilpotent subgroups, for in any peripheral structure $\mathcal{P}$, some peripheral group $H \in \mathcal{P}$ must be quasi-isometric to P/K_α by [BDM09, Theorem 4.8].

Finally, if $\alpha \neq \beta$ then G_α is not quasi-isometric to G_β by [BDM09, Theorem 4.8] *as P/K_α and P/K_β are not quasi-isometric.*

3. Separation of parabolic points and horoballs

In this section we study how the boundaries of peripheral subgroups are separated in $\partial_\infty X$. We also establish a preliminary result on quasi-isometrically embedding copies of $\mathbb{H}^2$.

3.1. Separation estimates.

We begin with the following lemma.

Lemma 3.1. *Let $(G, \mathcal{P}_1)$ and $(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ be relatively hyperbolic groups, where all peripheral subgroups in $\mathcal{P}_2$ are hyperbolic groups ($\mathcal{P}_2$ is allowed to be empty), and set $X = X(G, \mathcal{P}_1)$. Let $\mathcal{H}$ denote the collection of the horoballs of X and the left cosets of the subgroups in $\mathcal{P}_2$; more precisely, the images of those left cosets under the natural inclusion $G \hookrightarrow X$.*

Then the collection of subsets $\mathcal{H}$ has the following properties.

1. *For each $r \geq 0$ there is a uniform bound $b(r)$ on $\operatorname{diam}(N_r(H) \cap N_r(H'))$ for each $H, H' \in \mathcal{H}$ with $H \neq H'$.*
2. *Each $H \in \mathcal{H}$ is uniformly quasi-convex in X.*
3. *There exists R such that, given any $H \in \mathcal{H}$ and any geodesic ray γ connecting w to $a \in \partial_\infty H$, the subray of γ whose starting point has distance d_H from w is entirely contained in $N_R(H)$.*

Proof. Claim (1) follows from the corresponding fact in the Cayley graph [DS05, Theorem 4.1(α_1)].

Let us show (2). Uniform quasi-convexity of the horoballs is clear. If H is a left coset of a peripheral subgroup in $\mathcal{P}_2$, then it is quasi-convex in the Cayley graph Γ of G [DS05, Lemma 4.15]. What is more, by [DS05, Theorem 4.1(α_1)], geodesics in Γ connecting points of H are transversal with respect to $\mathcal{P}_1$. Therefore, by Lemma 2.8 they are quasi-geodesics in $X(G)$. We conclude that H is quasi-convex in X since pairs of points of H can be joined by quasi-geodesics (with uniformly bounded constants) which stay uniformly close to H.

We now show (3). As $a \in \partial_\infty H$ and H is quasi-convex, there exists $C = C(X(G, \mathcal{P}))$ so that $d(x, H) \leq C$ for each $x \in \gamma$ with $d(x, w)$ large enough. Fix such an x.

Let $y \in \gamma$ be such that $d(y,w) = d_H$. Let $p \in H$ be chosen so that $d(w,p) \leq d_H + 1$. By the defining property of y and the thinness of the geodesic triangle with vertices w, p, x, we have $d(y,H) \leq d(y,p) \leq C'(\delta_X)$. The desired property now follows from (2). □

From this lemma, we can deduce separation properties for the boundaries of sets in $\mathcal{H}$.

Lemma 3.2. *We make the assumptions of Lemma 3.1. Then there exists $C = C(X)$ so that for each $H, H' \in \mathcal{H}$ with $H \neq H'$ and $d_H \geq d_{H'}$ we have*

$$\rho(\partial_\infty H, \partial_\infty H') \geq e^{-\epsilon d_H}/C.$$

Proof. Let $a \in \partial_\infty H, a' \in \partial_\infty H'$. We have to show that $(a|a') \lesssim d_H$. Let γ, γ' be rays connecting w to a, a', respectively. For each $p \in \gamma$ such that $d(w,p) \leq (a|a')$ there exists $p' \in \gamma'$ such that $d(p',w) = d(p,w)$ and $d(p,p') \leq C'(\delta_X)$. With $b(r)$ and R as found by Lemma 3.1, set $B = b(R + C')$.

Suppose that $(a|a') \geq d_H + B + 1$. Consider $p \in \gamma$, $p' \in \gamma$ such that $d(p,w) = d(p',w) = d_H$. By Lemma 3.1(3), we have $p \in N_R(H)$ and $p' \in N_R(H')$, as $d_H \geq d_{H'}$. So, $p \in N_{R+C'}(H) \cap N_{R+C'}(H')$. The same holds also when $p \in \gamma$ is such that $d(p,w) = d_H + B + 1$. Therefore we have $\operatorname{diam}(N_{R+C'}(H) \cap N_{R+C'}(H')) > B$, contradicting Lemma 3.1(1). Hence $(a|a') < d_H + B + 1$, and we are done. □

Conversely, we show that separation properties of certain points in the boundary $\partial_\infty X$ have implications for the intersection of sets in X.

Lemma 3.3. *We make the assumptions of Lemma 3.1.*

Let γ be a geodesic line connecting w to $a \in \partial_\infty X$. Suppose that for some $H \in \mathcal{H}$ and $r \geq 1$ we have $d(a, \partial_\infty H) \geq e^{-\epsilon d_H}/r$. Then γ intersects $N_L(H)$ in a set of diameter bounded by $K + 2L$, for $K = K(r, X)$ and any $L \geq 0$.

In order to prove the lemma we will use [GdlH90, Lemma 2.12], which states that any finite configuration of points and geodesics in a δ_X-hyperbolic space X is approximated by a tree, up to an additive error that depends only on δ_X and the number of points and geodesics involved. So, we only need to show a result in the case of trees, and (up to controlled error) we will have obtained the same result for all hyperbolic spaces. In the notation of Figure 1, notice that $\beta_c(x,w) = d(x,p) - d(w,p)$ (the point p can be chosen to be the point on a geodesic from x to w such that $d(w,p) = (c|x)$; this makes sense in all hyperbolic spaces, not just trees.)

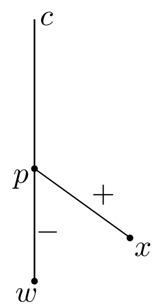

FIGURE 1. How to compute the Busemann function in a tree.

Proof of Lemma 3.3. We will treat the horoball case and the left coset case separately, beginning with the latter.

We can assume that H is not bounded, for otherwise the lemma is trivially true. Due to quasi-convexity, each point on H is at uniformly bounded distance from a geodesic line connecting points a_1, a_2 in $\partial_\infty H$, and thus also at uniformly bounded distance, say $C_1 = C_1(X)$, from either a ray connecting w to a_1 or a ray connecting w to a_2.

Let $c \in \partial_\infty H$ and let γ' be a ray connecting w to c. We have that $(a|c) \leq d_H + r' + C_2$, for $r' = \log(r)/\epsilon$ and $C_2 = C_2(\delta_X, \epsilon, C_0)$.

There exists $C_3 = C_3(C_1, X)$ so that any point x on γ such that $d(x, w) \geq (a|c) + L + C_3$ has the property that $d(x, \gamma') \geq L + C_1$. This applies to all rays connecting w to some $c \in \partial_\infty H$, and so $d(x, H) > L$.

Also, if $x \in \gamma$ and $d(x, w) < d_H - L$ then clearly $d(x, H) > L$. Thus if $\gamma \cap N_L(H) \neq \emptyset$ we have $(a|c) + L + C_3 \geq d_H - L$, and

$$\mathrm{diam}(\gamma \cap N_L(H)) \leq (a|c) + L + C_3 - (d_H - L) \lesssim_{C_2+C_3} r' + 2L.$$

We are left to deal with the horoball case. Let c and γ' be as above. Once again, $(a|c) \leq d_H + r' + C_2$, for $r' = \log(r)/\epsilon$. In an approximating tree for γ and γ' (see Figure 1), if x is any point on γ such that $d(x, w) > 2(a|c) - d_H + L$, one has $d(x, H) > L$ because

$$\beta_c(x, w) = d(x, p) - d(p, w) = d(x, w) - 2(a|c) > -d_H + L.$$

Therefore, using Lemma 2.2 and [GdlH90, Lemma 2.12], one finds $C_4 = C_4(X)$ so that if $d(x, w) > 2(a|c) - d_H + L + C_4$ then $d(x, H) > L$. Arguing as before, one sees that, for $C_5 = 2C_2 + C_4$,

$$\mathrm{diam}(\gamma \cap N_L(H)) \leq 2(a|c) - d_H + L + C_4 - (d_H - L) \lesssim_{C_5} 2r' + 2L. \qquad \square$$

3.2. **Embedded planes.** We finish this section by noting that in order to find a quasi-isometrically embedded copy of $\mathbb{H}^2$ in a relatively hyperbolic group, we need only to embed a quadrant of $\mathbb{H}^2$ into our model space X, provided that the embedding does not go too far into the horoballs. (Compare with [BK05].) As we see later, this means that we do not need to embed a quasi-circle into the boundary of our group, merely a quasi-arc.

Definition 3.4. *The* standard quadrant *in* $\mathbb{H}^2$ *is the set* $Q = \{(x, y) : 0 \leq x, y < 1\}$ *in the Poincaré disk model for* $\mathbb{H}^2$*.*

Let G, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{H}$ and $X = X(G, \mathcal{P}_1)$ be as in Lemma 3.1.

