Introduction

Topology is an important area of pure mathematics and an understanding in (at least) the basics can be very helpful for many different research directions. For example, algebraic geometry, differential geometry, algebraic topology etc. That is not to say topology and its many close siblings are not an interesting and important area of active research themselves. It can be hard at first to see the reason behind some definitions and concepts. However, a somewhat useful idea to keep in mind is that topology is a sort of geometry. In topology we see a triangle and a circle as the same because we can pull and stretch a circle into a triangle. Of course, they are not the same in Euclidean geometry, where we may only translate, flip and rotate a shape before we have changed it.

We also introduce a group structure to our topological spaces to form topological groups. Although we only have time to briefly introduce them, they are incredibly well studied and show up in lots of areas of mathematics. For example, locally compact groups (whose definition we will see) have a natural measure on them, called the Haar measure. This allows one to do integration and Fourier analysis on such groups. This is applied in the study of automorphic forms, an important area of modern Number Theory.

This half of the course is intended to serve two purposes. Firstly, we introduce the fundamental definitions and ideas of topological spaces and topological groups to give a flavour of this interesting area. Secondly, we include everything needed for the second half of the course on hyperbolic geometry, one of the many areas of mathematics that uses topology.

Admin

These notes accompany the first seven lectures (and two problem classes) for the course Topics in Modern Geometry. There may be some pictures, explanation and/or nonsense said in the lectures that will not be present in these notes. However, they are designed as a good skeleton with all definitions, propositions, theorems etc. required for the exam. Please contact me (adam.thomas@bristol.ac.uk) with any corrections, suggestions or questions you have concerning these notes.

Throughout the notes there are results and claims without proof, many with the phrase “prove it” or “check this”. These are important for the reader to check their understanding. Many of these make up the exercises in Problem Sheet 1, 2 and 3. If there is still something you cannot prove or check then please ask me and we can try together!
1 Topological Spaces

Naturally, we start with the definition of a topological space.

Definition 1.1. A pair \((X, \tau)\), where \(X\) is a set and \(\tau\) is a collection of subsets of \(X\) is a topological space if:

1. \(\emptyset \in \tau\) and \(X \in \tau\),
2. if \(\{U_\alpha \mid \alpha \in A\} \subseteq \tau\) then \(\bigcup_{A} U_\alpha \in \tau\) for any set \(A\),
3. if \(\{U_i \mid 1 \leq i \leq n\} \subseteq \tau\) then \(\bigcap_{i=1}^{n} U_i \in \tau\).

The set \(\tau\) above is called the topology and the members of \(\tau\) are called the open sets. Sometimes when the topology is clear from the context we will just say \(X\) is a topological space. And we may use space to shorten topological space (yes, we are that lazy!). Before we go any further let us look at some examples to familiarise ourselves with the definition.

Example 1.2. Let \(X\) be any set.

1. Define \(\tau_1\) to be the power set of \(X\), i.e. the set of all subsets of \(X\). Then \((X, \tau_1)\) is a topological space. Indeed, this is easy to check: \(\emptyset\) and \(X\) are both in \(\tau\), and the union or intersection of a subset of \(X\) is again a subset of \(X\). This topology is called the discrete topology on \(X\).
2. Now we define a second topology on \(X\). Let \(\tau_2 = \{\emptyset, X\}\). Then it is again clear that \((X, \tau_2)\) is also a topological space. This is called the indiscrete topology (or trivial topology) on \(X\).
3. We define a third topology on \(X\). Let \(\tau_3 = \{\emptyset\} \cup \{U \subseteq X \mid X \setminus U\text{ is finite}\}\). This is called the cofinite topology on \(X\). Check it really is a topology on \(X\)! (This will be on Problem Sheet 1)

These first examples are quite simple, but they can be an important source of (counter) examples and occur naturally in many areas of mathematics. Let us introduce a hopefully self-explanatory definition.

Definition 1.3. A finite topological space is a topological space \((X, \tau)\) with \(X\) a finite set.

We next describe a large class of examples you have seen before. Let \((X, d)\) be a metric space. We will put a topology on \(X\), showing that all metric spaces are topological spaces. We define \(U \subseteq X\) to be open (i.e. \(U \in \tau\)) if it has the following property: for all \(x \in U\), there exists some \(\epsilon > 0\) such that the open ball \(B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}\) is contained in \(U\). It is a standard exercise to show that \((X, \tau)\) is a topological space. Let us see an example of this for the metric space \(\mathbb{R}\) given the usual Euclidean metric.

Example 1.4. Let \(X = \mathbb{R}\). Then \(B_\epsilon(x) = (x - \epsilon, x + \epsilon)\). It easily follows that a subset \(U \subseteq \mathbb{R}\) is open if and only if it is the union of open intervals (open in the analytic sense!), including \((-\infty, a)\) and \((b, \infty)\). This is called the standard topology (or Euclidean topology) on \(\mathbb{R}\). Similarly, we say the standard topology on \(\mathbb{R}^n\) is the topology induced from the Euclidean metric (defined by \(d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}\)).

Note that if we do not mention the topology on \(\mathbb{R}^n\) explicitly then we mean the standard topology. We have closed intervals in \(\mathbb{R}\), what about in topological spaces?

Definition 1.5. Let \((X, \tau)\) be a topological space. A subset \(V\) of \(X\) is called closed if its complement in \(X\) is open, i.e. \(X \setminus V \in \tau\).

The complement of \([a, b] \subseteq \mathbb{R}\) is \((-\infty, a) \cup (b, \infty)\), which is an open set. Therefore, the definition agrees with what we think of as a closed interval!
A word of warning is in order. Subsets of a topological space can be neither open nor closed! For example, consider the subset \((a, b] \subseteq \mathbb{R}\). On the other hand, subsets can be both open and closed. The easiest example of this is \(\emptyset\) and \(X\) in any topological space \((X, \tau)\). We will come back to sets which are both open and closed when we discuss connectedness.

Using De-Morgan’s laws, it follows that (prove it!) an arbitrary intersection of closed sets is again closed and a finite union of closed sets is again closed. Let us see some definitions and results that should seem very familiar from the metric spaces course. The proof of the lemma is an exercise.

