On the average least prime negative Hecke eigenvalue

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Motivation

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The Legendre symbol is defined as follows. For an odd prime p and an integer a,

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue,} \\ -1, & \text{if } a \text{ is a quadratic non-residue,} \\ 0, & \text{if } a \text{ is a multiple of p.} \end{cases}$$

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Trivially, 0 and 1 will always be quadratic residues.

Question

For each prime p, when is $\left(\frac{\cdot}{p}\right)$ first negative?

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Theorem (Erdős, 1961)

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} n_2(p) = \sum_{k=1}^{\infty} \frac{p_k}{2^k}.$$

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This result has been extended in many directions such as modulo general m, for k^{th} powers modulo p, for general Dirichlet character, etc.

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Then $f \in S_k^{\text{new}}(N)$ is a **newform** if it is normalised and a Hecke eigenform. The space of newforms is finitely generated. We denote a finite generating set as $\mathbf{H}_{\mathbf{k}}^{*}(\mathbf{N})$.

Newforms

A newform $f \in H_k^*(N)$ has the Fourier expansion,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \quad e(nz) = e^{2\pi i nz}.$$

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So we may associate an angle $\theta_f(p) \in [0, \pi]$ such that,

$$\lambda_f(p) = 2\cos(\theta_f(p))$$

Analogous question

Let $f \in H_k^*(N)$, and $\lambda_f(n)$ be the n^{th} Fourier coefficeint of f.

Question

When on average is the first sign change of $\lambda_f(p)$ for prime p, $p \nmid N$?

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When on average is the first sign change of $\lambda_f(p)$ for prime p, $p \nmid N$?

Let p_f denote the first prime such that $\lambda_f(p_f) < 0$.

Theorem (V. soon)

Let N be prime, and $k \geq 2$ be even and fixed. Then,

$$\lim_{N \to \infty} \frac{1}{|H_k^*(N)|} \sum_{f \in H_k^*(N)} p_f = \sum_{i=1}^{\infty} \frac{p_i}{2^i}$$

Where p_i denotes the i^{th} prime.

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- (Standard methods) The signs of $\lambda_f(n)$ change infinitely often
- (Kohnen and Sengupta, 2006) $n_f \ll kN \exp\left(c\sqrt{\log N / \log\log 3N}\right) \log^{27} k$
- (Iwaniec, Kohnen and Sengupta, 2007) $n_f \ll Q^{29/60}$
- (Kowalski, Lau, Sound., Wu 2010) $n_f \ll Q^{9/20}$
- (Matomaki, 2012) $n_f \ll Q^{3/8}$

Evidence

Recall $\lambda_f(p) = 2\cos(\theta_f(p))$ for some $\theta_f(p) \in [0, \pi]$.

The Sato-Tate conjecture

For $f \in H_k^*(N)$, the associated $\theta_f(p)$ are equidistributed with respect to the Sato-Tate measure,

$$\mu_{ST} = \frac{2}{\pi} \sin^2 \theta \mathrm{d}\theta.$$

Proved 2011, Barnet-Lamb, Geraghty, Harris, Taylor, ...

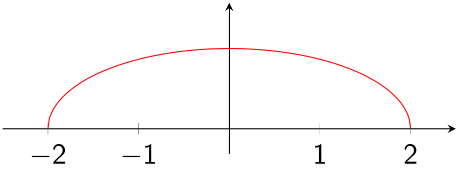

Figure: Distribution of $\lambda_f(p)$

Sketch proof

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Split the sum into two parts,

$$\sum_{f\in H_k^*(N)} p_f = \sum_{\substack{f\in H_k^*(N)\\ p_f\leq z}} p_f + \sum_{\substack{f\in H_k^*(N)\\ p_f > z}} p_f \,.$$

$z = c\sqrt{(\log kN)(\log\log kN)}$.

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Recall $\lambda_f(p) = 2\cos(\theta_f(p))$ for some $\theta_f(p) \in [0, \pi]$.
For the first sum we use the Sato-Tate distribution of the $\theta_f(p)$.

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The z-product of Chebyshev functions,

$$X_n(\theta) = \frac{\sin((n+1)\theta)}{\sin\theta} \quad \theta \in [0, \pi].$$

form an orthnoromal basis of $L^2([0, \pi]^z, \mu_{ST}^{\otimes z})$.