Proposition 3.5. *Let* $f : Q \to X$ *be a* (λ, λ)*-quasi-isometric embedding of the standard quadrant of* $\mathbb{H}^2$ *into* X*, with* $f((0, 0)) = w$*. Then there exists* $r = r(\lambda, X) > 0$ *with the following property. Suppose that for each* $a \in \partial_\infty Q$*,* $H \in \mathcal{H}$ *we have*

$$d(f(a), \partial_\infty H) \geq e^{-\epsilon d_H}/r.$$

Then,

(1) *there exists a quasi-isometric embedding* $g : \mathbb{H}^2 \to X$ *such that each point in* $g(\mathbb{H}^2)$ *is at uniformly bounded distance from* $\Gamma(G)$*.*
(2) *there exists a transversal (with respect to* $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$*) quasi-isometric embedding* $\hat{g} : \mathbb{H}^2 \to \Gamma(G)$ *at finite distance from* g*.*

Proof. (1) Each point $x \in Q \setminus \{(0, 0)\}$ lies on a unique geodesic γ_x connecting $(0, 0)$ to some point a_x in $\partial_\infty Q$. As $f(\gamma_x)$ is a λ-quasi-geodesic and X is hyperbolic, $f(\gamma_x)$ lies within distance $C_1 = C_1(\lambda, \delta_X)$ from a geodesic γ'_x from w to $f(a_x)$. By Lemma 3.3 all points on γ'_x are $C_2 = C_2(r, X)$ close to a point in $\Gamma(G)$, and therefore points in $f(Q)$ are $C_1 + C_2$ close to points in $\Gamma(G)$.

Notice that Q contains balls $\{B_n\}$ of $\mathbb{H}^2$ of arbitrarily large radius, each of which admits a (λ, λ)-quasi-isometric embedding $f_n : B_n \to X$ so that each point in $f_n(B_n)$ is a distance $C_1 + C_2$ close to a point in $\Gamma(G)$. In particular, translating those embeddings appropriately using the action of G on X we can and do assume that the center of B_n is mapped at uniformly bounded distance from w. As X is proper, we can use Arzelà-Ascoli to obtain the required (λ', λ')-quasi-isometric embedding g as the limit of a subsequence of $\{f_n\}$ (more precisely $\{f_n|_N\}$, where N is a maximal 1-separated net in $\mathbb{H}^2$), for $\lambda' = \lambda'(\lambda, C_1, C_2)$.

(2) By Lemma 3.3, f is transversal, so g and any map at finite distance from g will also be transversal. By (1) we can define $\hat{g} : \mathbb{H}^2 \to \Gamma(G)$ so that for each $x \in \mathbb{H}^2$ we have $d(\hat{g}(x), g(x)) \leq C_3$, for $C_3 = C_3(C_1, C_2)$.

Pick $x, y \in \mathbb{H}^2$. Notice that

$$d_G(\hat{g}(x), \hat{g}(y)) \geq d(\hat{g}(x), \hat{g}(y)) \gtrsim_{2C_3} d(g(x), g(y)) \gtrsim_{\lambda'} d(x, y)/\lambda',$$

so it is enough to show that for some λ'',

$$d_G(\hat{g}(x), \hat{g}(y)) \leq \lambda'' d(g(x), g(y)) + \lambda''.$$

Let γ be the geodesic in $\mathbb{H}^2$ connecting x to y. Let γ' be a geodesic in X from $\hat{g}(x)$ to $\hat{g}(y)$, which is at Hausdorff distance at most $C_4 = C_4(\lambda', C_3, \delta_X)$ from $\hat{g}(\gamma)$. In order to proceed we need the following.

Remark 3.6. *There exists some increasing function θ such that*

$$d_G(x_1, x_2) \leq \theta(d(x_1, x_2))$$

for each $x_1, x_2 \in \Gamma(G)$. Also, $\theta(x)/x \to 1$ as $x \to 0$.

Each maximal subpath $\beta \subseteq \gamma'$ contained in some $H \in \mathcal{H}$ has length at most $C_5 = C_5(C_4)$. Choose $M \geq \theta(C_5)$ such that $M \geq \sup\{\theta(x)/x : x \in (0, 1]\}$. We can substitute β by a subpath in $\Gamma(G)$ of length at most $Ml(\beta)$, where $l(\beta)$ is the length of β. Let α be the path in $\Gamma(G)$ obtained from γ' in this way.

Therefore $l(\alpha) \leq Ml(\gamma')$ and

$$d_G(\hat{g}(x), \hat{g}(y)) \leq l(\alpha) \leq Md(\hat{g}(x), \hat{g}(y)).$$

The desired estimate follows as $d(\hat{g}(x), g(x)), d(\hat{g}(y), g(y)) \leq C_3$. □

4. Boundaries of relatively hyperbolic groups

In this section we study the geometry of the boundary of a relatively hyperbolic group, endowed with a visual metric ρ as in Section 2.

Before we begin, we summarize some known results about the topology of such boundaries.

Theorem 4.1 (Bowditch)**.** *Suppose $(G, \mathcal{P})$ is a one-ended relatively hyperbolic group which does not split over a subgroup of a conjugate of some $P \in \mathcal{P}$, and every group in $\mathcal{P}$ is finitely presented, one or two ended, and contains no infinite torsion subgroup. Then $\partial_\infty(G, \mathcal{P})$ is connected, locally connected and has no global cut points.*

Proof. Connectedness and local connectedness follow from [Bow99c, Prop. 10.1], [Bow99a, Thm. 0.1].

Any global cut point must be a parabolic point [Bow99b, Thm. 0.2], but the splitting hypothesis ensures that these are not global cut points either [Bow99a, Prop. 5.1, Thm. 1.2]. □

Recall that a point p in a connected, locally connected topological space Z is *not a local cut point* if for every connected neighbourhood U of p, the set $U \setminus \{p\}$ is also connected. If, in addition, Z is compact, then Z is locally path connected, so p is not a local cut point if and only if every neighbourhood U of p contains an open V with $p \in V \subset U$ and $V \setminus \{p\}$ path connected.

More generally, we have the following definition, used in the statement of Theorem 1.3.

Definition 4.2. *A closed set V in a connected, locally connected topological space Z* does not locally disconnect *Z if for any $p \in V$ and connected neighbourhood U of p, the set $U \setminus V$ is also connected.*

For relatively hyperbolic groups, we note the following.

Proposition 4.3. *Suppose $(G, \mathcal{P})$ is relatively hyperbolic with connected and locally connected boundary. Let p be a parabolic point in $\partial_\infty(G, \mathcal{P})$ which is not a global cut-point. Then p is a local cut point if and only if the corresponding peripheral group has more than one end.*

Proof. The lemma follows, similarly to the proof of [Dah05, Proposition 3.3], from the fact that the parabolic subgroup P corresponding to p acts properly discontinuously and cocompactly on $\partial_\infty(G, \mathcal{P}) \setminus \{p\}$, which is connected and locally connected. Let us make this precise.

Choose an open set K_0 with compact closure in $\partial_\infty(G, \mathcal{P}) \setminus \{p\}$, so that $PK_0 = \partial_\infty(G, \mathcal{P}) \setminus \{p\}$. Then define $K_1 = \bigcup_{q \in P : d(q,e) \leq 1} qK_0$. As $\partial_\infty(G, \mathcal{P}) \setminus \{p\}$ is connected and locally path connected, and $\overline{K_1}$ is compact, one can easily find an open, path connected K so that $K_1 \subset K \subset \overline{K} \subset \partial_\infty(G, \mathcal{P}) \setminus \{p\}$.

Now suppose that P has one end. Let U be a neighbourhood of p. As P acts properly discontinuously on $\partial_\infty(G, \mathcal{P}) \setminus \{p\}$, there exists R so that if $d_P(e, g) > R$, then $gK \subset U$. Let Q be the unbounded connected component of $P \setminus B(e, R)$. Then QK is path connected as for $g, h \in P$, if $d_P(g, h) \leq 1$, $gK \cap hK \neq \emptyset$. Finally, observe that $V = QK \cup \{p\} \subset U$ is a neighbourhood of p so that $V \setminus \{p\} = QK$ is connected.

Conversely, suppose that p is not a local cut-point. Let D be so that if $qK \cap rK \neq \emptyset$ then $d_P(q, r) \leq D$. Suppose we are given $R > 0$. We can find a connected neighbourhood U of p in $\partial_\infty(G, \mathcal{P})$ so that $U \setminus \{p\}$ is path connected and $gK \cap U = \emptyset$ for all $g \in B(e, R + D) \subset P$. Let $R' \geq R + D$ be chosen so that $gK \cap U \neq \emptyset$ for all $g \in P \setminus B(e, R')$. Given $g, h \in P \setminus B(e, R')$ we can find a path in $U \setminus \{p\}$ connecting gK to hK. So, there exists a sequence $g = g_0, g_1, \ldots, g_n = h$ in $P \setminus B(e, R + D)$ so that $g_i K \cap g_{i+1} K \neq \emptyset$ for all $i = 0, \ldots, n-1$. Thus as $d_P(g_i, g_{i+1}) \leq D$, we have that g and h can be connected in P outside $B(e, R)$. As R was arbitrary, P is one ended. □

4.1. **Doubling.**

Definition 4.4. *A metric space* (X, d) *is* N-doubling *if every ball can be covered by at most* N *balls of half the radius.*

Every hyperbolic group has doubling boundary, but this is not the case for relatively hyperbolic groups.

Proposition 4.5. *The boundary of a relatively hyperbolic group* $(G, \mathcal{P})$ *is doubling if and only if every peripheral subgroup is virtually nilpotent.*

Recall that all relatively hyperbolic groups we consider are finitely generated, with $\mathcal{P}$ a finite collection of finitely generated peripheral groups.

Proof. By [DY05, Theorem 0.1], every peripheral subgroup is virtually nilpotent if and only if $X = X(G, \mathcal{P})$ has bounded growth at all scales: for every $0 < r < R$ there exists some N so that every radius R ball in X can be covered by N balls of radius r.

If X has bounded growth at some scale then $\partial_\infty X$ is doubling [BS00, Theorem 9.2].