**Definition 1.6.** Let \((X, \tau)\) be a topological space. Let \(x \in X\) and \(U \subseteq X\).

1. The **complement** (in \(X\)) of \(U\) is \(U^c = X \setminus U\).
2. A **neighbourhood** of \(x\) is any subset \(N \subseteq X\) such that \(N\) contains an open set \(V\) with \(x \in V\).
3. An **open neighbourhood** of \(x\) is any neighbourhood which is itself open.
4. The **interior** of \(U\), denoted \(\text{Int}(U)\) is the union of all open sets contained in \(U\). (This is an open set by definition)
5. The **closure** of \(U\), denoted \(\overline{U}\), is the intersection of all closed sets containing \(U\). (This is a closed set by the previous remark)

**Lemma 1.7.** Let \((X, \tau)\) be a topological space. A set \(U\) is open if and only if \(U = \text{Int}(U)\) if and only if for all \(u \in U\) there exists a neighbourhood of \(u\) contained in \(U\). A set \(U\) is closed if and only if \(U = \overline{U}\). Finally, the interior of a subset can be computed from closures and complements, namely \(\text{Int}(U) = X \setminus (X \setminus U)\).

The following equivalent definitions (prove they are equivalent!) shows some of the interplay between open and closed sets.

**Definition 1.8.** A subset \(U\) of a topological space \(X\) is **dense** if \(\overline{U} = X\). Equivalently, a subset \(U\) is dense if every non-empty open set of \(X\) has non-empty intersection with \(U\).

For example, \(\mathbb{Q}\) is dense in \(\mathbb{R}\) (prove this!). We have seen that we can put more than one topology on a set. Can we compare these?

**Definition 1.9.** Let \(X\) be a space and \(\tau_1\) and \(\tau_2\) be two topologies on \(X\). If \(\tau_1 \subseteq \tau_2\) then we say that \(\tau_1\) is **coarser** then \(\tau_2\) and that \(\tau_2\) is **finer** than \(\tau_1\). If the containment is proper we say **strictly finer** or **strictly coarser**, as appropriate.

Note that the discrete topology is always the finest topology and the indiscrete topology is always the coarsest topology on a set \(X\).

**Example 1.10.** Consider \(X = \mathbb{R}\). Then we have already seen four topologies on \(X\), namely the discrete, standard, cofinite and indiscrete topologies. Can we compare the standard topology, say \(\tau_1\), and the cofinite topology, \(\tau_2\)? We claim that \(\tau_2 \subseteq \tau_1\) and hence the standard topology is finer than cofinite topology. Suppose \(U \in \tau_2\). Then, by definition, \(X \setminus U\) is finite, and let us call the points \(x_1, \ldots, x_n\), with \(x_1 < \cdots < x_n\). Then \(U = (-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, \infty)\). So \(U\) is a union of open sets and hence \(U \in \tau_1\). To see the containment is proper, consider the open set \(V = (0, 1) \in \tau_1\). Then \(X \setminus V\) is certainly not finite and hence \(V \not\in \tau_2\).

Finally, we note that given more than one topology on a set we can intersect them to form another topology (check this is a topology). This topology will be coarser than the original two topologies. There are many examples to show that the union of two topologies need not be a topology.
2 Subspaces, Continuity and Homeomorphisms

In this lecture we consider the question of when two topological spaces are “the same”. To do this we need to define maps between topological spaces that in some way preserve the structure, which in this case is the topology. Before we do this let us see how to put a topology on subsets of topological space; this will give us more examples for later on.

**Definition 2.1.** Let \((X, \tau)\) be a topological space and suppose \(Y \subseteq X\). Then we define the **subspace topology** on \(Y\) to be \(\tau_Y = \{ U \cap Y \mid U \in \tau \}\), i.e. open sets of \(Y\) are the intersection of open sets of \(X\) with \(Y\).

**Example 2.2.** Let \(X = \mathbb{R}\) with the standard topology.

1. Let \(Y = (a,b)\). Then the open sets of the subspace topology on \(Y\) are intersections of unions of open intervals with \((a,b)\). These are just unions of open intervals contained in \((a,b)\).

2. Let \(Y = [a,b)\). Then as before unions of open intervals contained in \(Y\) are open. But in this case \([a,c) = (-\infty, c) \cap Y\) and hence \([a,c)\) is open for all \(a < c \leq b\).

3. The subspace topology on \(\mathbb{Z}\) is discrete (prove this!).

4. We can define the \(n\)-sphere as follows: \(S^n = \{ x \in \mathbb{R}^{n+1} \mid d(x,0) = 1 \}\) and we endow it with the subspace topology from the standard topology on \(\mathbb{R}^{n+1}\). What are the open sets of \(S^1\)?

We now move on to looking at functions between topological spaces. We do not want to allow any old maps, just as for vector spaces we concentrate on linear maps and for groups we concentrate on homomorphisms. For topological spaces we have continuous functions:

**Definition 2.3.** Suppose that \(X\) and \(Y\) are topological spaces. A function \(f : X \rightarrow Y\) is **continuous** if for every open set \(U \subseteq Y\), the preimage \(f^{-1}(U) = \{ x \in X \mid f(x) \in U \}\) is open.

We could have made the same definition replacing open with closed in both places. Prove this is an equivalent definition. You should also check that the composition of continuous functions is again continuous and prove the following lemma, which uses continuity to give an alternative definition for the subspace topology.

**Lemma 2.4.** Let \(X\) be a topological space and \(Y \subseteq X\). The subspace topology is the coarsest topology on \(Y\) such that the inclusion map \(Y \hookrightarrow X\) is continuous.

You have already seen a definition of continuous functions between metric spaces. Precisely, if \((X, d_X)\) and \((Y, d_Y)\) are metric spaces, a function \(f : X \rightarrow Y\) is continuous at \(x \in X\) if for all \(\epsilon > 0\) there exists \(\delta > 0\) such that whenever \(z \in X\) with \(d_X(x, z) < \delta\) we have \(d_Y(f(x), f(z)) < \epsilon\). A function \(f : X \rightarrow Y\) is continuous if it is continuous at every point \(x \in X\). It is a good exercise to show that this definition coincides with our definition for topological spaces, when we consider \(X\) and \(Y\) as topological spaces, as in Example [1.4]. This allows us to immediately conclude many of our favourite functions from \(\mathbb{R}^n \rightarrow \mathbb{R}^m\) (or subsets thereof) are continuous. For example, polynomials, exponentials, trigonometric functions etc.

The next definition seems similar at first to that of a continuous function, however it is not the same!