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$$\prod_{\substack{p \le z \\ (p,N)=1}} X_{n_p}(\theta_f(p)) = \lambda_f\Big(\prod_{\substack{p \le z \\ (p,N)=1}} p^{n_p} \Big).$$

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Then we can show,

$$\sum_{f \in H_k^*(N)} \omega_f \prod_{\substack{p \leq z \\ (p,N)=1}} Y_p(\theta_f(p)) = 0 + \text{error}.$$

Where ω_f is a suitable weight, $Y_p(\theta) = \sum_{j=1}^{s} y_p(j) X_j(\theta)$.

Then for finite z,

$$\frac{|f \in H_k^*(N) : \varepsilon_p \lambda_f(p) \geq 0, p \leq z, p \nmid N|}{|H_k^*(N)|} = \frac{|A(z)|}{|H_k^*(N)|} \to \mu_{ST}([0, \pi/2])^{\pi_N(z)}$$

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$$\sum_{\substack{f \in H_k^*(N) \\ p_f \leq z}} p_f = \sum_{p_i \leq z} p_i |A(p_i)|$$

$$\leq |H_k^*(N)| \sum_{i=1}^{\infty} \frac{p_i}{2^i} + o(|H_k^*(N)|),$$

provided z grows slow enough, hence the choice of z:

$$z = c\sqrt{(\log kN)(\log\log kN)}$$

Second sum

For the second sum we use large sieve type inequalities.

One of the sieves (Lau, Wu) - simplified

For $\{b_p\}_p$ a sequence of real numbers indexed by primes with $|b_p| \leq B$, $B > 0$.

$$\sum_f \left| \sum_{P<p\leq Q} b_p \frac{\lambda_f(p)}{p} \right|^{2j} \ll k\varphi(N) \left(\frac{384B^2 j}{P \log P} \right)^j + (kN)^{10/11} \left(\frac{10BQ^{1/10}}{\log P} \right)^{2j}.$$

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The dependence on k is from the bound $\langle f, f \rangle \ll_k N(\log N)^3$, used in a different large sieve variant.

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Apply the large sieve with $b_p = 2\varepsilon_p$ for $p \in (P, 2P]$, $p \nmid N$, and 0 otherwise.

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Using $\lambda_f(p)^2 = 1 + \lambda_f(p^2)$,

$$LHS \geq \sum_{f \in \mathcal{E}} \left(\sum_{P < p \leq 2P} \frac{1}{p} - \Big| \sum_{P < p \leq 2P} \frac{\lambda_f(p^2)}{p} \Big| \right)^{2j}$$

(Note: all sums over p have the condition $p \nmid N$)

Let $\mathcal{E}' := \{f \in H_k^*(N) : \left| \sum_{P<p\leq 2P} \frac{\lambda_f(p^2)}{p} \right| \geq \frac{\delta}{2\log P}\}$, where δ is from $\sum_{a<p\leq 2a} 1/p \geq \delta/\log a$ via standard calculations.

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$$LHS \geq \sum_{f\in\mathcal{E}\setminus\mathcal{E}'} \left(\sum_{\substack{P<p\leq 2P \\ p\nmid N}} \frac{1}{p} - \frac{\delta}{2\log P} \right)^{2j}.$$

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We bound the inner sum,

$$\geq \sum_{P<p\leq 2P} \frac{1}{p} - \sum_{\substack{P<p\leq 2P \\ p|N}} \frac{1}{p} - \frac{\delta}{2\log P} \geq \frac{\delta}{2\log P} - \frac{\omega(N)}{P} \geq \frac{\delta - 2}{2\log P}.$$

If N is prime.

Let $\mathcal{E}' := \{f \in H_k^*(N) : \left|\sum_{P<p\leq 2P} \frac{\lambda_f(p^2)}{p}\right| \geq \frac{\delta}{2\log P}\}$, where δ is from $\sum_{a<p\leq 2a} 1/p \geq \delta/\log a$ via standard calculations. Then,

$$LHS \geq \sum_{f\in\mathcal{E}\setminus\mathcal{E}'} \left(\sum_{\substack{P<p\leq 2P \\ p\nmid N}} \frac{1}{p} - \frac{\delta}{2\log P} \right)^{2j}.$$

We bound the inner sum,

$$\geq \sum_{P<p\leq 2P} \frac{1}{p} - \sum_{\substack{P<p\leq 2P \\ p|N}} \frac{1}{p} - \frac{\delta}{2\log P} \geq \frac{\delta}{2\log P} - \frac{\omega(N)}{P} \geq \frac{\delta-2}{2\log P}.$$

If N is prime. So we have,

$$|\mathcal{E}\setminus\mathcal{E}'| \ll kN \left(\frac{1536 j \log P}{(\delta-2)^2 P}\right)^j + (kN)^{10/11} P^j$$

Thank you for listening!
Any questions?