On the other hand, if $\partial_\infty X$ is doubling, then $\partial_\infty X$ quasi-symmetrically embeds into some $\mathbb{S}^{n-1}$ (see [Ass83], or [Hei01, Theorem 12.1]). Therefore, X quasi-isometrically embeds into some $\mathbb{H}^n$ [BS00, Theorems 7.4, 8.2], which has bounded growth at all scales. We conclude that X has bounded growth at all scales (for small scales, the bounded growth of X follows from the finiteness of $\mathcal{P}$, and the finite generation of G and all peripheral groups). □

4.2. **Partial self-similarity.** The boundary of a hyperbolic group G with a visual metric ρ is self-similar: there exists a constant L so that for any ball $B(z, r) \subset \partial_\infty G$, with $r \leq \operatorname{diam}(\partial_\infty G)$, there is a L-bi-Lipschitz map f from the rescaled ball $(B(z, r), \frac{1}{r}\rho)$ to an open set $U \subset \partial_\infty G$, so that $B(f(z), \frac{1}{L}) \subset U$. (See [BK11, Proposition 3.3] or [BL07, Proposition 6.2] for proofs that omit the claim that $B(f(z), \frac{1}{L}) \subset U$. The full statement follows from the lemma below.)

Relatively hyperbolic groups do not have the same self-similarity. However, a kind of partial self-similarity holds that will be essential in this and the following section.

The following lemma follows [BK11, Proposition 3.3] closely, adapted to the situation where G does not act cocompactly on X.

Lemma 4.6. *Suppose* X *is a* δ_X*-hyperbolic, proper, geodesic metric space with base point* $w \in X$*. Let* ρ *be a visual metric on the boundary* $\partial_\infty X$ *with parameters* C_0, ϵ*. Suppose a group* G *acts isometrically on* X*, with* Gw *the orbit of* w*.*

For each $D > 0$ there exists $L_0 = L_0(\delta_X, \epsilon, C_0, D) < \infty$ with the following property.

For all $z \in \partial_\infty X$ and $r \leq \operatorname{diam}(\partial_\infty X)$, let $y \in [w, z) \subset X$ be the point, if it exists, with $d(w, y) = -\frac{1}{\epsilon}\log(2rC_0) - \delta_X - 1$. Suppose we take $y' \in [w, y]$ so that $d(Gw, y') \leq D$, and set $r' = re^{\epsilon d(y,y')}$. Then there exists a L_0-bi-Lipschitz map f from the rescaled ball $(B(z, r'), \frac{1}{r'}\rho)$ to an open set $U \subset \partial_\infty G$, so that $B(f(z), \frac{1}{L_0}) \subset U$.

(If such a y does not exist, the same conclusion holds upon setting $r' = r$.)

Proof. Up to changing the constant L_0, we may assume that the point y exists, for if it does not, then r is bounded away from zero, so we may set f to be the identity map.

We now assume that y exists and y' as in the statement is fixed. We will use the following equalities:

$$-\frac{1}{\epsilon}\log(2r'C_0) = -\frac{1}{\epsilon}\log(2rC_0) - d(y, y') = d(w, y') + \delta_X + 1. \tag{4.7}$$

For every $z_1, z_2 \in B(z, r')$, and every $p \in (z_1, z_2)$, one has

$$d(y', [w, p]) \leq 3\delta_X. \tag{4.8}$$

This is easy to see: $\rho(z_1, z_2) \leq 2r'$, so $d(w, (z_1, z_2)) \geq -\frac{1}{\epsilon}\log(2r'C_0)$. Let $y_1 \in [w, z_1)$ be so that $d(y_1, w) = d(y', w)$, and notice that $d(y_1, (z_1, z_2)) > \delta_X$ by (4.7). For any $p \in (z_1, z_2)$, the thinness of the geodesic triangle w, z_1, p implies that $d(y_1, [w, p]) \leq \delta_X$. In particular, for $z_2 = p = z$, we have $d(y_1, [w, z)) \leq \delta_X$, so $d(y_1, y') \leq 2\delta_X$, and the general case follows.

Now choose $g \in G$ so that $d(g^{-1}w, y') \leq D$. For any $z_1, z_2 \in B(z, r')$, by (4.7), (4.8) we have

$$|d(g^{-1}w, (z_1, z_2)) - d(w, (z_1, z_2)) - \frac{1}{\epsilon}\log(2r'C_0)| \leq D + 7\delta_X + 1.$$

As $d(g^{-1}w, (z_1, z_2)) = d(w, (gz_1, gz_2))$, this gives that

$$L_0^{-1}\frac{\rho(z_1, z_2)}{r'} \leq \rho(gz_1, gz_2) \leq L_0\frac{\rho(z_1, z_2)}{r'},$$

for a choice of $L_0 \geq 2C_0^3 e^{\epsilon(D+7\delta_X+1)}$.

Thus the action of g on $B(z, r')$ defines a L_0-bi-Lipschitz map f with image U, which is open because g is acting by a homeomorphism. It remains to check that $B(f(z), \frac{1}{L_0}) \subset U$.

Suppose that $z_3 \in B(f(z), \frac{1}{L_0})$. Then $d(w, (f(z), z_3)) > \frac{-1}{\epsilon}\log(C_0/L_0)$, but $d(w, (f(z), z_3)) = d(g^{-1}w, (z, g^{-1}z_3))$. So, for large enough L_0, we have

$$\begin{aligned} d(w, (z, g^{-1}z_3)) &\geq \frac{-1}{\epsilon} \log(C_0/L_0) + d(w, y') - C_1(\delta_X, D) \\ &> \frac{-1}{\epsilon} \log(C_0/L_0) - \frac{1}{\epsilon} \log(2r'C_0) - C_2(C_1, \delta_X) \\ &> \frac{-1}{\epsilon} \log(r'/C_0), \end{aligned}$$

where the last equality follows from increasing L_0 by an amount depending only on ϵ, C_0, C_2. We conclude that $\rho(z, g^{-1}z_3) < r'$. $\square$

4.3. **Linear Connectedness.** Under the hypotheses of Theorem 4.1, we saw that $\partial_\infty(G, \mathcal{P})$ was connected and locally connected. In this section we show that $\partial_\infty(G, \mathcal{P})$ will satisfy a quantitatively controlled version of this property.

Definition 4.9. *We say a complete metric space* (X, d) *is* L-linearly connected *for some* $L \geq 1$ *if for all* x, $y \in X$ *there exists a compact, connected set* $J \ni x$, y *of diameter less than or equal to* $Ld(x, y)$. *We can assume that* J *is an arc (up to slightly increasing* L*).*

For more discussion of this property, including the proof that we can assume J is an arc, see [Mac08, Page 3975].

Proposition 4.10. *If* $(G, \mathcal{P})$ *is relatively hyperbolic,* $\partial_\infty(G, \mathcal{P})$ *is connected and locally connected with no global cut points, then* $\partial_\infty(G, \mathcal{P})$ *is linearly connected.*

If $\mathcal{P}$ is empty then G is hyperbolic, and this case is already known by work of Bonk and Kleiner [BK05, Proposition 4]. Lemma 4.6 gives an alternate proof of this result, which we include to warm up for the proof of Proposition 4.10. Both proofs rely on the work of Bestvina and Mess, and Bowditch and Swarup cited in the introduction.

Corollary 4.11 (Bonk-Kleiner)**.** *If the boundary of a hyperbolic group* G *is connected, then it is linearly connected.*

Proof. Let $X = \Gamma(G)$ by a Cayley graph of G with visual metric ρ, and let L_0 be chosen by Lemma 4.6 for $D = 1$. The boundary of G is locally connected [BM91, Bow98, Bow99b, Swa96], so for every $z \in \partial_\infty G$, we can find an open connected set V_z satisfying $z \in V_z \subset B(z, \frac{1}{2L_0})$. The collection of all V_z forms an open cover for the compact space $\partial_\infty G$, and so this cover has a Lebesgue number $\alpha > 0$.

Suppose we have $z, z' \in \partial_\infty G$. Let $r = \frac{2L_0}{\alpha}\rho(z, z')$. If $r > \mathrm{diam}(\partial_\infty X)$, we can join z and z' by a set of diameter $\mathrm{diam}(\partial_\infty X) < \frac{2L_0}{\alpha}\rho(z, z')$.

Otherwise, by Lemma 4.6 (with $y = y'$), there exists an L_0-bi-Lipschitz map $f : (B(z,r), \frac{1}{r}\rho) \to U$. Since $\rho(f(z), f(z')) \leq L_0 \frac{\rho(z,z')}{r} = \frac{\alpha}{2}$, we can find a connected set $J \subset B(f(z), \frac{1}{L_0}) \subset U$ that joins $f(z)$ to $f(z')$. Therefore $f^{-1}(J) \subset B(z,r)$ joins z to z', and has diameter at most $2r = \frac{4L_0}{\alpha}\rho(z,z')$. So $\partial_\infty G$ is $4L_0/\alpha$-linearly connected. □

The key step in the proof of Proposition 4.10 is the construction of chains of points in the boundary.

Lemma 4.12. *Suppose $(G, \mathcal{P})$ is as in Proposition 4.10. Then there exist K_1 so that for each pair of points $a, b \in \partial_\infty(G, \mathcal{P})$ there exists a chain of points $a = c_0, \dots, c_n = b$ such that*

(1) *for each $i = 0, \dots, n$ we have $\rho(c_i, c_{i+1}) \leq \rho(a,b)/2$, and*
(2) *$\operatorname{diam}(\{c_0, \dots, c_n\}) \leq K_1 \rho(a,b)$.*

We defer the proof of this lemma.

Proof of Proposition 4.10. Given $a, b \in \partial_\infty(G, \mathcal{P})$, apply Lemma 4.12 to get a chain of points $J_1 = \{c_0, \dots, c_n\}$. For $j \geq 1$, we define J_{j+1} iteratively by applying Lemma 4.12 to each pair of consecutive points in J_j, and concatenating these chains of points together. Notice that

$$\operatorname{diam}(J_{j+1}) \leq \operatorname{diam}(J_j) + \frac{2K_1}{2^j}\rho(a,b).$$

This implies that the diameter of $J = \overline{\bigcup J_j}$ is linearly bounded in $\rho(a,b)$, and J is clearly compact and connected as desired. □

We require two further lemmas before commencing the proof of Lemma 4.12. The first is an elementary lemma on the geometry of infinite groups.