**Definition 2.5.** Suppose that \(X\) and \(Y\) are topological spaces. A function \(f : X \rightarrow Y\) is **open** (resp. **closed**) if for every open (resp. closed) set \(U \subseteq X\), the image \(f(U) \subseteq X\) is open (resp. closed).

We want to think of two topological spaces as the same if we can get from one to the other and back again via stretching and bending. The customary example is a coffee cup and a doughnut. If you haven’t seen
it there is an animation here [https://en.wikipedia.org/wiki/Homeomorphism]

**Definition 2.6.** Suppose that $X$ and $Y$ are topological spaces. A function $f : X \to Y$ is a *homeomorphism* if it is bijective, continuous and its inverse $f^{-1}$ is also continuous. We say that $X$ and $Y$ are *homeomorphic* if there exists a homeomorphism from $X$ to $Y$, and we sometimes write $X \cong Y$.

Note that a continuous bijection $f$ is a homeomorphism if and only if $f$ is open (or closed). Before we see some examples of homeomorphisms let us remark two easy consequences of the definition. The first lemma is an easy exercise.

**Lemma 2.7.** Let $X, Y, Z$ be topological spaces. Suppose $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms. Then $g \circ f : X \to Z$ is a homeomorphism.

**Lemma 2.8.** The relation $X \cong Y$ if and only if $X$ is homeomorphic to $Y$ is an equivalence relation on the class of topological spaces.

*Proof.* We need to prove that the relation is (i) reflexive ($X \cong X$), (ii) symmetric ($X \cong Y$ then $Y \cong X$), and (iii) transitive ($X \cong Y$ and $Y \cong Z$ then $X \cong Z$). For (i), note that the identity map is a homeomorphism $X \to X$. For (ii), note that $f : X \to Y$ is a homeomorphism then $f^{-1} : Y \to X$ is a homeomorphism. Finally, (iii) follows from the previous Lemma 2.8.

Here are some examples of homeomorphic spaces.

**Example 2.9.** 1. $\mathbb{R}$ with the standard topology and $\mathbb{R}_{>0}$ with the subspace topology

The function in this case is the exponential function $f(x) = e^x$. This is a bijection and since $\mathbb{R}$ and $\mathbb{R}_{>0}$ are metric spaces, we can appeal to real analysis to see that the exponential function is continuous. Moreover, the inverse of $f$ is the logarithm function $f^{-1}(x) = \ln(x)$, which is also continuous. This example shows a topological space can be homeomorphic to a proper subspace!

2. Any two finite open intervals with the subspace topology from $\mathbb{R}$

By transitivity, we need only show that any open interval is homeomorphic to $(0,1)$. Consider the following function: $f : (0,1) \to (a,b)$, with $f(x) = a + x(b-a)$. Then $f$ is clearly bijective and since it is a polynomial in $x$ it is continuous. The inverse function is $f^{-1} : (a,b) \to (0,1)$, with $f^{-1}(x) = \frac{(x-a)}{(b-a)}$. Again this is continuous since it is a polynomial function.

3. $\mathbb{R}$ with the standard topology and $(0,1)$ with the subspace topology

Show that $f_1 : (0,1) \to \mathbb{R}$, with $f_1(x) = \tan(\pi(x - \frac{1}{2}))$ is a homeomorphism.

To show that two spaces $X$ and $Y$ are not homeomorphic can be difficult. For example, how do you show that $(0,1)$ with the standard topology and $(0,1)$ with the trivial topology are not homeomorphic? Or how do you know/show $(0,1)$ and $(0,1]$ are not homeomorphic? (The answer will be given in lecture 5 when we discuss connectedness)

For this reason, we want many properties about topological spaces to be invariant under homeomorphism, i.e. if $X$ has the property, and $X$ is homeomorphic to $Y$ then $Y$ has the property.

**Definition 2.10.** A *topological property* is a property invariant under homeomorphism.

For a list of topological properties far longer than those covered in the course, see [https://en.wikipedia.org/wiki/Topological_property](https://en.wikipedia.org/wiki/Topological_property) for examples.
3 Bases, Countability, the Hausdorff condition and Manifolds

In this lecture we discuss notions that tell us we do not have “too many” open sets but also “enough”! We will use our new definitions to be describe what we mean by a topological manifold. We don’t have much time to discuss them but they are incredibly important in mathematics, especially when one adds more structure to them, for example differentiable manifolds or Lie groups.

When discussing open sets of \( \mathbb{R} \) it is somewhat cumbersome to say “unions of open intervals”, especially when we then consider the subspace topology on intervals. Instead we would like to just say open intervals are the “building blocks” of all open sets. This leads to the definition of a base.

**Definition 3.1.** Let \( X \) be a set and suppose \( \sigma \) is a collection of subsets of \( X \). Then the topology generated by \( \sigma \) is the intersection of all topologies which contain \( \sigma \).

**Definition 3.2.** Let \((X, \tau)\) be a topological space. Then a subset \( \sigma \subseteq \tau \) is a basis for \( \tau \) if every set in \( \tau \) can be written as a union of sets in \( \sigma \).

Note that the empty set is the union of the “empty collection” of sets in \( \sigma \).

**Example 3.3.** The open balls, \( B_\epsilon(x) \), form a base for the induced topology on any metric space (check this!). Hence the open intervals form a base for the standard topology on \( \mathbb{R} \).

In fact, we can make a smaller base for \( \mathbb{R}^n \) by taking open balls around rational points (each coordinate is rational) with rational radii.

**Definition 3.4.** A topological space \( X \) is said to be second countable if it has a basis consisting of countably many open sets, i.e. it has a countable base.

This shows that the standard topology on \( \mathbb{R}^n \) is second countable. The notion of second countability puts restraints on the size of our topology. There are other, weaker conditions we could impose. These are first countability, separable and Lindelöf. We will not need the precise details but all of them are satisfied in a second countable space.

Now we turn our attention to a notion of having “enough” open sets, a so called separation condition. There are many separation conditions one can impose on a topological space. We will cover the most common one, which is the Hausdorff condition.

**Definition 3.5.** A topological space \((X, \tau)\) is Hausdorff if for every pair \( x \neq y \in X \) there exists neighbourhoods \( N_x \) of \( x \) and \( N_y \) of \( y \) such that \( N_x \cap N_y = \emptyset \).

We want to check that the Hausdorff condition is a topological property.