Lemma 4.13. *Let P be an infinite group (with a fixed, finite system of generators). Then for each $p, q \in P$ there exists a geodesic ray α starting from p and such that $d(q, \alpha) \geq d(p,q)/3$.*

Proof. As P is infinite, there exists a geodesic line γ through p, which can be subdivided into geodesic rays α_1, α_2 starting from p. We claim that either α_1 or α_2 satisfies the requirement. In fact, if that was not the case we would have points $p_i \in \alpha_i \cap B_{d(p,q)/3}(q)$. Notice that $d(p_i, p) \geq 2d(p,q)/3$. Now, on one hand

$$d(p_1, p_2) \leq d(p_1, q) + d(q, p_2) \leq 2d(p,q)/3,$$

but on the other

$$d(p_1, p_2) = d(p_1, p) + d(p, p_2) \geq 4d(p,q)/3,$$

a contradiction. □

The next lemma describes the geometry of geodesic rays passing through a horoball.

Lemma 4.14. *Fix $O \in \mathcal{O}$ and $a, b \in \partial_\infty(G, \mathcal{P}) \backslash \{a_O\}$. Let γ_a, γ_b be geodesics from a_O to a, b and let q_a, q_b be the last points in $\gamma_a \cap \overline{O}, \gamma_b \cap \overline{O}$, which we assume to be both non-empty. Also, let γ be a geodesic from w to a_O and let q be the first point in $\gamma \cap \overline{O}$ (so that $d_O \approx d(w, q)$). Then there exists $E = E(X) < \infty$ so that the following holds.*

(1) *If $(a|a_O) \geq d_O$ then*
$$(a|a_O) \approx_E d(q_a, q)/2 + d_O.$$
(2) *If $(a|a_O), (b|a_O) \in [d_O, (a|b)]$ then*
$$(a|b) \gtrsim_E 2(a|a_O) - d_O - d(q_a, q_b)/2 \approx_E d(q_a, q) + d_O - d(q_a, q_b)/2.$$
Moreover, if $d(q_a, q_b) \geq E$ then $\approx_E$ holds.

(3) *If $(a|a_O) < d_O$ then $d(q_a, q) \approx_E 0$.*

(4) *Let z be a point on O with $d(z, q) \geq E$, let γ_0 be a geodesic ray starting at z and such that $\gamma_0 \cap O = \{z\}$ and let γ_1 be a geodesic ray from z to a_O. Then the concatenation of γ_1 and γ_0 lies within distance E from a bi-infinite geodesic.*

Proof. As in Lemma 3.3 we only need consider the case of trees, illustrated by Figure 2.

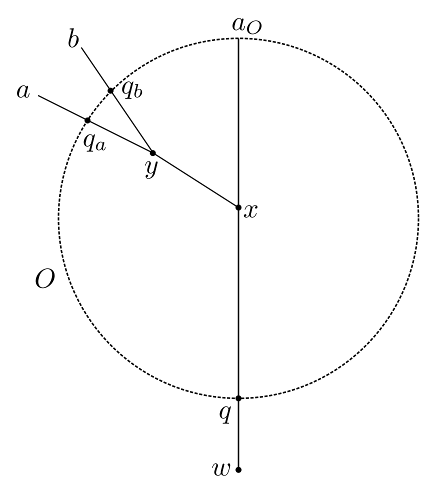

FIGURE 2. Geodesics passing through a horoball

(1) $(a|a_O) = d(w, q) + d(q, x) = d_O + d(q_a, q)/2$, as $d(x, q) = d(x, q_a)$.

(2) The figure illustrates one of the cases to consider, the other being when q_a, q_b are between x and y.

$(a|b) = d(w, q) + 2d(q, x) - d(q_a, q_b)/2 = d_O + 2((a|a_O) - d_O) - d(q_a, q_b)/2.$

$$(a|b) = d(w,q) + d(q,q_a) - d(q_a,q_b)/2 = d_O + d(q,q_a) - d(q_a,q_b)/2.$$

In the other case we obtain a strict inequality, but also $q_a = q_b$.

(3) In this case γ_a diverges from γ before q.

(4) In the corresponding situation in a tree, the said concatenation actually is a bi-infinite geodesic. □

For each peripheral subgroup $P \in \mathcal{P}$ we denote by d_P the path metric on any left coset of P.

Proof of Lemma 4.12. We need to find chains of points joining distinct points $a, b \in \partial_\infty(G,\mathcal{P})$, as described in the statement of the lemma.

Let $p_{a,b} \in [w,a)$ denote the centre of the quasi-tripod w, a, b, i.e. the point on $[w,a) \subset X = X(G,\mathcal{P})$ such that $d(w,p_{a,b}) = (a|b)$.

Let $R = R(X)$ be a large constant to be determined by Case 1 below. All constants may depend tacitly on C_0, ϵ, δ_X.

Case 1: We first assume that there exists $O \in \mathcal{O}$ such that $p_{a,b} \in O$ and $d(p_{a,b}, gP) > R$, where gP is the left coset of the peripheral subgroup P corresponding to O.

Suppose that $b = a_O$. Let γ_a be a geodesic from w to a and let q_a be the last point in $\gamma_a \cap \overline{O}$. Let γ be a geodesic from w to a_O and let q be the first point in $\gamma \cap \overline{O}$. Notice that q, q_a lie on gP, and by Lemma 4.14(1)

$$(a|a_O) \approx_E d(q_a,q)/2 + d_O \geq R + d_O. \tag{4.15}$$

By Lemma 4.13, there exists a geodesic ray α in gP starting at q_a such that $d_P(q,\alpha) \geq d_P(q_a,q)/3$. Therefore, by Lemma 2.6 and (4.15),

$$d(q,\alpha) \approx_A 2\log(d_P(\alpha,q)) \geq 2\log(d_P(q_a,q)/3) \approx_{A+3} d(q_a,q) \geq 2R. \tag{4.16}$$

Let $q_a = q_0, \ldots, q_n, \ldots$ be the points of $\alpha \cap gP$. For each i consider the point $c_i \in \partial_\infty(G,\mathcal{P})$ which is the endpoint of $q_i q_0^{-1}\gamma_a$.

By Lemma 4.14(4) and (4.16), for $R \geq R_0(E,A)$, a geodesic ray γ_{c_i} from w to c_i will pass a point within E from q_i, and hence by Lemma 4.14(3) we can assume that $(c_i|a_O) \geq d_O$ for each i. Note that $d(q_{c_i}, q_i) \leq C_1$, where q_{c_i} is the last point in $\gamma_{c_i} \cap \overline{O}$, and $C_1 = C_1(E)$.

Using Lemma 4.14(1) and (4.16), there exists $C_2 = C_2(C_1, A)$ so that

$$(c_i|a_O) \gtrsim_E d(q_{c_i},q)/2 + d_O \gtrsim_{C_2} d(q_a,q)/2 + d_O \approx_E (a|a_O).$$

And consequently there exists $C_3 = C_3(C_2,E)$ so that for each i

$$\rho(c_i, a_O) \leq C_3 \rho(a, a_O). \tag{4.17}$$

By Lemma 4.14(2), (4.15) and (4.16), we have for $C_4 = C_4(E, C_1, A)$

$$\begin{aligned}(c_i|c_{i+1}) &\gtrsim_{C_4} d(q_i, q) + d_O - d(q_i, q_{i+1})/2 \\ &\gtrsim_{C_4} d(q_a, q) + d_O \gtrsim_E (a|a_O) + R.\end{aligned}$$

So, taking $R \geq R_1(C_4, E, R_0)$, we have

$$\rho(c_i, c_{i+1}) \leq \rho(a, a_O)/2. \tag{4.18}$$

For each i, by Lemmas 2.6 and 4.14(1), $(c_i|a_O) \approx_{E+A} \log(d_P(q_i, q)) + d_O$, so for N large enough we have $\rho(c_N, a) \leq \rho(a, a_O)/2$.

Therefore the chain of points $a = c_0, \dots, c_N, a_O = b$ satisfies our requirements by (4.17), (4.18).

Now suppose that $b \neq a_O$, but that $\rho(a, b) \leq \rho(a, a_O)/S, \rho(b, a_O)/S$, for some large enough S to be determined. Let $S' = \log(S/C_0^2)/\epsilon$, and note that $(a|b) - (a|a_O) \geq S'$ and $(a|b) - (b|a_O) \geq S'$.

Let γ_a, q_a be as above and let γ_b, q_b be defined analogously. As $R \geq 2E$, by Lemma 4.14(3) we have $(a|a_O), (b|a_O) \geq d_O$. Using Lemma 4.14(1) and the approximate equality case of Lemma 4.14(2), we have

$$\begin{aligned}d(q_a, q) &\approx_E 2((a|a_O) - d_O) \\ &\geq 2((a|a_O) - d_O)) + 2(S' - ((a|b) - (a|a_O))) \\ &= 2(2(a|a_O) - (a|b) - d_O + S') \approx_{2E} d(q_a, q_b) + 2S'.\end{aligned} \tag{4.19}$$

Let α be a geodesic in gP connecting q_a to q_b. Similarly to the previous case, let $q_a = q_0, \dots, q_n = q_b$ be the points of $\alpha \cap gP$ and consider the points $c_i \in \partial_\infty(G, \mathcal{P})$ which are the endpoints of $q_i q_0^{-1} \gamma_a$. Notice that

$$2\log(d_P(q, q_a)/d_P(q_a, q_b)) \approx_{2A} d(q, q_a) - d(q_a, q_b) \gtrsim_{3E} 2S'$$

by Lemma 2.2 and (4.19), so for $S = S(E, A)$ large enough,

$$2\log(d_P(q, q_a)/d_P(q_a, q_b) - 1) \geq S',$$

thus

$$\begin{aligned}d(q, q_i) &\approx_A 2\log(d_P(q, q_i)) \geq 2\log(d_P(q, q_a) - d_P(q_a, q_i)) \\ &\geq 2\log(d_P(q, q_a) - d_P(q_a, q_b)) \gtrsim_A S' + d(q_a, q_b).\end{aligned}$$

In particular, if $S = S(E, A)$ is large enough we have $(c_i|a_O) \geq d_O$ for each i, by Lemma 4.14(3). From Lemma 4.14(2) we have, for $C_5 = C_5(E, C_1)$

$$\begin{aligned}(a|c_i) &\gtrsim_{C_5} 2(a|a_O) - d_O - d(q_a, q_i)/2 \\ &\gtrsim_{2A} 2(a|a_O) - d_O - d(q_a, q_b)/2 \quad \approx_E (a|b),\end{aligned}$$

giving the diameter bound $\rho(a, c_i) \leq C_6 \rho(a, b)$, for $C_6 = C_6(C_5, E, A)$.

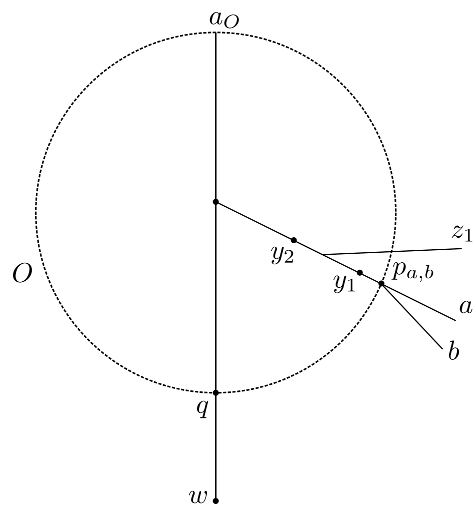

FIGURE 3. Lemma 4.12, Case 2b

Since $(c_i|a_O) \gtrsim \min\{(c_i|a),(a|a_O)\} \gtrsim (a|a_O)$, with error $C_7 = C_7(C_6)$, the bound $\rho(c_i, c_{i+1}) \leq \rho(a,b)/2$ follows from

$$\begin{aligned}(c_i|c_{i+1}) &\gtrsim_{C_4} 2(c_i|a_O) - d_O - d(q_i,q_{i+1})/2 \gtrsim_{2C_7+1} 2(a|a_O) - d_O \\ &\approx_E (a|b) + d(q_a,q_b)/2 \geq (a|b) + R,\end{aligned}$$

for R large enough, depending on C_7 and E.

Finally, if $\rho(a,b) \geq \rho(a,a_O)/S$ or $\rho(a,b) \geq \rho(b,a_O)/S$, we can find chains connecting a to a_O and a_O to b and concatenate them.

Case 2: We assume that $d(p_{a,b}, \Gamma(G)) < R$.

Let $L_0 > 1$ and $\alpha > 0$ be chosen as in the proof of Corollary 4.11 on setting $D = R$.

Let $r_1 = 2\frac{L_0}{\alpha}\rho(a,b)$ and let $y_1 \in [w,a)$ be chosen so that $d(w,y_1) = -\frac{1}{\epsilon}\log(2r_1C_0) - \delta_X - 1$. (We can assume, as in the proof of Corollary 4.11, that such a y_1 exists.) Let $t \gg 0$ be a large constant to be determined later.

Case 2a: If there exists $y' \in [w,y_1]$ so that $d(y_1,y') \leq 3t$ and $d(y',\Gamma(G)) \leq D$, then we can apply Lemma 4.6 as in the proof of Corollary 4.11 to show that a and b can be joined by a set of diameter at most $\frac{4L_0}{\alpha}e^{\epsilon 3t}\rho(a,b)$.

Case 2b: If no such y' exists, we are in the situation of Figure 3. Let $y_2 \in [w,y_1)$ be chosen so that $d(y_1,y_2) = t$ and let $O \in \mathcal{O}$ be the horoball containing $[y_2,y_1]$, which corresponds to the coset gP. Let $d_O = d(w,O)$, and let $p_1 \in gP$ be chosen so that $d(p_1,p_{a,b}) < R$. (In the figure, $p_1 = p_{a,b}$.)

Let ρ_1 be a visual metric on $\partial_\infty(G, \mathcal{P})$ based at p_1. We can assume that $(\partial_\infty(G, \mathcal{P}), \rho)$ and $(\partial_\infty(G, \mathcal{P}), \rho_1)$ are isometric, with the isometry induced by the action of p_1. In the metric ρ_1, we have that a, b and a_O are points separated by at least $\delta_0 = \delta_0(R)$.

The boundary $(\partial_\infty(G, \mathcal{P}), \rho) = (\partial_\infty(G, \mathcal{P}), \rho_1)$ is compact, locally connected and connected. Consequently, given a point $c \in \partial_\infty(G, \mathcal{P})$ that is not a global cut point, and $\delta_0 > 0$, there exists $\delta_1 = \delta_1(\delta_0, c, \partial_\infty(G, \mathcal{P})) > 0$ so that any two points in $\partial_\infty(G, \mathcal{P}) \setminus B(c, \delta_0)$ can be joined by an arc in $\partial_\infty(G, \mathcal{P}) \setminus B(c, \delta_1)$.

In our situation, δ_1 may be chosen to be independent of the choice of (finitely many) $c = a_O$ satisfying $d(O, p_1) = 0$, so $\delta_1 = \delta_1(\delta_0, \partial_\infty(G, \mathcal{P}))$. Therefore, a and b can be joined by a compact arc J in $(\partial_\infty(G, \mathcal{P}), \rho_1)$ that does not enter $B_{\rho_1}(a_O, \delta_1)$. So geodesic rays from p_1 to points in J are at least $2\delta_X$ from the geodesic ray $[p_{a,b}, a_O)$ outside the ball $B(p_1, t)$, for $t = -\frac{1}{\epsilon}\log(\delta_1) + C_8$, for $C_8 = C_8(C_0, \epsilon, \delta_X)$.

Translating this back into a statement about $(\partial_\infty(G, \mathcal{P}), \rho)$, we see that geodesics from w to points in J must branch from $[w, p_{a,b}]$ after y_2, that is, the set J lies in the ball $B(a, C_9 e^{\epsilon t} d(a, b))$, for $C_9 = C_9(C_0)$.

From these connected sets of controlled diameter, it is easy to extract chains of points satisfying the conditions of the lemma. □

5. Avoidable sets in the boundary

In order to build a hyperbolic plane that avoids horoballs, we need to build an arc in the boundary that avoids parabolic points. In Theorem 1.3, we also wish to avoid the specified hyperbolic subgroups. We have topological conditions such as the no local cut points condition which help, but in this section we find more quantitative control.

Given $p \in X$, and $0 < r < R$, the *annulus* $A(p, r, R)$ is defined to be $\overline{B}(p, R) \setminus B(p, r)$. More generally, we have the following.

Definition 5.1. *Given a set V in a metric space Z, and constants $0 < r < R < \infty$, we define the* annular neighbourhood

$$A(V, r, R) = \{z \in Z : r \leq d(z, V) \leq R\}.$$

If an arc passes through (or close to) a parabolic point in the boundary, we want to reroute it around that point. The following definition will be used frequently in the following two sections.

Definition 5.2 ([Mac08])**.** *For any x and y in an embedded arc A, let $A[x, y]$ be the closed, possibly trivial, subarc of A that lies between them.*

*An arc B ι-*follows *an arc A if there exists a (not necessarily continuous) map $p : B \to A$, sending endpoints to endpoints, such that* *for all $x, y \in B$, $B[x,y]$ is in the ι-neighbourhood of $A[p(x), p(y)]$; in particular, p displaces points at most ι.*

We now define our notion of avoidable set, which is a quantitatively controlled version of the no local cut point and not locally disconnecting conditions.

Definition 5.3. *Suppose (X, d) is a complete, connected metric space. A set $V \subset X$ is L-*avoidable on scales below δ *for $L \geq 1$, $\delta \in (0, \infty]$ if for any $r \in (0, \delta/2L)$, whenever there is an arc $I \subset X$ and points $x, y \in I \cap A(V, r, 2r)$ so that $I[x, y] \subset N(V, 2r)$, there exists an arc $J \subset A(V, r/L, 2rL)$ with endpoints x, y so that J $(4rL)$-follows $I[x, y]$.*

The goal of this section is the following proposition.

Proposition 5.4. *Let $(G, \mathcal{P}_1)$ and $(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ be relatively hyperbolic groups, where all groups in $\mathcal{P}_2$ are proper infinite hyperbolic subgroups of G ($\mathcal{P}_2$ may be empty), and all groups in $\mathcal{P}_1$ are finitely presented and one ended. Let $X = X(G, \mathcal{P}_1)$, and let $\mathcal{H}$ be the collection of all horoballs of X and left cosets of the subgroups of $\mathcal{P}_2$. (As usual we regard G as a subspace of X.)*

Suppose that $\partial_\infty X$ is connected and locally connected, with no global cut points. Suppose that $\partial_\infty P$ does not locally disconnect $\partial_\infty X$ for each $P \in \mathcal{P}_2$. Then there exists $L \geq 1$ so that for every $H \in \mathcal{H}$, $\partial_\infty H \subset \partial_\infty X$ is L-avoidable on scales below $e^{-\epsilon d(w,H)}$.

This proposition is proved in the following two subsections.

5.1. **Avoiding parabolic points.** We prove Proposition 5.4 in the case H is a horoball. This is the content of the following proposition.