**Theorem 3.6.** The Hausdorff condition is a topological property.

**Proof.** It suffices to prove that if \( f : X \to Y \) is a homeomorphism and \( Y \) is Hausdorff then \( X \) is Hausdorff. So let \( x_1, x_2 \in X \) be two disjoint points. Then \( f(x_1) \neq f(x_2) \) since \( f \) is a bijection. By definition, there exists open sets \( U, V \) such that \( f(x_1) \in U \) and \( f(x_2) \in V \) with \( U \cap V = \emptyset \). We take the preimages under \( f \) of \( U \) and \( V \), which since \( f \) is a homeomorphism is equivalent to taking the image of \( U \) and \( V \) under the continuous map \( f^{-1} \). By construction we have \( x_1 \in f^{-1}(U) \) and \( x_2 \in f^{-1}(V) \). The intersection \( f^{-1}(U) \cap f^{-1}(V) \) is empty since \( U \cap V \) is empty. Moreover, \( f^{-1}(U) \) and \( f^{-1}(V) \) are open since \( f \) is continuous. Therefore, we have found our two disjoint neighbourhoods of \( x_1 \) and \( x_2 \) and hence \( X \) is Hausdorff. \( \square \)
Note that different authors use separated or $T_2$ instead of Hausdorff. Although our neighbourhoods are not assumed to be open, they contain an open set. So replacing ‘neighbourhood’ with ‘open neighbourhood’ does not change the definition. It is not something that needed much discussion in metric spaces. This is because all metric spaces are Hausdorff, as you will prove on Problem Sheet 2. In fact, this helps us prove something you may have wondered about: are all topological spaces just metric spaces? It now suffices to find an example of a topological space which is not Hausdorff (you can find a topology on $X = \{0, 1\}$ that does this). In light of this we make the following definition.

**Definition 3.7.** A topological space $(X, \tau)$ is **metrisable** if it is homeomorphic to a metric space, i.e. there exists a metric $d$ on $X$ such that the topology induced by $d$ is $\tau$.

As you may expect, it turns out to be quite a hard problem to decide if an arbitrary topological space is metrisable and we will not cover that in this course. The next lemma shows that the Hausdorff condition is inherited by subspaces, when given the subspace topology. Whenever we talk about a subspace of a topological space we are endowing the subspace with the subspace topology unless mentioned otherwise.

**Lemma 3.8.** Let $X$ be a Hausdorff topological space and suppose $Y \subseteq X$. Then $Y$ is a Hausdorff topological space when endowed with the subspace topology.

**Proof.** Let $y_1, y_2$ be disjoint points of $Y$. Then since $X$ is Hausdorff there exist disjoint open neighbourhoods $U_1, U_2$ of $y_1, y_2$, respectively. Define $V_i = U_i \cap Y$. Then $V_i$ is an open neighbourhood of $y_i$ in $Y$, noting that the definition of subspace topology ensures that $V_i$ is open in $Y$. Finally, $V_1 \cap V_2 = (U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \cap Y = \emptyset \cap Y = \emptyset$. Therefore, $Y$ is Hausdorff.

We end this lecture by giving the definition of a topological $n$-manifold. The idea is that we want our topological space to behave nicely and look locally like $\mathbb{R}^n$ (and $\mathbb{R}^n$ is the archetypal topological manifold!).

**Definition 3.9.** A topological space $M$ is **locally Euclidean of dimension $n$** if every point $x \in M$ has a neighbourhood $N_x$ such that $N_x$ is homeomorphic to an open subset of $\mathbb{R}^n$ (with the standard topology).

**Definition 3.10.** A topological space $M$ is an **$n$-dimensional topological manifold** is a second countable Hausdorff space that is locally Euclidean of dimension $n$.

Some authors allow manifolds without a certain dimension $n$ attached, that is to say different points can have neighbourhoods homeomorphic to open subsets of $\mathbb{R}^m$ for different $m$.

**Example 3.11.**

1. $\mathbb{R}^n$.
2. Any open subset of an $n$-manifold.
3. $S^n$ is an $n$-dimensional manifold (hence the notation $S^n \subset \mathbb{R}^{n+1}$)
4. Any countable discrete topological space is a 0-manifold.

You may be wondering about non-examples. A good one to think about is the union of the $x$-axis and the $y$-axis in $\mathbb{R}^2$. The problem here is that the origin has no neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$ (although we can’t prove it yet).
4 Product Spaces and Topological Groups

We have seen that given a topological space \((X, \tau)\) and any subset \(Y \subset X\) we can get a “new” topological space using the subspace topology. New is in quotation marks because \((0,1)\) is homeomorphic to \(\mathbb{R}\), for example, so it isn’t really new in that case. In this section we look at a further construction of new spaces from old ones. Once we have done that we turn our attention to adding extra structure, specifically a group structure, to our topological spaces in a compatible way.

Firstly, we consider how to combine many topological spaces into a new one. Of course, we do this in many parts of mathematics. For example consider direct sums of vector spaces, direct products of groups, or the Cartesian product of sets. We will define a topology on a possibly infinite Cartesian product of topological spaces.

**Definition 4.1.** Suppose \((X_i, \tau_i)\) are topological spaces for each \(i \in I\). Let \(X = \prod_{i \in I} X_i\) be the Cartesian product of the sets and let \(p_i : X \to X_i\) be the canonical projection maps for each \(i\). Then the **product topology** on \(X\) is the coarsest topology for which all \(p_i\) are continuous.

It follows that the open sets in the product topology are unions of sets of the form \(\prod_{i \in I} U_i\) where \(U_i \in \tau_i\) and \(U_i \neq X_i\) for only finitely many \(i\). If \(I\) is finite, sets of the form \(\prod_{i \in I} U_i\) for \(U_i \in \tau_i\) is a base for the product topology.

The **box topology** on a Cartesian product of topological spaces is the topology with base \(\prod_{i \in I} U_i\) where \(U_i \in \tau_i\). Of course, this agrees with the product topology on finite products, but is in general finer. There are many reasons for choosing the product topology as the right topology to put on an infinite Cartesian product. One of which is that Theorem 5.14, presented later, is not true if you endow the Cartesian product with the box topology.

**Example 4.2.** Take \(n\) copies of \(\mathbb{R}\) with the standard topology. Then, as we would hope, the product topology on \(\mathbb{R}^n\) agrees with the standard topology. Indeed, we can check that the topology generated from a base consisting of all products of open intervals is the same as the Euclidean topology.

Now we have everything we need to define a topological space that also has a group structure.

**Definition 4.3.** A **topological group** \(G\) is a triple \((G, \tau, \cdot)\) where \((G, \cdot)\) is a group, \((G, \tau)\) is a topological space and the functions \(\iota : G \to G\), with \(\iota(g) = g^{-1}\) and \(m : G \times G \to G\) (where \(G \times G\) is endowed with the product topology), with \(m(g_1, g_2) = g_1 \cdot g_2\) are continuous maps.

If the group operation is clear from the context we will drop the \(\cdot\) from the notation and similarly if the topology is clear from the context we will drop the \(\tau\). Let us see some examples.

**Example 4.4.** 1. Any group can be made into a topological group by endowing it with either the discrete topology or the indiscrete topology. If the topology is discrete we say that \(G\) is a **discrete group**.

2. The standard topology on \(\mathbb{R}^n\) considered with usual coordinate wise addition is an (abelian) topological group.

3. Many matrix groups over \(\mathbb{R}\) or \(\mathbb{C}\) are topological groups. For example, take \(G = \text{GL}_n(\mathbb{C})\), which is the group of \(n \times n\) matrices with entries in \(\mathbb{C}\) of non-zero determinant. Then \(G\) is a topological space when considered as a subspace of \(\mathbb{C}^{n^2}\). We need to check that matrix multiplication and inversion are continuous. Cramer’s rule shows that inversion is a rational map in each entry and is therefore continuous. Similarly, matrix multiplication is defined by polynomials in each entry and is also continuous.
4. Any subgroup $H$ of a topological group $G$ is a topological group itself, when endowed with the subspace topology (check this!).

5. $\text{SL}_n(\mathbb{C})$, $\text{Sp}_{2n}(\mathbb{C})$, $\text{SL}_n(\mathbb{R})$, 

6. $\text{SL}_2(\mathbb{Z})$ is an important example. It is a discrete subgroup of $\text{SL}_2(\mathbb{R})$.

Example 4.5 (A non-example!). Consider the group $G = \mathbb{Z}/2\mathbb{Z}$ of order 2. Define a topology $\tau = \{\emptyset, \{0\}, \{0, 1\}\}$. Check that $(G, \tau, +)$ is not a topological group. (Hint: is the addition map continuous?)

We want to be able to say what it means for two topological groups to be the same. As one might expect, we demand that they are the same as topological spaces and as groups.

Definition 4.6. Let $G, H$ be topological groups. A map $f : G \to H$ is a topological homomorphism if it is a continuous map and also a group homomorphism.

Definition 4.7. Let $G, H$ be topological groups. We say that $G$ and $H$ are topologically isomorphic if there exists a map $f : G \to H$ which is both a homeomorphism and a group isomorphism.

Now we know what it means for two topological groups to be different, we can show that two isomorphic groups can be topologically non-isomorphic.

Example 4.8. Let $X = \mathbb{R}$ and consider $X$ as a group under addition. We endow $X$ with two different topologies. We let $\tau_1$ be the discrete topology and we let $\tau_2$ be the standard topology. We want to show that $G_1 = (X, \tau_1, +)$ is not topologically isomorphic to $G_2 = (X, \tau_2, +)$. In $G_1$, singletons $\{x\}$ are open whereas in $G_2$ they are not (the only open sets are unions of open intervals). Any homeomorphism $f$ between $G_1$ and $G_2$ would send singletons to singletons (it is bijective) and send open sets to open sets (the inverse of $f$ is continuous) therefore $f$ does not exist.

Example 4.9. Given a finite number of topological groups $G_1, \ldots, G_n$ we can form the product topological group $G = G_1 \times \cdots \times G_n$. As a set it is the Cartesian product, the group structure is the direct product and the topological structure is the product topology.

Groups act on themselves in a number of ways, left multiplication, right multiplication, inversion and conjugation. The same is true about topological groups and the action is in fact a homeomorphism of the topological space.

Lemma 4.10. Let $G$ be a topological group. Then $L_g : G \to G, x \mapsto gx$ and $R_g : G \to G, x \mapsto xg$ are homeomorphisms.

Proof. Let $i_g : G \to G \times G, x \mapsto (g, x)$, which is continuous and let multiplication be denoted by $m$. Then $L_g = m \circ i_g$ and is hence continuous. Moreover, $L_g$ is bijective (check this!). It remains to check that the inverse of $L_g$ is continuous. But $L_{g^{-1}} \circ L_g = \text{Id}_G$ and hence the inverse of $L_g$ is just $L_{g^{-1}}$, which is continuous.
5 Compact and Locally Compact Spaces

In this lecture we see two important examples of topological properties. Compactness will sound familiar from the metric spaces course. However, since a topological space is more general we will need a new definition. We should keep in mind that we are trying to generalise closed and bounded subsets of a metric space in the context of topological spaces. For example, we will show that any real-valued continuous map from a compact space is bounded and attains its bounds (Extreme Value Theorem). We start with some preliminary definitions.

**Definition 5.1.** Let \((X, \tau)\) be a topological space and suppose \(Y \subseteq X\). We say that \(\{U_\alpha\}_{\alpha \in A}\) is a **cover** of \(Y\) if \(Y \subseteq \bigcup_{\alpha \in A} U_\alpha\). The cover is called an **open cover** if \(U_\alpha \in \tau\) for all \(\alpha \in A\). Given a cover \(\{U_\alpha\}_{\alpha \in A}\) and a subset \(B \subseteq A\) then \(\{U_\beta\}_{\beta \in B}\) is a **subcover** if \(Y \subseteq \bigcup_{\beta \in B} U_\beta\).

Note that since a cover is defined to be set we do not have to worry about repetitions!

**Definition 5.2.** Let \((X, \tau)\) be a topological space. We say that \(X\) is **compact** if every open cover of \(X\) has a finite subcover, i.e. if \(X = \bigcup_{\alpha \in A} U_\alpha\) with \(U_\alpha \in \tau\) for all \(\alpha \in A\), then there exists a finite subset \(B \subseteq A\) such that \(X = \bigcup_{\beta \in B} U_\beta\).

**Definition 5.3.** Let \((X, \tau)\) be a topological space. We say that a subset \(Y \subseteq X\) is compact if \(Y\), endowed with the subspace topology, is a compact space. Immediately, that says that given \(Y \subseteq \bigcup_{\alpha \in A} U_\alpha\) with \(U_\alpha \in \tau\) for all \(\alpha \in A\), then there exists a finite subset \(B \subseteq A\) such that \(Y \subseteq \bigcup_{\beta \in B} U_\beta\).