Proposition 5.5. *Suppose $(G, \mathcal{P})$ is relatively hyperbolic, $\partial_\infty(G, \mathcal{P})$ is connected and locally connected with no global cut points, and all peripheral subgroups are one ended and finitely presented. Then there exists $L \geq 1$ so that for any horoball $O \in \mathcal{O}$, $a_O = \partial_\infty O \in \partial_\infty(G, \mathcal{P})$ is L-avoidable on scales below $e^{-\epsilon d(w,O)}$.*

The reason for restricting to this scale is that this is where the geometry of the boundary is determined by the geometry of the peripheral subgroup. Recall from Proposition 4.3 that such parabolic points are not local cut points.

The first step is the following simple lemma about finitely presented, one ended groups.

Lemma 5.6. *Suppose P is a finitely generated, one ended group, given by a (finite) presentation where all relators have length at most M, and* *let* $\Gamma(P)$ *be its Cayley graph. Then any two points* $x, y \in \Gamma(P)$ *such that* $2M \le r_x \le r_y$*, where* $r_x = d(e,x)$ *and* $r_y = d(e,y)$*, can be connected by an arc in* $A(e, r_x/3, 3r_y) \subset \Gamma(P)$*.*

Proof. By Lemma 4.13, we can find an infinite geodesic ray in $\Gamma(P)$ from x which does not pass through $B(e, r_x/3)$. Let x' be the last point on this ray satisfying $d(e, x') = 2r_y$. Do the same for y, and let y' denote the corresponding point. Note that x' and y' lie on the boundary of the unique unbounded component of $\{z \in \Gamma(P) : d(e,z) \ge 2r_y\}$, which we denote by Z. We prove the lemma by finding a path from x' to y' contained in $A(e, 2r_y - M, 2r_y + M)$.

Let β_1 be a path joining x' and y' in $\Gamma(P) \cap Z$. We can assume that β_1 is an arc, and $\beta_1 \cap \overline{B}(e, 2r_y) = \{x', y'\}$. Let p be the first point of $[y', e]$ that meets $[e, x']$ in $\Gamma(P)$. Then the concatenation of $\beta_1, [y', p]$ and $[p, x']$ forms a simple, closed loop β_2 in $\Gamma(P)$.

As β_2 represents the identity in P, there exists a diagram $\mathcal{D}$ for β_2: a connected, simply connected, planar 2-complex $\mathcal{D}$ together with a map of $\mathcal{D}$ into the Cayley complex $\Gamma^2(P)$ sending cells to cells and $\partial\mathcal{D}$ to β_2.

Let $\mathcal{D}' \subset \mathcal{D}$ be the union of closed faces $B \subset \mathcal{D}$ which have a point $u \in \partial B$ with $d(u, e) = 2r_y$. Let $\mathcal{D}''$ be the connected component of x' in $\mathcal{D}'$. Let $\gamma \subset \partial\mathcal{D}''$ be the simple closed curve in $\mathbb{R}^2$ bounding $\mathcal{D}''$.

If either β_1 or $[y', p] \cup [p, x']$ live in $A(p, 2r_y - M, 2r_y + M)$, we are done. Otherwise, as we travel around γ from x', in one direction we must take a value $> 2r_y$, and in the other a value $< 2r_y$, thus there is a point $v \in \gamma \setminus \{x'\}$ with $d(e, v) = 2r_y$. If v is in the interior of $\mathcal{D}$, the adjacent faces are in $\mathcal{D}'$, giving a contradiction. So $v \in \beta_2$, and v must be y'. Thus there is a path from x' to y' in $\mathcal{D}' \subset A(e, 2r_y - M, 2r_y + M)$. □

We can now prove the proposition.

Proof of Proposition 5.5. By Proposition 4.3, parabolic points in the boundary are not local cut points.

We claim that there exists an $L \ge 1$ so that for any parabolic point a_O, any $r \le e^{-\epsilon d(w,O)}/2L$, and any $x, y \in A(a_O, r, 2r)$, there exists an arc $J \subset A(a_O, r/L, 2rL)$ joining x to y.

This claim suffices to prove the proposition, because the $4rL$-following property is automatic since $\operatorname{diam}(B(a_O, 2rL)) \le 4rL$.

Suppose the claim is false. Then there exist annuli $A(a_{O_n}, r_n, 2r_n) \subset \partial_\infty G$ centered on parabolic points associated to horoballs $O_n \subset X$, which contain points a_n, b_n that cannot be joined in $A(a_{O_n}, r_n/n, 2r_n n)$.

Using Lemma 4.6, we may assume that $d(w, O_n) \leq C_1(X)$, and further that every $O_n = O$, for some horoball O. Moreover, we may assume that $r_n \to 0$.

Let gPg^{-1} be the corresponding parabolic subgroup, with $p_O \in gP$ a closest point to w. Fix some $n \geq N$, where N is to be determined below, and let $a = a_n$ and $b = b_n$, $r = r_n$. Let q_a, q_b be the last points of $(a_O, a), (a_O, b)$ contained in O. We will write $x \asymp_C y$ if the quantities x, y satisfy $x/C \leq y \leq Cx$. Notice that

$$d(q_a, p_O) \approx 2((a|a_O) - d_O) \approx 2(-\frac{1}{\epsilon}\log(r_n) - d_O) \approx 2\log(r_n^{-1/\epsilon}),$$

with error $C_2 = C_2(C_1, E)$, so by Lemma 2.6 we have, for $C_3 = C_3(C_2, A)$,

$$d_{gP}(q_a, p_O) \asymp_{C_3} r_n^{-1/\epsilon},$$

and the same for q_b. Let r_P be the smaller of $d_{gP}(q_a, p_O), d_{gP}(q_b, p_O)$, and notice that their ratio is controlled by C_3^2 independent of n.

By Lemma 5.6 there is a chain of points $q_a = q_0, q_1, \dots, q_n = q_b$ in gP joining q_a to q_b in gP so that, in the path metric on gP, $q_i \in A(p_O, r_P/3, 3r_PC_3^2)$.

We now follow part of the proof of Lemma 4.12, Case 1. Notice that by Lemma 4.14(4) each geodesic (a_O, c_i) passes close to q_i, where $c_i = q_iq_0^{-1}a$. Note that $(q_i|q_{i+1}) \gtrsim_{C_4} (a|a_O) + ((a|a_O) - d_O) \approx_{C_4} 2(a|a_O)$, with $C_4 = C_4(E, C_1)$. If we choose N so that $r_n \leq r_*$, for $r_* = r_*(C_3)$, we have $(q_i|a_O) \approx_{C_5} (a|a_O)$ for $C_5 = C_5(\delta_X)$ and all i, so $\rho(c_i, a_O) \asymp_{C_6} r_n$, for $C_6 = C_6(C_5)$. Moreover,

$$\frac{\rho(c_i, c_{i+1})}{\rho(c_i, a_O)} \leq C_7 \frac{e^{-\epsilon 2(a|a_O)}}{e^{-\epsilon(a|a_O)}} = e^{-\epsilon(a|a_O)} \asymp_{C_7} r_n,$$

for some $C_7 = C_7(C_4, C_5)$. By Proposition 4.10, $\partial_\infty G$ is L-linearly connected. Since r_n goes to zero as $n \to \infty$, for $n \geq N = N(C_3, C_6)$, we have $\rho(c_i, c_{i+1}) \leq \rho(c_i, a_O)/(2LC_6)$. Therefore we can join together the chain of points $c_0 = a, c_1, \dots, c_n = b$ inside $A(a_O, r_n/2C_6, (C_6+1)r_n)$. This gives us a contradiction. □

5.2. **Avoiding hyperbolic subgroups.** In this section we complete the proof of Proposition 5.4 for $H = gP$, where $P \in \mathcal{P}_2$. By assumption, $\partial_\infty H \subset \partial_\infty X$ does not locally disconnect $\partial_\infty X$.

First, a preliminary lemma.

Lemma 5.7. *There exists L_1 independent of H so that $\partial_\infty H \subset \partial_\infty X$ is L_1-porous on scales below $e^{-\epsilon d(w,H)}$, i.e. for any $a \in \partial_\infty H$ and $r \leq e^{-\epsilon d(w,H)}$, there exists $b \in B(a, r) \subset \partial_\infty X$ so that $\rho(b, \partial_\infty H) \geq r/L_1$.*

Proof. If not, we can find a sequence of cosets $H_n = g_n P_n$, points $a_n \in \partial_\infty H_n$ and scales r_n so that $N(\partial_\infty H_n, r_n/n) \supset B(a_n, r_n)$.

As in the proof of Proposition 5.5, we may assume that $d(w, H_n)$ is bounded, using the action of G and Lemma 4.6, and therefore, after taking subsequences, that $H_n = H$ is constant. Likewise, using the action of $g_n P_n g_n^{-1}$ and Lemma 4.6, we can assume that r_n is bounded away from zero.

Using compactness, we may also assume that a_n converges to some $a \in \partial_\infty H$. So, we have that a is in the interior of $\partial_\infty H \subset \partial_\infty X$, since $\partial_\infty H$ is closed in $\partial_\infty X$. This is a contradiction because $\partial_\infty H$ is not all of $\partial_\infty G$ (proper peripheral subgroups of a relatively hyperbolic group are of infinite index), so if a is a point of $\partial_\infty H$, one can use the action of H to find points in $\partial_\infty G \setminus \partial_\infty H$ that are arbitrarily close to a. □

We continue with the proof of Proposition 5.4.

Suppose that the conclusion to the proposition is false. That is, there exists a sequence of configurations

$$\{(H_n = g_n P_n, x_n, y_n, I_n, r_n)\}_{n \in \mathbb{N}}$$

where $I_n \subset N(\partial_\infty H_n, 2r_n) \subset \partial_\infty X$ is an arc with endpoints $x_n, y_n \in A(\partial_\infty H_n, r_n, 2r_n)$, and $r_n \leq e^{-\epsilon d(w, H_n)}/2n$, but there is no arc

$$J \subset A(\partial_\infty H_n, r_n/n, 2r_n n)$$

with endpoints x_n and y_n that $4r_n n$-follows $I_n[x_n, y_n]$.