**Example 5.4.** The Heine-Borel Theorem (as you have seen in the Metric Spaces course) states that a subset of \(\mathbb{R}^n\) is compact if and only if it is closed and bounded. Therefore \(\mathbb{R}^n\) itself is not compact. Neither is \((0, 1) \subseteq \mathbb{R}\) nor \([0, \infty) \subseteq \mathbb{R}\).

**Example 5.5.** A trivial example of a compact space is any non-empty subset \(Y\) of an indiscrete topological space \(X\). Indeed, the only open cover of \(Y\) will be \(\{Y\}\), which is already finite!

**Theorem 5.6.** Let \(X, Y\) be topological spaces and \(f : X \to Y\) be a continuous map. If \(X\) is compact then so is \(f(X)\), i.e. the continuous image of a compact space is compact.

**Proof.** Suppose that \(\{U_\alpha\}_{\alpha \in A}\) is an open cover of \(f(X)\). Then \(\{f^{-1}(U_\alpha)\}_{\alpha \in A}\) is an open cover of \(X\). Since \(X\) is compact, there exists a finite subset \(B \subseteq A\) such that \(\{f^{-1}(U_\beta)\}_{\beta \in B}\) is a finite subcover. Hence \(\{U_\beta\}_{\beta \in B}\) is still a cover of \(f(X)\) and hence a finite subcover, as required. \(\square\)

**Corollary 5.7.** Compactness is a topological property.

We now use Corollary 5.7 to prove the Extreme Value Theorem.

**Theorem 5.8** (Extreme Value Theorem). Suppose \(X\) is a compact topological space. Let \(f : X \to \mathbb{R}\) be a continuous map. The image of \(f\) is bounded and \(f\) attains the bounds.

**Proof.** The continuous image of a compact space is compact. So by the Heine-Borel Theorem \(f(X)\) is closed and bounded. The result now easily follows. \(\square\)

We can slightly relax the compactness condition as follows.
Definition 5.9. Let \((X, \tau)\) be a topological space. We say that \(X\) is \textit{locally compact} if every point \(x \in X\) has a compact neighbourhood.

We will prove in Problems Class 2 that local compactness is a topological property. Let us turn our attention to topological groups. We mentioned locally compact groups in the introduction. We now have all of the parts to define them: they are locally compact Hausdorff topological groups. It is easier to check local compactness (and many other local conditions) for topological groups. It follows from a property known as the homogeneity (of the action of \(G\) on itself via left multiplication).

Proposition 5.10. A topological group \(G\) is locally compact if and only if the identity has a compact neighbourhood.

Proof. If a topological group is locally compact then every point has a compact neighbourhood so the identity certainly does. For the reverse implication, consider the map \(L_g : G \to G\) defined by \(x \mapsto gx\). This map is continuous as seen in Lemma 4.10. By Corollary 5.7, the continuous image of a compact set is compact. Therefore, given any point \(g \in G\) we obtain a compact neighbourhood of \(g\) by taking the image under \(L_g\) of the compact neighbourhood of the identity.

Example 5.11. The topological group \(\mathbb{R}^n\) is locally compact, even though it is not compact. Indeed, we have the following compact neighbourhood of the origin \(\prod_{i=1}^{n} [-\epsilon, \epsilon]\) for any \(\epsilon > 0\).

On Problem Sheet 3 you will prove that \(\text{GL}_2(\mathbb{R})\) is also locally compact.

The following result shows that subsets of a compact space are forced to be compact whenever they are closed.

Lemma 5.12. Let \((X, \tau)\) be a compact space and let \(V \subseteq X\) be a closed subset. Then \(V\) is compact.

Proof. Let \(\{U_\alpha\}_{\alpha \in A}\) be an open cover of \(V\). Then \(X \setminus V\) is by definition open and hence \(\{U_\alpha\}_{\alpha \in A} \cup (X \setminus V)\) is an open cover of \(X\). Since \(X\) is compact, there exists a finite subcover \(\{U_\beta\}_{\beta \in B} \cup (X \setminus V)\). Hence \(\{U_\beta\}_{\beta \in B}\) is a finite subcover of \(V\), as required.

Theorem 5.13. Let \(X\) be a Hausdorff space and let \(Y \subseteq X\) be a compact subset. Then \(Y\) is closed.

Proof. We need to show that \(C = X \setminus Y\) is open. Fix \(c \in C\). Since \(X\) is Hausdorff, for every \(y \in Y\) there exist disjoint open sets \(U_y\) and \(V_y\) such that \(c \in U_y\) and \(y \in V_y\). Therefore, \(\{V_y \mid y \in Y\}\) is an open cover of \(Y\) and the compactness of \(Y\) asserts that there exists a finite subcover \(\{V_y \mid y \in F\}\) for some finite subset \(F \subseteq Y\).

It now follows that \(U = \bigcap_{y \in F} U_y\) is an open neighbourhood of \(c\) (noting that is a \textit{finite} intersection hence open) disjoint from \(Y\), i.e. contained in \(C\). Therefore \(C\) contains an open neighbourhood of every point in \(C\), and is hence open by Lemma 1.7.

We finish the lecture by stating a result on products of compact spaces. The proof is reasonably straightforward for a finite number, or even a countable number of compact spaces and you it will appear on an exercise sheet.

Theorem 5.14 (Tychonoff’s Theorem). Suppose \((X_i, \tau_i)\) are compact topological spaces for each \(i \in I\). Then \(X = \prod_{i \in I} X_i\), endowed with the product topology is compact.
6 Connected and Path Connected Spaces

In this lecture we study two more topological properties. The notion of path connectedness is very natural, when can you join two points in a topological space. But there is a weaker condition, that of connectedness which is slightly less intuitive but very important!

**Definition 6.1.** Let $X$ be a topological space. Then $X$ is **connected** if the only sets that are both open and closed are $\emptyset$ and $X$. If $X$ is not connected then it is **disconnected**.

There are many equivalent definitions of a connected space. We summarise a few of them in the following lemma.

**Lemma 6.2.** Let $X$ be a topological space. The following conditions are equivalent.

1. $X$ is connected.
2. There does not exist non-empty disjoint open sets $U$ and $V$ with $X = U \cup V$.
3. All continuous functions from $X$ to $\{0,1\}$ are constant, where $\{0,1\}$ is given the discrete topology.

**Proof.** We will show 2 implies 3, via the contrapositive. The rest are similar and left as an exercise.

Let $f : X \rightarrow \{0,1\}$ be a non-constant continuous function. Define $U = f^{-1}(0)$ and $V = f^{-1}(1)$. Then $U$ and $V$ are non-empty since $f$ is not constant, their union is clearly $X$ and they are both open since $f$ is continuous.

Let us see some examples to check our understanding of the definitions.

**Example 6.3.** Let $X$ be a set.

1. Endow $X$ with the discrete topology. Assuming $|X| > 1$ then $X$ is disconnected. This is because there exists a non-empty set $U$ with $V = X \setminus U$ also non-empty. Clearly $U$ and $V$ are disjoint and they are both open since every subset is open.
2. Endow $X$ with the indiscrete topology. Then $X$ is connected since the only open sets are $\emptyset$ and $X$ and hence the only open and closed sets are $\emptyset$ and $X$.
3. Suppose $X$ is infinite and endow $X$ with the cofinite topology. Suppose $U$ is a non-empty open proper subset of $X$. Then $X \setminus U$ is finite and non-empty, therefore not open. We have shown that the only open and closed sets are $\emptyset$ and $X$, hence $X$ is connected.

**Example 6.4.** An interval in $\mathbb{R}$ is connected. We will see this in Problem Class 2. In fact the only connected subsets of $\mathbb{R}$ are intervals (and singletons being $[x,x]$).

**Definition 6.5.** Let $X$ be a topological space and let $x, y \in X$. A path from $x$ to $y$ is a continuous function $f : [0,1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$.

**Definition 6.6.** A topological space $X$ is **path connected** if there exists a path between any two points in $X$.

**Example 6.7.** Let $X = \mathbb{R}^n$ with the standard topology. Let $x, y \in \mathbb{R}^n$. Then consider the function $f : [0,1] \rightarrow \mathbb{R}^n$ given by $f(i) = x + i(y - x)$. Then $f$ is continuous, $f(0) = x$ and $f(1) = y$. Therefore, $\mathbb{R}^n$ with the standard topology is path connected.
We would hope that $\mathbb{R}^n$ with the standard topology is also connected. The following proposition allows us to conclude that.

**Proposition 6.8.** Let $X$ be a path connected topological space. Then $X$ is connected.

**Proof.** Suppose $X$ is not connected, looking for a contradiction. Then there exists non-empty, disjoint, open sets $U$ and $V$ such that $X = U \cup V$. Let $u \in U$ and $v \in V$. Since $X$ is path connected, there exists a continuous function $f : [0,1] \to X$ with $f(0) = u$ and $f(1) = v$. Consider the preimages $f^{-1}(U)$ and $f^{-1}(V)$. They are open since $f$ is continuous, and they are disjoint because $U$ and $V$ are disjoint. Moreover, since $X = U \cup V$ we must have $[0,1] = f^{-1}(U) \cup f^{-1}(V)$. But this shows that $[0,1]$ is a disconnected, a contradiction by Example 6.4.

The reverse implication is not true in general. The “Topologist’s Sine Curve” is the customary example of a connected space which is not path connected. It is a subset of $\mathbb{R}^2$ given by $\{(0) \times [-1,1] \cup \{(x, \sin(\frac{1}{x}) | x \in (0,1]\}$. We won’t prove that it is indeed a connected but not path connected space - proofs can be found in many books/websites. However, in $\mathbb{R}^n$ if a subset is connected and open then it is path connected.

We have discussed connectedness and path connectedness but we need to show that they are both topological properties if they are going to be much use to us.

**Lemma 6.9.** Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be a continuous function. Then if $X$ is connected (resp. path connected) then $f(X)$ is connected (resp. path connected), i.e. the continuous image of a connected (resp. path connected) space is connected (resp. path connected).

**Proof.** We give the proof for path connectedness, the other is an exercise.

Suppose $y_1, y_2 \in f(X)$. Then there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$ and there exists a continuous function $g : [0,1] \to X$ with $g(0) = x_1$ and $g(1) = x_2$. The composition $f \circ g : [0,1] \to Y$ is a continuous function with $f \circ g(0) = f(x_1) = y_1$ and $f \circ g(1) = y_2$. Therefore $f(X)$ is path connected.

The following corollary is an immediate consequence.

**Corollary 6.10.** Connectedness and path connectedness are topological properties.

**Example 6.11.** Corollary 6.10 shows that $\mathbb{R}^n$ is not homeomorphic to $(0,1) \cup (3,4) \subset \mathbb{R}$ for example. But we can also show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for all $n \geq 2$. We will do this in Problem Class 2 but it follows in a similar way to the next example.

**Example 6.12.** In this example we show that $X = (0,1)$ and $Y = (0,1]$ are not homeomorphic (as subspaces of $\mathbb{R}$). Both $X$ and $Y$ are connected so that isn’t going to help us. But what if we remove a point from $X$ and $Y$? Suppose $f : X \to Y$ is a homeomorphism. Then let $f^{-1}(1) = x$. So $A = X \setminus \{x\} = (0,x) \cup (x,1)$ is now disconnected. But the restriction of $f$ to $A$ is a homeomorphism between $A$ and $(0,1)$ (check this!). So we now have a homeomorphism between a disconnected space and a connected space, a contradiction.
7 Quotient Spaces and Quotient Groups

For the final lecture we turn our attention to another way of creating new topological spaces from old ones. We want to be able to identify points of our space, sort of gluing them together, to obtain a new space. An idea to keep in mind is gluing together two ends of a piece of string to form a circle - we will make this rigorous in an example later.

The rigorous notion for identifying points in our set will be an equivalence relation. We are then “quotienting” out by this relation i.e. we are saying that all equivalent points should be considered as a single point in our new set. This leads us to the following definition.

**Definition 7.1.** Let \((X, \tau)\) be a topological space and suppose \(\sim\) is an equivalence relation on \(X\). The quotient space \(X/\sim\) is the set of equivalence classes of \(X\) under \(\sim\), i.e. \(X/\sim = \{[x] : x \in X\}\) where \([x] = \{y \mid x \sim y\}\).

Now we have a set, the quotient space, we need to put a topology on it.