Note that, as in Lemma 5.7, we may assume that $d(w, H_n)$ is bounded, and in fact that $H_n = H$ is constant, and that $\rho(x_n, y_n)$ is bounded away from zero. This is because we can use the action of G to shift H close to the origin, and the action of $g_n P_n g_n^{-1}$ to "zoom in" on x_n and y_n, both using Lemma 4.6.

We fix n, and build our desired arc J from x_n to y_n in stages. Let $L_1 \geq 2$ be as in Lemma 5.7. We can find a chain of points $x_n = z_0, z_1, z_2, \ldots, z_m = y_n$ in $A(\partial_\infty H_n, r_n/L_1, 2r_n)$ so that for every i, $\rho(z_i, z_{i+1}) \leq 4r_n$, and so that $\{z_i\}$ $2r_n$-follows $I_n[x_n, y_n]$. (The definition of ι-follows is extended from arcs to chains in the obvious way.)

This is done as follows: extract from $I_n[x_n, y_n]$ a chain of points $\{z_i'\}$ that 0-follows $I_n[x_n, y_n]$ and so that $\rho(z_i', z_{i+1}') \leq r_n$. Then whenever $\rho(z_i', \partial_\infty H) \leq r_n/L_1$, use Lemma 5.7 to find a point z_i at most $r_n/L_1 + r_n$ away from z_i', and outside $N(\partial_\infty H, r_n/L_1)$. Otherwise let $z_i = z_i'$. Adjacent points in this chain are at most $r_n + 2(r_n/L_1 + r_n) \leq 4r_n$ separated, and the new chain $2r_n$-follows the previous chain.

We claim there exists $L_2 = L_2(L_1, X)$, independent of n and H, so that any two points $u, v \in A(\partial_\infty H, r_n/L_1, 2r_n)$ satisfying $\rho(u, v) \leq 4r_n$

can be joined by an arc

$$K \subset A(\partial_\infty H, r_n/L_2, 2L_2 r_n)$$

of diameter at most $2L_2r_n$.

In fact, the compactness of $\partial_\infty X$ and the fact that $\partial_\infty H \subset \partial_\infty X$ does not locally disconnect $\partial_\infty X$ give the existence of a constant L_2 depending on n satisfying the requirement. An argument based on self-similarity of $\partial_\infty X$ around points of $\partial_\infty H$, i.e. an application of Lemma 4.6 as in Corollary 4.11, lets us rescale r_n to size at least $C(X, H, L_0) > 0$, and hence choose L_2 uniformly in n.

Using this, we find arcs J_i joining each z_i and z_{i+1} together in our chain. From this, we extract an arc J by cutting out loops: travel along J_0 until you meet J_j for some $j \geq 1$, and at that point cut out the rest of J_0 and all J_k for $1 \leq k < j$. Concatenate the remainders of J_0 and J_j together, and continue along J_j.

The resulting arc J will $2L_2r_n$-follow the chain $\{z_i\}$, and so it will $4L_2r_n$-follow $I_n[x_n, y_n]$.

This provides a contradiction for large enough n. □

6. Quasi-arcs that avoid obstacles

In this section we build quasi-arcs in a metric space that avoid specified obstacles.

Definition 6.1. *Let (Z, ρ) be a compact metric space. Let $\mathcal{V}$ be a collection of compact subsets of Z provided with some map $D : \mathcal{V} \to (0, \infty)$, which we call a* scale function*.*

The (modified) relative distance function $\Delta : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ *is defined for $V_1, V_2 \in \mathcal{V}$ as*

$$\Delta(V_1, V_2) = \frac{\rho(V_1, V_2)}{\min\{D(V_1), D(V_2)\}}.$$

*We say $\mathcal{V}$ is L-*separated *if for all $V_1, V_2 \in \mathcal{V}$, if $V_1 \neq V_2$ then $\Delta(V_1, V_2) \geq \frac{1}{L}$.*

As we saw in Section 5, we often only have control on topology on a sufficently small scale. The purpose of the scale function is to determine the size of the neighbourhood of each $V \in \mathcal{V}$ on which we have this control. An example of a scale function is $D(V) = \mathrm{diam}(V)$, if every $V \in \mathcal{V}$ has $|V| > 1$. In this case, Δ is the usual relative distance.

The goal of this section is the following result.

Theorem 6.2. *Let (Z, ρ) be a compact, N-doubling and L-linearly connected metric space. Suppose $\mathcal{V}$ is a collection of compact subsets of Z with scale function $D : \mathcal{V} \to (0, \infty)$.*

Suppose $\mathcal{V}$ is L-separated, and each $V \in \mathcal{V}$ is both L-porous and L-avoidable on scales below $D(V)$. Then for a constant $\lambda = \lambda(N, L)$ there exists a λ-quasi-arc γ in Z which satisfies $\operatorname{diam}(\gamma) \geq \frac{1}{2}\operatorname{diam}(Z)$*, and* $\rho(\gamma, V) \geq \frac{1}{\lambda}D(V)$ *for each $V \in \mathcal{V}$.*

Before continuing, we recall some results from [Mac08].

Recall that for any x and y in an embedded arc A, we denote by $A[x, y]$ the closed, possibly trivial, subarc of A that lies between them. We say that an arc A in a doubling and complete metric space is an *ι-local λ-quasi-arc* if $\operatorname{diam}(A[x, y]) \leq \lambda d(x, y)$ for all x, $y \in A$ such that $d(x, y) \leq \iota$. (See Definition 5.2 for the notion of ι-following.)

Proposition 6.3 ([Mac08, Proposition 2.1])**.** *Given a complete metric space (Z, ρ) that is L-linearly connected and N-doubling, there exist constants $s = s(L, N) > 0$ and $S = S(L, N) > 0$ with the following property: for each $\iota > 0$ and each arc $A \subset X$, there exists an arc J that ι-follows A, has the same endpoints as A, and satisfies*

$$\forall x, y \in J,\ \rho(x, y) < s\iota \implies \operatorname{diam}(J[x, y]) < S\iota. \tag{6.4}$$

Lemma 6.5 ([Mac08, Lemma 2.2])**.** *Suppose (Z, ρ) is an L-linearly connected, N-doubling, complete metric space, and let s, S, ϵ and δ be fixed positive constants satisfying $\delta \leq \min\{\frac{s}{4+2S}, \frac{1}{10}\}$. Now, if we have a sequence of arcs $J_1, J_2, \ldots, J_n, \ldots$ in Z, such that for every $n \geq 1$*

- *J_{n+1} $\epsilon\delta^n$-follows J_n, and*
- *J_{n+1} satisfies (6.4) with $\iota = \epsilon\delta^n$ and s, S as fixed above,*

then the Hausdorff limit $J = \lim_{\mathcal{H}} J_n$ exists, and is an $\epsilon\delta^2$-local $\frac{4S+3\delta}{\delta^2}$-quasi-arc. Moreover, the endpoints of J_n converge to the endpoints of J, and J ϵ-follows J_1.

Proof of Theorem 6.2. Let $r = r(L, N) > 0$ be fixed sufficiently small as determined later in the proof. Let $D_0 = \sup\{D(V) : V \in \mathcal{V}\}$; if $\mathcal{V} = \emptyset$, set $D_0 = \operatorname{diam}(Z)$. Observe that as every $V \in \mathcal{V}$ is L-porous, we have $D_0 \leq L \operatorname{diam}(Z)$. (We assume $L \geq 10$.)

We filter $\mathcal{V}$ according to size. For $i \in \mathbb{N}$, let $\mathcal{V}_n = \{V \in \mathcal{V} : r^n < D(V)/D_0 \leq r^{n-1}\}$ and $\mathcal{C}_n(K) = \{N(V, D_0r^n/K) : V \in \mathcal{V}_i\}$. As $\mathcal{V}$ is L-separated, for any $K > 2L$, each $C_n(K)$ consists of disjoint neighbourhoods.

We start with any arc J_0 in Z joining points separated by $\operatorname{diam}(Z)$, and build arcs J_n in Z by induction on n.

Assume we have been given an arc J_{n-1}. We will modify J_{n-1} independently inside the (disjoint) sets in $\mathcal{C}_n(4L)$. Let $r'_n = \frac{D_0r^n}{16L^2}$, and observe that for any $V \in \mathcal{V}_n$,

$$A(V, r'_n/L, 2r'_n/L) \subset A(V, r'_n/L^2, 3r'_nL) \subset N(V, D_0r^n/4L) \in \mathcal{C}_n(4L).$$

First, we modify the arc J_{n-1} to ensure that its endpoints lie outside $N(V, 2r'_n/L)$, for any $V \in \mathcal{V}_n$. If an endpoint lies within $2r'_n/L$ of V, use porosity to find a point in $A(V, 2r'_n/L, 2r'_n)$ within a distance of $2r'_n + 2r'_n/L$ of this endpoint, and connect these points inside $N(V, 3r'_nL)$ using linear connectedness. Now perform suitable cancellation to obtain a new arc J'_n which $3r'_nL$-follows J_{n-1}.

Second, given $V \in \mathcal{V}_n$, each time J'_n meets $N(V, r'_n/L^2)$, the arc J'_n travels through $A = A(V, r'_n/L, 2r'_n/L)$ both before and after meeting $N(V, r'_n/L^2)$. For each such meeting, we use that V is L-avoidable with "r" equal to r'_n/L to find a detour path in $A(V, r'_n/L^2, 4r'_n)$. After doing so, we concatenate the paths found into an arc J''_n, as at the end of the proof of Proposition 5.4. Note that J''_n will $4r'_n$-follow J'_n.

Now apply Proposition 6.3 to J''_n with $\iota = r'_n/2L^2$. Call the resulting arc J_n: it ι-follows J''_n, so it $(D_0r^n/4L)$-follows J_{n-1}. Observe that J_n avoids the neighbourhood of V of size $r'_n/2L^2 = D_0r^n/32L^4$.

We now find the limit of the arcs J_n. For every n, J_n $(D_0r^n/4L)$-follows J_{n-1}. Let $s' = s/8L^3$, where s is given by Proposition 6.3, then observe that J_n satisfies (6.4) with $\iota = D_0r^n/4L$, and where s is replaced by s'.

We can assume that $r \leq \min\left\{\frac{s'}{4+2S}, \frac{1}{10}\right\}$, where S is given by Proposition 6.3, since s' and S depend only on L and N.

Now apply Lemma 6.5 to the arcs J_n to find a quasi-arc γ. Provided $r \leq \frac{1}{2}$, the endpoints of γ are at most

$$\sum_{n=1}^{\infty} \frac{D_0r^n}{4L} = \frac{D_0r}{4L(1-r)} \leq \frac{D_0r}{2L} \leq \frac{\operatorname{diam}(Z)}{4},$$

from those of J_0, so $\operatorname{diam}(\gamma) \geq \frac{1}{2}\operatorname{diam}(Z)$.

Finally, for each n, γ lies in a neighbourhood of J_n of size at most

$$\frac{D_0r^{n+1}}{4L} + \frac{D_0r^{n+2}}{4L} + \cdots = \frac{D_0r^{n+1}}{4L(1-r)} \leq \frac{D_0r^n}{64L^4},$$

where this last inequality holds for $r \leq 1/32L^3$. We conclude by observing that for any $V \in \mathcal{V}_n \subset \mathcal{V}$, we have

$$\rho(\gamma, V) \geq \frac{D_0r^n}{64L^4} \geq \frac{r}{64L^4}D(V). \qquad \square$$

7. Building hyperbolic planes

In this section we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.3. Let $\mathcal{V} = \{\partial_\infty H : H \in \mathcal{H}\}$, where $\mathcal{H}$ is the collection of all horoballs of $X = X(G)$ and left cosets of the subgroups of $\mathcal{P}_2$. Define the scale function $D : \mathcal{V} \to (0, \infty)$ by $D(\partial_\infty H) = e^{-\epsilon d(w,H)}$ for each $H \in \mathcal{H}$.

The boundary $\partial_\infty(G, \mathcal{P}_1)$ is N-doubling, for some N, by Proposition 4.5.

Theorem 4.1 implies that $\partial_\infty(G, \mathcal{P}_1)$ is connected and locally connected, with no global cut points. By Proposition 4.10 $\partial_\infty(G, \mathcal{P}_1)$ is L_2-linearly connected for some $L_2 \geq 1$, while by Proposition 5.4 there exists $L_3 \geq 1$ so that for every $H \in \mathcal{H}$, $\partial_\infty H$ is L_3-avoidable on scales below $e^{-\epsilon d(w,H)}$. Let L_1 be the porosity constant set by Lemma 5.7; note that points in a connected space are automatically porous. Lemma 3.2 implies that $\mathcal{V}$ is L_4-separated, for some L_4.

We set L to be the maximum of L_1, L_2, L_3 and L_4. We apply Theorem 6.2 to build a quasi-arc γ in $\partial_\infty(G, \mathcal{P}_1)$.

Denote the standard quadrant in $\mathbb{H}^2$ by Q, and observe that $\partial_\infty Q$, with the usual visual metric, is bi-Lipschitz to the interval $[0, 1]$. Therefore by [TV80, Theorem 4.9] there is a quasi-symmetric map $f : \partial_\infty Q \to \gamma \subset \partial_\infty(G, \mathcal{P})$. By [BS00, Theorems 7.4, 8.2], this extends to give a quasi-isometrically embedded hyperbolic quadrant in $X(G, \mathcal{P}_1)$, with boundary γ.

Finally, the separation condition on γ and Proposition 3.5 give us a transversal, quasi-isometric embedding of $\mathbb{H}^2$ in $X(G, \mathcal{P})$. □

It only remains to prove Theorem 1.2.

Proof of Theorem 1.2. The only if direction of the proof is elementary, for if G admits a graph of groups decomposition with finite edge groups and virtually nilpotent vertex groups, then any quasi-isometrically embedded copy of $\mathbb{H}^2$ must map, up to finite distance, into (a left coset of) a vertex group, which is impossible.

We now reduce the if direction of Theorem 1.2 to a special case of Theorem 1.3. We can assume that $\mathcal{P}$ does not contain virtually cyclic subgroups as those can be removed from the list of peripheral subgroups preserving relative hyperbolicity. As the peripheral subgroups are finitely presented, G is finitely presented as well (see for example [Osi06]), therefore it admits a graph of groups decomposition where all edge groups are finite and all vertex groups have at most one end [Dun85, Theorem 5.1]. As all edge groups are finite, the vertex groups are undistorted and therefore they are themselves relatively hyperbolic with the obvious peripheral structure [DS05, Theorem 1.8]. By the assumptions of the theorem, one of the vertex groups, say G', is properly relatively hyperbolic, that is the peripheral subgroups are proper (virtually nilpotent) subgroups.

Bowditch shows [Bow01, Theorem 1.4] that G' admits a maximal peripheral splitting, and so we can find a vertex group G'' of this splitting so that G'' does not admit a splitting over parabolic subgroups. Such a vertex group is one ended, hyperbolic relative to one-ended virtually nilpotent subgroups $\mathcal{P}''$ (we can remove all two-ended ones, if any). Therefore, all hypotheses of Theorem 1.3 are fulfilled for G_3 with $\mathcal{P}_1 = \mathcal{P}''$ and $\mathcal{P}_2 = \emptyset$. Unfortunately G'' may be distorted in G', but the statement of Theorem 1.3 gives the existence of a *transversal* quasi-isometric embedding of $\mathbb{H}^2$. As the coned-off graph of a vertex group isometrically embeds (given a suitable choice of generators) in the coned-off graph of the whole group, an easy argument based on Proposition 2.10 shows that a transversal quasi-isometric embedding of $\mathbb{H}^2$ in a vertex group gives a quasi-isometric embedding in the whole group. □

8. Application to 3-manifolds

In this final section, we consider which 3-manifold groups contain a quasi-isometrically embedded copy of $\mathbb{H}^2$.

Lemma 8.1. *Let M be a closed graph manifold. Then $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$.*

Proof. All fundamental groups of closed graph manifolds are quasi-isometric [BN08, Theorem 2.1], so we can choose M. Consider a splitting of the closed genus 2 surface S into an annulus A and a twice-punctured torus S', as in Figure 4 below.

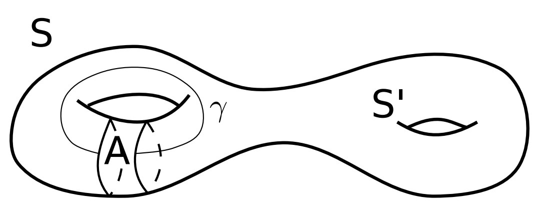

FIGURE 4. The surfaces S, S', A and the path γ.

Let M be obtained from two copies M_1, M_2 of $S' \times S^1$, in such a way that the gluings interchange the S^1 factor and the surface factor.

Now, embed S in the following way. Let γ be path connecting the boundary components of S' depicted in Figure 4, and embed A in M_1 as $\gamma \times S^1$. We can assume, up to changing the gluings, that there exists p such that $A \cap M_2 \subseteq S' \times \{p\}$. Embed S' in M_2 as $S' \times \{p\}$.

Now, we claim that S is a retract of M. If this is true then first of all $\pi_1(S)$ injects in $\pi_1(M)$, and this gives us a map $f : \mathbb{H}^2 \to \widetilde{M}$.

Also, $\pi_1(S)$ is undistorted in $\pi_1(M)$ and therefore f is a quasi-isometric embedding.

So, we just need to prove the claim. Define $g_2 : M_2 \to S' \times \{p\}$ simply as $(x,t) \mapsto (x,p)$. It is easy to see that there exists a retraction $g' : S' \to \gamma$ such that each boundary component of S' is mapped to an endpoint of γ. Let $g_1 : M_1 \to \gamma \times S^1$ be $(x,t) \mapsto (g'(x),t)$. There clearly exists a retraction $g : M \to S$ which coincide with g_i on M_i. □

Theorem 8.2. *Let M be a connected orientable closed 3-manifold. Then $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if M does not split as the connected sum of manifolds each with geometry $S^3, \mathbb{R}^3, S^2 \times \mathbb{R}$ or* Nil.

Proof. We will use the geometrisation theorem [Per02, Per03, KL08, MT07, CZ06]. It is easily seen that $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if the fundamental group of one of its prime summands does. So, we can assume that M is prime. Suppose first that M is geometric. We list below the possible geometries, each followed by yes/no according to whether or not it contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ in that case and the reason for the answer.

- S^3, no, it is compact.
- $\mathbb{R}^3$, no, it has polynomial growth.
- $\mathbb{H}^3$, yes, obvious.
- $S^2 \times \mathbb{R}$, no, it has linear growth.
- $\mathbb{H}^2 \times \mathbb{R}$, yes, obvious.
- $\widetilde{SL_2\mathbb{R}}$, yes, it is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$ (see, for example, [Rie01]).
- Nil, no, it has polynomial growth.
- Sol, yes, it contains isometrically embedded copies of $\mathbb{H}^2$.

If M is not geometric, then we have 2 cases:

- M is a graph manifold. In this case we can apply Lemma 8.1 to find the quasi-isometrically embedded $\mathbb{H}^2$.
- M contains a hyperbolic component N. As $\pi_1(N)$ is one-ended and hyperbolic relative to copies of $\mathbb{Z}^2$, by Theorem 1.3 (or by [MZ08, MZ11], upon applying Dehn filling to the manifold.) it contains a quasi-isometrically embedded copy of $\mathbb{H}^2$. This is also quasi-isometrically embedded in $\pi_1(N)$ since $\pi_1(N)$ is undistorted in $\pi_1(M)$, because there exists a metric on M such that $\widetilde{N}$ is convex in $\widetilde{M}$ (see [Lee95]). □

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