**Definition 7.2.** Let \((X, \tau)\) be a topological space and suppose \(\sim\) is an equivalence relation on \(X\). We define the quotient map to be the function \(q : X \to X/\sim\) sending \(x\) to \([x]\). Let \(U\) be a subset of \(X/\sim\). Then \(U\) is defined to be open if and only if \(q^{-1}(U)\) is an open subset of \(X\).

Equivalently, the quotient topology is the finest topology on \(X/\sim\) such that the quotient map is continuous.

Let us record the immediate but important result.

**Lemma 7.3.** The quotient map is continuous.

Examples of quotient spaces are very easy to come by - we just define any equivalence relation we like on the underlying set \(X\).

**Example 7.4.** Firstly let us consider the quotient space of certain sets by an equivalence relation.

1. Suppose \(X = \mathbb{R}\) and let \(x \sim y\) if and only if \(x - y\) is an integer. So what are the equivalence classes here? Well a representative of each one can be given by a real number in the interval \([0, 1)\), for example. So each equivalence class is \([i] = i + \mathbb{Z}\) for \(i \in [0, 1)\).

2. Suppose \(X = \mathbb{R}^2\) and let \((x_1, x_2) \sim (y_1, y_2)\) if and only if \(x_1^2 + x_2^2 = y_1^2 + y_2^2\). Then the points in our quotient space are circles of a given radius. So \([x_1, x_2] = \text{circle of radius } \sqrt{x_1^2 + x_2^2}\).

However, once we put the quotient topology on our quotient space can we identify these somewhat mystical quotient spaces with well known spaces? The following theorem will help us to do that.

**Theorem 7.5.** Let \((X, \tau)\) be a compact space and \((Y, \mu)\) be a Hausdorff space. Suppose that \(f : X \to Y\) is a continuous bijection. Then \(f\) is a homeomorphism.

**Proof.** We need to prove that the inverse of \(f\) from \(Y\) to \(X\), call it \(g\), is continuous. (note that \(g\) is well-defined since \(f\) is bijective) We use the equivalent definition of continuous that says the preimage of a closed set is closed. Let \(U\) be a closed set of \(X\). We wish to show that \(g^{-1}(U)\) is closed. But \(g^{-1}(U) = f(U)\) and so we are really just showing that \(f\) is a closed map, i.e. takes closed sets to closed sets.

Let \(V \subseteq X\) be closed. Firstly, \(V\) is compact by Lemma 5.12. Secondly, Corollary 5.7 shows that \(f(V)\) is a compact subspace of \(Y\). Finally, a compact subspace of a Hausdorff space is closed by Theorem 5.13, hence \(f(V)\) is closed, as required.

\(\square\)
Example 7.6. Let $I = [0, 1]$ and define an equivalence relation via $x \sim y$ if and only if $x = y$ or $\{x, y\} = \{0, 1\}$. We claim that $I/\sim$ is homeomorphic to $S^1$. We define a map $f : I \to S^1$ by $x \mapsto (\cos(2\pi x), \sin(2\pi x))$. This is a continuous map. Moreover, it is not bijective but it is surjective and the only point in $S^1$ with a preimage larger than one point is $s = (1, 0) \in \mathbb{R}^2$. The preimage of $s$ is $\{0, 1\}$, which are exactly the points we are gluing together in our quotient space. Therefore, if we define $\tilde{f} : I/\sim \to S^1$ by $[x] \mapsto f(x)$, we actually have a well-defined continuous map, which is also bijective. Now we apply Theorem 7.5 noting that $I$ is compact, the quotient map is continuous and thus $I/\sim$ is also compact.

7.1 Topological Quotient Groups

This final part is non-examinable.

Now we have seen some quotient spaces, can we combine them with the construction of a quotient group to make quotient topological groups? Recall the definition of a normal subgroup.

Definition 7.7. A subgroup $H$ of a group $G$ is called normal if $g^{-1}Hg = H$ for all $g \in G$.

We can quotient a group by a normal subgroup.

Lemma 7.8. Let $G$ be a group and let $N$ be a normal subgroup. Then the set of cosets $gN$ form a group under the multiplication law $gN \cdot hN = ghN$. We call this the quotient group $G/N$.

We also need an equivalence relation.

Definition 7.9. Let $G$ be a topological group and let $N$ be a normal subgroup. Define an equivalence relation on $G$ by $g \sim h$ if and only if $gh^{-1} \in N$, i.e. $g$ is equivalent to $h$ if and only if $g$ and $h$ are in the same coset. In this case we write $G/N$ for $G/\sim$.

It turns out that the quotient map in this case is open.

Lemma 7.10. Let $G$ be a topological group and let $N$ be a normal subgroup. The quotient map $q : G \to G/N$ is open.

Proof. Let $U$ be an open subset of $G$. We need to show that $q(U)$ is open in $G/N$. Thus, by definition of the quotient topology, we need to show $q^{-1}(q(U))$ is open in $G$. Suppose $x \in q^{-1}(q(U))$. Then $xN \in q(U)$ which is true if and only if there exists $u \in U$ and $n \in N$ with $x = un$. Therefore

$$q^{-1}(q(U)) = \bigcup_{n \in N} Un$$

and we are done if we show that $Un$ is open for all $n \in N$. The identity $e$ is in $N$ and so $Ue = U$ is open. Now, $Un$ is the image of $U$ under the map $R_n$, which is a homeomorphism by Lemma 4.10 and we are done.

Theorem 7.11. Let $G$ be a topological group and let $N$ be a normal subgroup. Then $G/N$ is a topological group, where the set is the cosets of $N$, the group structure is the quotient group $G/N$ and the topology is the quotient topology with the equivalence relation $g \sim h$ if and only if $gh^{-1} \in N$.

The proof of the Theorem is not too difficult but we don’t have room for it in the course. It uses the fact that the quotient map is continuous which we have just proved. It also requires a result proving that to check whether a homomorphism between two topological groups $G_1$ and $G_2$ is continuous, it suffices to check whether the preimage of an open neighbourhood of the identity of $G_2$ is open.

We finish with a brief example.
Example 7.12. Let $G = \mathbb{R}$ (under addition and with the standard topology) and let $N = \mathbb{Z}$. Since $G$ is abelian all subgroups are normal. So we can form the quotient group $\mathbb{R}/\mathbb{Z}$. In fact, $\mathbb{R}/\mathbb{Z}$ is topologically isomorphic to $S^1$. 